On relaxation and transport in gyrokinetic drift wave turbulence with zonal flow

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We present a theory for relaxation and transport in phase space for gyrokinetic drift wave turbulence with zonal flow. The interaction between phase space eddys and zonal flows is considered in two different limits, namely for $K \gg 1$ and $K \simeq 1$ where K is the Kubo number. For $K \gg 1$, the growth of an isolated coherent phase space structure is calculated, including the associated zonal flow dynamics. For $K \simeq 1$, mean field relaxation dynamics is considered in the presence of phase space granulations and zonal flows. In both limits, it is shown that the evolution equations for phase space structures are structurally similar to a corresponding Charney-Drazin theorem for zonal momentum balance in a potential vorticity conserving, quasi-geostrophic system. The transport flux in phase space is calculated in the presence of phase space density granulations and zonal flows. The zonal flow exerts a dynamical friction on ion phase space density evolution, which is a fundamentally new zonal flow effect. [doi:10.1063/1.3662428]

I. INTRODUCTION

Turbulent relaxation and transport are important issues for fusion plasmas. Conventionally, the relaxation process is thought to begin with free energy stored in plasma inhomogeneity being released by linear instability of drift waves. Transport or mean field evolution due to the instability is usually described via a quasilinear calculation,[1] which assumes a spectrum of eigenmodes only, and thus treats turbulence as an ensemble of waves.[2] In terms of dimensionless numbers, this conventional approach is valid for Kubo number $K \equiv \tilde{v}\tau_c/\Delta_c \ll 1$ where $\tilde{v}$ is the typical velocity, τ_c is the correlation time, Δ_c is the correlation length. Despite usual practice, the conventional approach is not compatible with the mixing length theory[2] predictions—which are standard estimates for saturation levels—since in the saturated state we expect from mixing length theory that $\tilde{v} \sim \Delta_c/\tau_c$, so turbulence is characterized by $K \sim 1$. Moreover, $K \gtrsim 1$ can result for turbulence with non-mode like fluctuations such as vortices, eddys, blobs, etc., with $\tau_L \gtrsim \tau_{cir}$ (Fig. 1), where $\tau_L \sim \tau_c$ is a life time of field pattern and $\tau_{cir} \sim \Delta_c/\tilde{v}$ is an eddy circulation time. Thus, since turbulence is often in a state with $K > 1$ or at least $K \sim 1$, turbulence driven relaxation and transport should be analyzed in such regimes. There exist attempts[3,4] to characterize transport in such a case; however, they usually study transport for a given, fixed spectrum of turbulence, without linking the structures inherent to $K \gtrsim 1$ with the turbulence *dynamics*. Since transport necessarily evolves profiles which in turn evolves turbulence, we need a self-consistent model of turbulent transport for $K \gtrsim 1$.

For $K \gtrsim 1$, kinetic plasma turbulence models, such as 1D Vlasov or gyrokinetic (GK) models used heavily in the fusion community, exhibit the existence of phase space structures. In the 1D Vlasov plasmas, $K \gtrsim 1$ corresponds to the state where the effect of particle trapping is important, since the Kubo number can be recast as $K = \tilde{v}\tau_c/\Delta_c \sim \tau_{ac}/\tau_b$ where $\tau_{ac}^{-1} = |d\omega/dk - \omega/k|\Delta k$ is the auto-correlation time of a packet and $\tau_b^{-1} \sim \tilde{v}/\Delta_c \sim k\sqrt{(q\phi/m)}$ is bounce frequency of particles trapped in potential trough (for 1D Vlasov models). Particle trapping leads to formation of phase space structures, such as Bernstein-Green-Kruskal (BGK) eddys,[5] phase space holes,[6–8] clumps or granulations,[7–9] etc, which can be important players for relaxation and transport. As argued by Dupree[10] and Kadomtsev,[11] phase space structures can emit a wake of waves via Cerenkov emission, an effect which necessarily appears as dynamical friction in the phase space density evolution equation (Fig. 2). In this view, the emission is treated much like that from a test particle[12] and the emitting structure is viewed as a kind of macro particles, albeit one with a finite lifetime. Thus, turbulence driven mean field evolution equation is altered from a pure diffusive type to a Lenard-Balescu type, with *both diffusion and dynamical friction entering the evolution of* $\langle f \rangle$ *(Table I)*. The mean field evolution has been applied to current carrying Vlasov plasma, and the origin of anomalous resistivity has been linked to momentum exchange mediated by structures,[10] in addition to the conventional approach based on waves.[13]

The formation of structures[7,8,14] and its effect on transport[15,16] are also discussed for inhomogeneous phase space turbulence, which is relevant to the problems in confined plasma transport. The ideas of phase space density granulation and dynamical friction are applied to collisionless ion-temperature-gradient (ITG) turbulence.[16] In that analysis, the authors argued that the relaxation process inherently produces ion phase space density granulations, which then experience dynamical friction via the wake effect due to Cerenkov emission. The dynamical friction due to dissipative electrons is found to cause anomalous transport of ion energy and particles. On the

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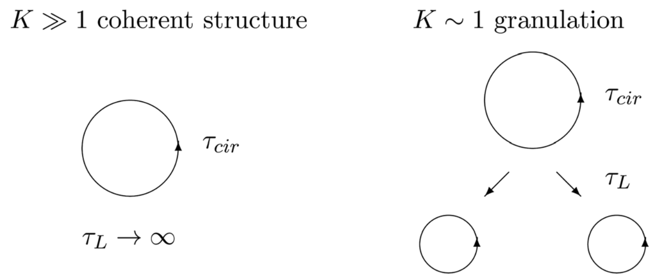

FIG. 1. $K \gg 1$ and $K \simeq 1$.

other hand, the dynamical friction due to polarization charge is not included in that analysis, as the authors naively thought that the mixing of ion guiding center phase space is not coupled to polarization charge mixing; $\langle \tilde{v}_r \delta f_i \rangle \sim \langle \tilde{v}_r \tilde{n}_{GC,i} \rangle \sim \langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \rangle$ and $\langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \rangle \sim \mathrm{Re} \sum_k ik_\theta k_\perp^2 |\tilde{\phi}|_k^2 \to 0$.

However, *we do know that the polarization charge flux is tied to Reynolds force via the identity,*[17,18] $\langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \rangle = \partial_r \langle \tilde{v}_r \tilde{v}_\theta \rangle$, *which usually is non-zero*. The analysis of the reference[16] overlooked the envelope and zonal flow scale. The issue may be clarified by noting that there are several spatial scales inherent to drift wave turbulence (Fig. 3), which are as follows:

- Mode fluctuation scale $l_c \sim k_r^{-1}$, where l_c is the typical correlation scale and k_r is the mode wave number.
- Fluctuation spatial envelope scale of fluctuation Δ_{env}, where $\Delta_{env} \sim (|\tilde{\phi}_k|^{2'}/|\tilde{\phi}_k|^2)^{-1}$.
- Avalanche size Δ_{ava}, where $\Delta_{ava} > l_c$. An avalanche involves intermittent interaction of several neighboring fluctuation envelopes.
- Profile scale, L_f where $L_f \equiv (\langle f \rangle'/\langle f \rangle)^{-1}$.

Usually $k_r^{-1} < \Delta_{env} \lesssim \Delta_{ava} < L_f$. Since the wake has a finite spatial extent (Fig. 2), it naturally introduces an effective envelope scale to the fluctuation dynamics. The envelope scale can also be set by mode propagation and absorption points, the excitation profile, plasma profile curvature, etc. The envelope dependence alters the radial variation of the fluctuations from $\partial_r \tilde{\phi} \sim ik_r \phi_k e^{ik_r r}$ to $\partial_r \tilde{\phi} \sim (ik_r + \partial_R)\phi_k\,(R) e^{ik_r r}$, where $|ik_r| \gg |\partial_R|$ and r, R are associated with fast fluctuation variation and slow envelope variation, respectively. Hence the envelope dependence replaces $k_r \to k_r - i\partial_R$, which effectively can be thought of as an $\mathrm{Im} k_r$. This leads to $\mathrm{Im} k_\perp^2 \neq 0$ and thus $\langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \rangle \neq 0$. This is plausible since the envelope variation also implies a non-zero Reynolds force, $\partial_r \langle \tilde{v}_r \tilde{v}_\theta \rangle$. Thus, by

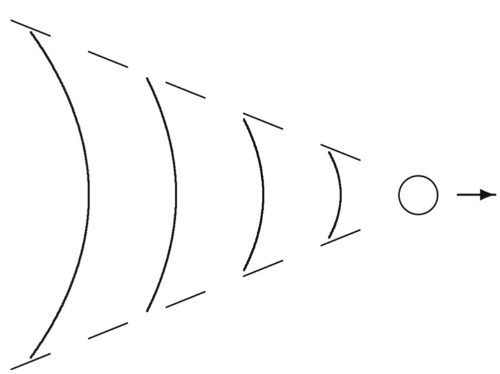

FIG. 2. Cerenkov emission and wake.

TABLE I. Turbulence with waves vs turbulence with structures.

	Turbulence with waves	Turbulence with structures
Fluctuations	Eigenmode	Non-mode like
	Drift wave,…	BGK eddy, clump, hole,…
Instability	Growth of mode	Growth of structure
Mean evolution	Quasilinear diffusion	Diffusion + Dynamical friction, drag

accounting for the slow envelope variation, dynamical friction appears from the polarization charge, which induces direct zonal flow coupling to relaxation and transport by phase space density granulation. We note that Δ_{ava} could also be a relevant scale for zonal flow variation;[19] however, we do not further consider avalanching here.

In this paper, we discuss relaxation and transport in inhomogeneous phase space turbulence with zonal flow in the limit $K \gtrsim 1$. We consider a simple model[20] for GK drift wave turbulence,

$$\partial_t f + v_d \partial_y f + \{\phi, f\} = C(f), \tag{1a}$$

$$\alpha_e(\phi - \langle \phi \rangle_y) - \rho^2 \nabla_\perp^2 \phi = \frac{2}{n_{eq}\sqrt{\pi}} \int_0^\infty dE \sqrt{E} f - 1, \tag{1b}$$

with heat flux Q matched according to

$$Q = -\chi_{coll} \langle T \rangle' + \frac{2}{\sqrt{\pi}} \int dE \sqrt{E} E \langle \tilde{V}_r \delta f \rangle, \tag{1c}$$

where χ_{coll} is a collisional thermal conductivity. The model describes basically 2D drift wave dynamics, with a trapped particle precession resonance. Parallel acceleration is annihilated by bounce averaging, which enables us to focus on pure spatial mixing due to $E \times B$ drift. Equation (1a) is bounce averaged kinetic equation for trapped ions. $v_d = v_{d,0} E/T_i$ is an energy dependent magnetic precession drift velocity.[21] The Poisson bracket accounts for $E \times B$ convection. Equation (1b) is GK Poisson equation,[22,23] which accounts for polarization charge. Although the model is very simplified as compared to the full GK description, it does contain a minimal representation of all the relevant effects we need here. Namely, polarization charge introduces zonal flow coupling to the model, since any mixing of f leads to the mixing of $\nabla^2 \phi$, which in turn leads to Reynolds forcing via the Taylor identity;[17] $\langle \tilde{v}_r \delta f_i \rangle \sim \langle \tilde{v}_r \tilde{n}_{GC,i} \rangle \sim \langle \tilde{v}_r \nabla_\perp^2 \phi \rangle = \partial_r \langle \tilde{v}_r \tilde{v}_\theta \rangle$. Also, $K \gtrsim 1$ is easily possible in the model, since the correlation time for particle and spectra can become long, i.e., since

$$\Delta(\omega - \omega_d) \cong \Delta k_\theta \left| \frac{d\omega}{dk_\theta} - \frac{\omega}{k_\theta} \right|, \tag{2}$$

then $\tau_{ac} \sim (|d\omega/dk_\theta - \omega/k_\theta| \Delta k_\theta)^{-1}$. Given the weakly dispersive nature of long wavelength drift wave turbulence, it is very easy to have long τ_{ac} and thus Kubo number $K = \tau_{ac}\tilde{v}/\Delta_c \gtrsim 1$ is possible. Hence phase space structure formation can be expected in this model.

In the remainder of the paper, we consider the two different limits of the model described above, $K \gg 1$ and $K \sim 1$. First we consider the strong resonant limit for $K \gg 1$, which is applicable when a single structure can form (Fig. 1). In this strongly resonant limit, we show that a phase space structure, as well as a wave, carries a pseudomomentum,[24] and can exchange it with the zonal flow. To see this, we consider the growth of a single structure in the ion phase space density, in the presence of electrons and polarization charges. There, based on the invariance of the total dipole moment, we show that a single ion phase space density structure cannot avoid zonal flow coupling. The pseudomomentum associated with structures is identified as the negative of the kinetic wave activity density, $-\int dE\sqrt{E}\langle\delta f_i^2\rangle/\langle f\rangle'|_0$. The structure growth equation with zonal flow is shown to be closely related to the Charney-Drazin, (C-D) theorem[18,25,26] for potential vorticity (PV) conserving quasigeostrophic (QG) system, which is the fundamental momentum constraint for a system with turbulence and zonal flows. This is due to the fact that even single structure dynamics is necessarily tightly coupled to zonal flows.

Then we move to the limit $K \sim 1$, where structures can form but also break up (Fig. 1), thus forming a statistical ensemble of granulations.[9,12,15,16] Granulations are similar to fluid eddys, albeit in phase space, and constitute an incoherent part of fluctuations which enters relaxation and transport process as dynamical friction. We consider the dynamics of granulation and its effect on transport, in the presence of zonal flow coupling. The dynamics is described by a statistical theory based on the 2 point correlation,

$$\left(\frac{\partial}{\partial t} + T(1,2)\right)\langle\delta f(1)\delta f(2)\rangle = P(1,2). \tag{3a}$$

Here $\langle\delta f(1)\delta f(2)\rangle$ is phase space density correlation function, called the "phasetrophy,"[12] since it is analogous to enstrophy in QG turbulence. $T(1,2)$ determines the life time of correlation due to relative streaming, $\tilde{v}_{E\times B}$ scattering, and collisions. $P(1,2) \sim -\langle\tilde{v}_r\delta f\rangle\langle f\rangle'$ is the production of phasetrophy due to the relaxation process. In the following analysis, we show that production due to polarization charge introduces zonal flow coupling to the statistical granulation dynamics. The granulation evolution with zonal flow is compared to the C-D theorem for the QG system. We argue that the granulation evolution equation with zonal flow takes the form of the prey equation in the general predator-prey system. Thus, as other drift wave-zonal flow (DW-ZF) turbulence systems, granulations and zonal flows also form a self-regulating system in phase space. We also derive the transport flux associated with the granulation induced relaxation process. The mean field evolution is extracted from $P(1,2)$ by noting that $df/dt = 0$ and $d/dt\langle\delta f^2\rangle = -\partial_t\langle f\rangle^2$. For drift turbulence,

$$\partial_t\langle f\rangle = -\partial_r\langle\tilde{v}_r\delta f\rangle = -\partial_r[-D\partial_r\langle f\rangle + F\langle f\rangle]. \tag{3b}$$

Here D is analogous to the familiar quasilinear diffusion term, and F is the dynamical friction term, which arises from granulation. We show that dynamical friction due to zonal flow is non-zero. Specifically, dynamical friction accounts for the contribution to $P(1,2)$ from the fluctuation Reynolds work on the zonal flow.

The remainder of the paper is organized as follows. In Sec. II, single structure growth is formulated with zonal flow and linked to the C-D momentum theorem. Section III treats the case of multiple structures, using statistical theory. Phase space density granulation evolution is formulated in the presence of zonal flow. The connection of the result to C-D momentum theorems, as well as the implications for transport, is discussed. Section IV presents conclusions and discussion.

II. SINGLE PHASE SPACE STRUCTURE AND ZONAL FLOW

In this section, we discuss the interaction between a single phase space structure and zonal flows, a situation which corresponds to the limit $K \gg 1$. In this limit, particle trapping becomes important and a coherent structure, such as a hole or a blob, emerges. Such a structure can grow when the background distribution has a gradient due to inhomogeneity, since total f must be conserved along a trajectory; for example, a hole in phase space can grow when propagating up a background mean gradient (Fig. 4). In the following, we consider the structure growth dynamics[6,14] and show that a structure is dynamically coupled to zonal flow since radial transport and thus growth of a structure necessitates a flux of polarization charge, so as to maintain charge balance.

Here, we consider the structure dynamics in the model[20] described above

$$(\partial_t + v_D\partial_y)f + \left\{\frac{c}{B}\phi, f\right\} = C(f), \tag{4a}$$

$$\frac{\delta n_i}{n_0} + \rho_s^2\nabla_\perp^2\frac{q\phi}{T_e} = \frac{\delta n_e}{n_0}. \tag{4b}$$

The first equation is a bounce kinetic equation for the guiding center ion distribution, with an energy dependent drift velocity[21] $v_D = \bar{v}_D\bar{E}$ where $\bar{E} \equiv E/T_i$ and the Poisson bracket $\{\phi, f\} \equiv \partial_x\phi\partial_y f - \partial_y\phi\partial_x f$. Note that magnetic trapping does not allow correlated particles to disperse in the parallel direction. The second equation is the Gyrokinetic Poisson equation,[22,23] which includes polarization charge. Given the weakly dispersive character of long wave length drift waves in the model, the correlation time $\tau_{ac} \sim (|d\omega/dk_\theta - \omega/k_\theta|\Delta k_\theta)^{-1}$ can become long. For example, if $\omega = \sqrt{\epsilon}\omega_*/(1+\rho_s^2k_\perp^2)$, we have

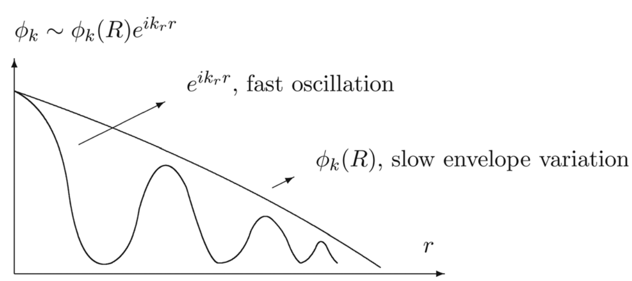

FIG. 3. Envelope modulation.

$$\tau_{ac} \sim \left(\frac{2\sqrt{\epsilon} k_\theta^2 \rho_s^2}{(1 + k_\perp^2 \rho_s^2)^2} v_* \Delta k_\theta \right)^{-1}. \tag{5}$$

Then we easily see that τ_{ac} can be long for drift waves with $k_\theta \rho < 1$, even if the k spectrum is broad, i.e., $\Delta k_\theta \rho \sim 1$. Hence, the Kubo number can be large $K = \tau_{ac} \tilde{v} / \Delta_c \gg 1$ for this model. In this limit with strongly coherent resonances, it seems likely that a coherent structure, such as hole, blob or clump, can form[7,8] (Fig. 1).

Given the possibility of structure formation, we consider the dynamics of a fluctuation $\delta f((x - x_0)/\Delta x, (E - E_0)/\Delta E)$, which is localized in a phase space point (x_0, E_0) with an extent Δx and ΔE, Fig. 4. Here x_0 is the location of a structure we are considering and E_0 is the energy at resonance, i.e., $\omega - \bar{\omega}_d \bar{E}_0 = 0$. $\Delta x \sim \Delta_c$ is the size of a structure in radial direction and the width in velocity space ΔE is estimated from $\bar{\omega}_d \Delta \bar{E} \sim \tau_{circ}^{-1} \sim \tilde{v}_E / \Delta_c$, so $\Delta \bar{E} \sim \tilde{v}_E / (\bar{\omega}_d \Delta_c)$.

A structure in phase space can grow as depicted in Fig. 4, whose dynamics is described by δf^2 evolution.[6,14] Since Eq. (4a) can be written as $df/dt = C(f)$, it implies $df^2/dt = 2fC(f)$. Writing $f = f_0 + \delta f$ and assuming $\partial_t \delta f / \delta f \gg \partial_t f_0 / f_0$, the δf^2 evolution is obtained as

$$\partial_t \int d^3 v \langle \delta f_i^2 \rangle = -2 \frac{d}{dt} \int d^3 v \langle \delta f_i f_{i,0} \rangle + 2 \langle f C(f) \rangle. \tag{6a}$$

Here, f_0 includes the background mean distribution $\langle f \rangle$ as well as the depletion due to the structure. δf is a perturbation. $\langle \ldots \rangle$ denotes an ensemble average, which corresponds to the zonal average in y direction here. As in Fig. 4, the distortion $\delta f((x - x_0)/\Delta x, (E - E_0)/\Delta E)$ is localized around a phase space point. Then we expand f_0 about that point as

$$f_{i,0} = f_{i,0}(x_0, E_0) + (x - x_0) \left. \frac{\partial \langle f_i \rangle}{\partial x} \right|_{(x_0, E_0)} + \cdots. \tag{6b}$$

Note that the expansion is only carried out in x here. This reflects the fact that in Eq. (4a) energy is not scattered by turbulence, since the bounce averaged $v_\parallel \tilde{E}_\parallel$ vanishes, i.e., $dE/dt = (e/m_i) \langle v_\parallel \tilde{E}_\parallel \rangle_b = 0$ where $\langle \ldots \rangle_b = \int dl / v_\parallel (\ldots) / \int dl / v_\parallel$ is a bounce average. Using the expansion, the structure evolution equation now becomes

$$\partial_t \int d^3 v \frac{\langle \delta f_i^2 \rangle}{2} = -\langle \tilde{v}_r \tilde{n}_i \rangle \left. \frac{\partial \langle f \rangle}{\partial x} \right|_0 + \langle f C(f) \rangle, \tag{6c}$$

where the subscript 0 denotes a location of the structure in phase space (x_0, E_0), $d(x - x_0)/dt = \tilde{v}_r$ and $\int d^3 v \delta f_i = \tilde{n}_i$ Eq. (6c) relates the distortion of the mean distribution to the growth of the perturbation. Note that the structure growth, Eq. (6c), differs from the drift hole growth derived in an earlier study[14] in that: (i) structure growth here is decoupled from velocity space scattering and (ii) structure growth here *is* coupled to zonal flow generation. The first difference is trivial, since we are dealing with a bounce kinetic equation here, so there is no parallel acceleration. The second, more physically relevant difference arises since the net dipole moment of the structure, including polarization charge, is conserved,[6] $\int dx \sum_\alpha q_\alpha n_\alpha(x) x$. In other words, since quasi-neutrality including polarization charge is maintained $\langle \tilde{v}_r \tilde{n}_i \rangle = \langle \tilde{v}_r \tilde{n}_e \rangle - \langle \tilde{v}_r \tilde{n}_{pol} \rangle$ and $\langle \tilde{v}_r \tilde{n}_{pol} \rangle \sim \langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \rangle \sim \partial_r \langle \tilde{v}_r \tilde{v}_\theta \rangle$ via the Taylor identity,[17] ion structure growth is intrinsically coupled to zonal flow growth via flux of polarization charge. This gives

$$\partial_t \int d^3 v \frac{\langle \delta f_i^2 \rangle}{2 \partial \langle f_i \rangle / \partial x|_0} = -\langle \tilde{v}_r \tilde{n}_e \rangle + \frac{n_0}{\omega_{c,i}} \partial_r \langle \tilde{v}_r \tilde{v}_\theta \rangle + \frac{\langle f C(f) \rangle}{\partial \langle f_i \rangle / \partial x|_0}, \tag{6d}$$

which clearly states that the single structure growth cannot decouple from zonal flow evolution. Note that zonal flow coupling here appears through polarization charge flux, for the same reason as zonal flow coupling in QG turbulence appears through vorticity flux.[17]

In Eq. (6d), the particle flux appears in the right hand side. Then, one may ask how we reconcile Eq. (6d) with heat flux drive. This may be resolved by going back to Eq. (6a) and by taking E moment of δf^2. This leads to

$$\partial_t \int d^3 v E \langle \delta f_i^2 \rangle = -2 \langle \tilde{v}_r \tilde{T}_i \rangle \left. \frac{\partial \langle f \rangle}{\partial x} \right|_0 + 2 \int d^3 v E \langle f C(f) \rangle, \tag{7}$$

where $\langle \tilde{v}_r \tilde{T}_i \rangle \equiv \int d^3 v E \langle \tilde{v}_r \delta f_i \rangle$. Equation (7), together with heat balance equation $Q_0 = -\chi_{neo} \langle T_i \rangle' + \int d^3 v E \langle \tilde{v}_r \delta f_i \rangle$, describes energetics of the system.

The structure evolution, Eq. (6d), reveals a momentum constraint for the interaction between a single phase space structure and zonal flow. Using the momentum balance $\partial_t \langle v_\theta \rangle + \partial_r \langle \tilde{v}_r \tilde{v}_\theta \rangle = -\nu \langle v_\theta \rangle$ gives

$$\frac{\partial}{\partial t} \left(\frac{n_0}{\omega_{c,i}} \langle v_\theta \rangle + \int d^3 v \frac{\langle \delta f_i^2 \rangle}{2 \langle f_i \rangle' |_0} \right) = -\langle \tilde{v}_r \tilde{n}_e \rangle - n_0 \frac{\nu}{\omega_c} \langle v_\theta \rangle + \frac{\langle f C(f) \rangle}{\partial \langle f_i \rangle / \partial x|_0}. \tag{8}$$

Hence, we see that up to constant factors $\int d^3 v \langle \delta f_i^2 \rangle / (\partial \langle f_i \rangle / \partial x)|_0$ can be thought of as a generalized momentum associated with fluctuation, namely the pseudomomentum of a single phase space structure which accounts for the zonal momentum of the structure. Equation (8) states that a structure growth in phase space is dynamically coupled to zonal flow to conserve zonal momentum. At stationary state, electron flux can sustain a flow against collisional drag, $\langle v_\theta \rangle = -(\omega_c / \nu) \langle \tilde{v}_r \tilde{n}_e \rangle / n_0$; localized ion structure scatters electrons and can pump zonal flow growth. The statement that *a single localized structure in phase space can drive zonal flow* should be regarded as interesting, in light of the total absence of the familiar wave interaction-based mechanisms of zonal flow generation, such as inverse cascade,[27] the Rhines mechanism,[28] modulational instability,[29] etc. Indeed, Eq. (8) supports the notion that PV transport or mixing and one direction of symmetry are all that is required for zonal flow generation.[30]

At this point, it is interesting to compare Eq. (8) to the Charney-Drazin momentum theorem for a potential vorticity conserving, quasigeostrophic turbulence. The theorem[18] is proved for the Hasegawa-Wakatani system[31]

$$\frac{\partial}{\partial t}\left\{\frac{\langle\delta q^2\rangle}{2\langle q\rangle'}+\langle v_\theta\rangle\right\}=-\langle\tilde{v}_r\tilde{n}_e\rangle-\nu\langle v_\theta\rangle -\frac{1}{\langle q\rangle'}\left(\frac{\partial}{\partial r}\left\langle\tilde{v}_r\frac{\delta q^2}{2}\right\rangle+D_0\langle(\nabla\delta q)^2\rangle\right). \quad (9)$$

Here $q=n-\rho_s^2\nabla_\perp^2(e\phi/T_e)$, $\langle\delta q^2\rangle/\langle q\rangle'$ is the wave activity density or the negative of pseudomomentum,[24] D_0 is a diffusivity of potential vorticity, and v is a collisional drag on flow. $Pr=1$ was assumed. As $\langle\delta q^2\rangle/\langle q\rangle'\sim-|v_*|^{-1}\sum_k(1+\rho_s^2k_\perp^2)^2|\hat{\phi}|_k^2\sim-\sum_k k_\theta(\mathcal{E}_k/\omega_k)$, which is recognizable as the negative of the wave momentum density, the theorem states a momentum constraint for self-regulating DW-ZF turbulence system. Now each term in Eq. (8) has its counterpart in Eq. (9). The kinetic analogue of wave activity density, $\langle\delta f_i^2\rangle/\langle f\rangle'|_0$, can be contrasted to wave activity density $\langle\delta q^2\rangle/\langle q\rangle'$. The both terms are related to the momentum of fluctuations, although $-\langle\delta f_i^2\rangle/\langle f_i\rangle'|_0$ is the momentum of a phase space structure in the presence of strong wave-particle interaction. Particle flux and zonal flow drag appear in both equations. The underlying physics which unifies the two different systems is the conservation of "potential vorticity," which consists of generalized or extended fluid vorticity; for QG system PV consists of fluid vorticity plus the "planetary" part related to β or v_*; for GK system it consists of polarization charge plus electron density. This ultimately follows from the Kelvin's theorem[32] for the conservation of total circulation, which underpins the Charney-Drazin theorem for flow momentum and a generalized pseudomomentum. See Table II for the comparison.

We note, though, Eq. (8) defines the pseudomomentum of phase space structure and is not a mere mathematical transcription of the Charney-Drazin theorem for QG system to another structurally similar system, i.e., the GK system. The extension of the theorem to the kinetic system is subtle, since kinetic system has a singularity from wave-particle resonance, with no counterpart in the QG system. The subtlety becomes apparent when we try to physically interpret the kinetic wave activity density, $\langle\delta f^2\rangle/\langle f\rangle'$. Of course, in the non-resonant limit, the kinetic wave activity density corresponds to the negative of pseudomomentum associated with waves, as $\delta f_k\simeq(-\tilde{v}_{r,k}\langle f\rangle')/(-i\omega_k)$ and $\int\sqrt{E}dE\langle\delta f^2\rangle/\langle f\rangle'\sim\int\sqrt{E}dE\sum_k\langle f\rangle'(k_\theta^2|\tilde{\phi}|_k^2)/\omega_k^2\sim-\sum_k(\mathcal{E}_k/\omega_k)k_\theta$ where $\mathcal{E}_k$ is the wave energy density. In the resonant limit, however, we cannot physically interpret the kinetic wave activity density based on the comparison between GK and QG turbulence. The above discussion of the ion structure growth reveals $-\int\sqrt{E}dE\langle\delta f_i^2\rangle/\langle f_i\rangle'|_0$ as the pseudomomentum associated with phase space structure, and extends the interpretation of the kinetic wave activity density $\int\sqrt{E}dE\langle\delta f^2\rangle/\langle f\rangle'$ to the case of resonant particles. This also suggests the robustness of the C-D momentum theorem: the momentum theorem holds, in both QG and GK systems, in the both non-resonant and resonant limit.

III. MULTI-STRUCTURES IN PHASE SPACE AND ZONAL FLOW

In this section, we move from the problem of a single structure to the problem of multi-structures and discuss aspects of relaxation and transport in phase space turbulence for $K\sim1$, where structures can form but also break, leading to formation of incoherent granular fluctuations in phase space. To formulate transport with such fluctuation, we first consider a model to characterize granularity of phase space density, which leads us to the calculation of phase space density correlation[9,16] $\langle\delta f(1)\delta f(2)\rangle$. Note that $\langle\ldots\rangle$ is defined as the average over $\mathbf{x}_+=(\mathbf{x}_1+\mathbf{x}_2)/2$ in this section. As discussed in Sec. II, a single structure in phase space interacts with zonal flow; hence it is plausible to expect multi-structure or granulation to also interacts with the zonal flow. In the following, we derive the time evolution of phase space density granulation and show that granulation is dynamically coupled to zonal flow via production. This coupling is due to zonal momentum exchange in the δf^2 production process and is *not* simply due to usual effects of shear suppression, cross phase modification, etc. Then we turn to transport calculation due to phase space density granulation. Since phase space density granulation interacts with zonal flow, the zonal flow leaves a footprint in granulation driven transport. In the following, we argue that zonal flow introduces a novel effect in transport via dynamical friction.

A. Model and its dielectric function

The model we utilize here[16] consists of bounce averaged kinetic equations for ions and electrons ($\sigma=i,e$),

$$\left(\frac{\partial}{\partial t}+v_D(E)\frac{\partial}{\partial y}+\mathbf{v}_E\cdot\nabla+\nu_{eff}^\sigma\right)\delta h_\sigma =\frac{\partial}{\partial t}\frac{q_\sigma\tilde{\phi}}{T_\sigma}\langle f_\sigma\rangle-\tilde{\mathbf{v}}_{ExB}\cdot\nabla\langle f_\sigma\rangle \quad (10a)$$

and the Gyrokinetic Poisson equation,[22,23]

TABLE II. Comparison of quasi-geostrophic system and gyrokinetic system.

	QG system	GK system
"Potential Vorticity"	PV, $q=\nabla_\perp^2 f+F(f,n)$	GK Poisson, pol charge $\int d^3v f+\rho_s^2\nabla_\perp^2\phi=g(\phi,n_e,\ldots)$
Conservation of PV	$dq/dt=\partial_t q+\{q,\phi\}=0$	$df/dt=\partial_t f+\{f,H\}=0$
Circulation	$\Gamma=(V+2\Omega a\sin\theta)dl$	$\Gamma=\mathbf{v}\cdot d\mathbf{x}$
Kelvin's Theorem	Yes	Yes (Lynden-Bell,'67)
C-D Theorem	Yes	Yes for non-resonant limit
		Yes for resonant limit

$$\frac{\delta n_e}{n_0} = \frac{\delta n_i}{n_0} + \rho_i^2 \nabla_\perp^2 \frac{q\tilde{\phi}}{T_i}. \tag{10b}$$

Here δh is the non-adiabatic part of distribution function fluctuation, $\delta f_\sigma = -(q_\sigma \tilde{\phi}/T_\sigma)\langle f_\sigma \rangle + \delta h_\sigma$, $v_D(E) = \bar{v}_D \bar{E}$ where $\bar{E} = E/T_i$ is the drift due to magnetic field curvature and inhomogeneity, and a Krook operator was used, $C(\delta f_\sigma) = -\nu_{eff}^\sigma \delta h_\sigma$. Electrons are assumed to be dissipative with the energy dependent collision frequency[21] $\nu_{eff}(E) = (\nu_e/\epsilon_0)\overline{E_e}^{-3/2}$ where $\bar{E}_e \equiv E/T_e$. ϵ_0 is the inverse aspect ratio. We note that a singularity associated with the electron collision frequency at $\bar{E} = 0$ does not cause any problem in the calculation performed in the manuscript, as its *inverse*, i.e., $\bar{E}^{3/2}(\nu_e/\epsilon_0)^{-1}$ appears as the relevant quantity, and is manifestly non-singular. Specifically, the electron frequency appears in $\mathrm{Im}\epsilon_e \propto \int \sqrt{\bar{E}} d\bar{E} \bar{E}^{3/2} e^{-\bar{E}} (\nu_e/\epsilon_0)^{-1}(\ldots)$ which can be integrated analytically. See Appendix A for the calculation. We also note that the strong electron collisions $(\nu_e/\epsilon_0 \gg \omega_{*e}, 1/\tau_c)$ smears out electron granulation formation, as the propagator for trapped electron becomes $g_{k\omega} = i(\omega - \omega_{D,e}\bar{E}_e + i/\tau_c + i\nu_e/\epsilon_0 \bar{E}_e^{-3/2})^{-1} \simeq (\nu_e/\epsilon_0)\bar{E}_e^{3/2}$. Thus the electrons are *laminar* in this model and we focus on the ion granulation dynamics. The model we use here is quite general—2D ion guiding center advection, polarization charge in GK Poisson equation, and precession drift resonance. It is arguably the simplest model of GK drift wave turbulence with wave-particle resonance effects.

In the later calculation, we need the plasma response function, or plasma dielectric for the model. This is given by (See Appendix A for the derivation.)

$$\hat{\epsilon}(k,\omega) = \frac{T_i}{T_e} + \rho_i^2 k_\perp^2 + 1 - P\int d^3v \frac{\omega - \omega_*^i(E)}{\omega - \bar{\omega}_D \bar{E} - k_\theta \langle v_E \rangle (r)} \langle f_i \rangle + i\mathrm{Im}\epsilon_i + i\mathrm{Im}\epsilon_e + i\mathrm{Im}\epsilon_{pol}, \tag{11a}$$

where $\omega_*^\sigma(E) \equiv (k_\theta c T_\sigma \langle f_\sigma(E)\rangle')/(q_\sigma B \langle f_\sigma(E)\rangle) = k_\theta v_*^\sigma (1 + \eta_\sigma (\bar{E} - 3/2))$, $v_*^i \equiv -(\rho_i/|L_n|)v_{thi}$, $v_*^e = (\rho_s/|L_n|)c_s$, and P denotes the principle part of the integral. The first term is from adiabatic electrons. The second term is from polarization charge. The second line includes ion contribution from adiabatic passing and trapped populations. The imaginary part of the dielectric is

$$\mathrm{Im}\epsilon_i = \sqrt{2\epsilon_0}\frac{2}{\sqrt{\pi}}\sqrt{\bar{E}_{res}}\pi \frac{\omega - \omega_*^i(\bar{E}_{res})}{|\bar{\omega}_D|} e^{-\bar{E}_{res}}, \tag{11b}$$

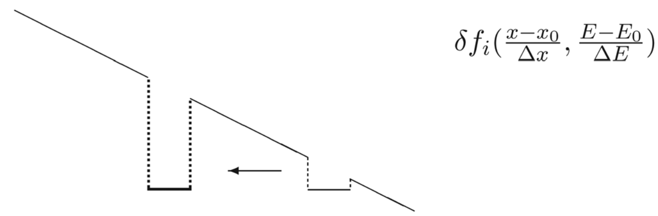

FIG. 4. The growth of hole. Since $df/dt = 0$, a hole can grow by moving against background gradient. Here we consider a localized hole around (x_0, E_0), with the extent Δx and ΔE.

$$\mathrm{Im}\epsilon_e = \frac{4}{\sqrt{\pi}} \frac{\sqrt{2\epsilon_0}}{\nu_e/\epsilon_0} \frac{T_i}{T_e} \left(\omega - \omega_*^e \left(1 + \frac{3}{2}\eta_e \right) \right), \tag{11c}$$

$$\mathrm{Im}\epsilon_{pol} = -2\rho_i^2 k_r \partial_r, \tag{11d}$$

where $\bar{E}_{res} \equiv (\omega - k_\theta \langle v_\theta \rangle' (r - r_0))/\bar{\omega}_D$. $\mathrm{Im}\epsilon_i$ arises from the resonance between waves and toroidal ion precession. Note that shear flow alters the resonant frequency $\bar{\omega}_D \bar{E} \to \bar{\omega}_D \bar{E} + k_\theta \langle v_\theta \rangle' (r - r_0)$. $\mathrm{Im}\epsilon_e$ comes from collisional dissipation in electrons. $\mathrm{Im}\epsilon_{pol}$ originates from an envelope coupling via modulation, i.e., $k_r \to k_r - i\partial_r$, where ∂_r captures the envelope variation of the fluctuation spectrum, which is slow compared to k_r (Fig. 3). As in Fig. 2, granulations produce a wake with a spatial extent, which contributes to the slow envelope variation. This envelope introduces two different scales in the fluctuation; a micro-scale which is characterized by mode wave number k_r and an envelope variation which is captured by the derivative acting on the slow scale envelope, ∂_r. The confluence of the two different radial scales leads to $k_r \to k_r - i\partial_r$ and $\epsilon_{pol} \sim k_\perp^2 \sim k_r^2 - 2ik_r\partial_r$, so $\mathrm{Im}\epsilon_{pol} \sim -2k_r\partial_r$. Noting the role of polarization charge in single structure dynamics and the dynamical friction $F \propto \mathrm{Im}\epsilon$, we will see that the envelope coupling term $\mathrm{Im}\epsilon_{pol}$ introduces zonal flow coupling to the mean field evolution via $F \sim \mathrm{Im}\epsilon_{pol}$. Note also that the plasma dielectric is now an operator, since $\mathrm{Im}\epsilon_{pol} \propto \partial_r$. Also, hereafter it is understood that $r - r_0 = x$.

B. Derivation of phasetrophy evolution

In this section, we derive the time evolution equation for phase space density correlation[9] $\langle \delta h(1)\delta h(2)\rangle$. Phase space density correlation can be compared to several physical quantities, Fig. 5. Phase space density correlation $\langle \delta h^2 \rangle$ can be thought of as "potential enstrophy" in phase space or "phasetrophy,"[12] since phase space density f in the GK system is similar to the potential vorticity q in the QG system (Table II). Following the analogue between the QG and the

Quasi-geostrophic system | Gyrokinetic system

Potential enstrophy $\langle \delta q^2 \rangle$ $\Leftrightarrow$ Phase space density correlation $\langle \delta f(1)\delta f(2)\rangle$ $\searrow$ Fluctuation entropy $\int \sqrt{E} dE \langle \delta f^2 \rangle / \langle f \rangle$ $\Rightarrow$ entropy budget

$\downarrow$ $\downarrow$

Wave activity density $\langle \delta q^2 \rangle / \langle q \rangle'$ $\Leftrightarrow$ Kinetic wave activity density $\int \sqrt{E} dE \langle \delta f^2 \rangle / \langle f \rangle' |_0$

$\to$ pseudomomentum | $\to$ pseudomomentum $\Rightarrow$ momentum budget

FIG. 5. Relation of "phasetrophy" $\langle \delta f(1)\delta f(2)\rangle$ to other physical quantities.

GK system, the phasetrophy gives the kinetic wave activity density, which is closely related to the fluctuation pseudomomentum. Alternatively, phase space density correlation evolution $\langle \delta f(1)\delta f(2)\rangle$ is also related to the fluctuation entropy in kinetics, i.e., $s = \int \sqrt{E}dE\langle \delta f^2\rangle/\langle f\rangle$.

$\langle \delta h(1)\delta h(2)\rangle$ evolution is derived as follows.[9,12,16] Upon multiplying $\delta h(2)$, adding the equation with 1 and 2 exchanged, introducing the relative coordinate $y_- \equiv y_1 - y_2$, and averaging over $\mathbf{x}_+ = (\mathbf{x}_1 + \mathbf{x}_2)/2$, Eq. (10a) for ions gives

$$\frac{\partial}{\partial t}\langle \delta h(1)\delta h(2)\rangle + v_{rel}\frac{\partial}{\partial y_-}\langle \delta h(1)\delta h(2)\rangle + T(1,2) + C(1,2) = P(1,2). \tag{12}$$

Here the terms in the left-hand side are

$$v_{rel} \equiv \bar{v}_D(\bar{E}_1 - \bar{E}_2) + \langle v_E\rangle'(x_1 - x_2), \tag{13a}$$

$$T(1,2) \equiv \langle \delta h(2)\tilde{\mathbf{v}}_{E\times B}(1)\cdot\nabla\delta h(1)\rangle + (1\leftrightarrow 2), \tag{13b}$$

$$C(1,2) \equiv \langle \delta h(2)\nu_{eff}(1)\delta h(1)\rangle + (1\leftrightarrow 2), \tag{13c}$$

$(1\leftrightarrow 2)$ denotes the term with the arguments 1 and 2 exchanged. v_{rel} is the relative velocity of particles at two different points in phase space. $T(1,2)$ is the triplet term which describes the decorrelation process due to nonlinear $\tilde{v}_{E\times B}$ scattering. $C(1,2)$ is the collisional cut-off. After a closure calculation[16] of the triplet term, the left-hand side of Eq. (12) reduces to

$$\left(\frac{\partial}{\partial t} + \bar{v}_D E_- \frac{\partial}{\partial y_-} + \langle v_E\rangle' x_- \frac{\partial}{\partial y_-} - \frac{\partial}{\partial \mathbf{x}_-}\cdot\mathbf{D}_{rel}\cdot\frac{\partial}{\partial \mathbf{x}_-}\right) \times \langle \delta h(1)\delta h(2)\rangle + C(1,2), \tag{14}$$

where $\mathbf{D}_{rel}$ is the relative diffusion matrix

$$\mathbf{D}_{rel} = \sum_{k\omega}\{1 - \cos(\mathbf{k}\cdot\mathbf{x}_-)\}\langle \tilde{\mathbf{v}}_{E\times B}\tilde{\mathbf{v}}_{E\times B}\rangle_{k\omega} \times \mathrm{Re}\,\frac{i}{\omega - k_\theta \bar{v}_D\bar{E} - k_\theta\langle v_E\rangle' x + i/\tau_c}. \tag{15}$$

The relative streaming of the magnetic drift, zonal flow shear, the relative diffusion, and $C(1,2)$ together determine an effective life time of correlation. In the limit of $1 \to 2$, since $v_{rel} \to 0$ and $\mathbf{D}_{rel} \propto k_\perp^2 x_-^2 \to 0$, the lifetime is determined by the collisional cut-off.

The right-hand side of Eq. (12) is

$$P(1,2) \equiv -\langle \tilde{v}^r_{E\times B}(1)\delta h(2)\rangle\langle f_i(1)\rangle' + \left\langle \delta h(2)\partial_t\frac{q\tilde{\phi}(1)}{T_i}\right\rangle\langle f_i(1)\rangle + (1\leftrightarrow 2). \tag{16a}$$

$P(1,2)$ is the production of phasetrophy due to transport and relaxation. Note that the first term in the production $P(1,2)$ has a generic form expected for production, namely flux $\langle \tilde{v}_r\delta f\rangle$ times gradient $\langle f\rangle'$. In terms of Fourier components, we have

$$P(1,2) = \mathrm{Re}\sum_{k\omega}(-i\omega + i\omega^i_*(1))\left\langle \frac{q\tilde{\phi}(1)}{T_i}\delta h^*(2)\right\rangle_{k\omega} \times e^{i\mathbf{k}\cdot\mathbf{x}_-}\langle f_i(1)\rangle + (1\leftrightarrow 2), \tag{16b}$$

where $\mathbf{x}_- \equiv \mathbf{x}_1 - \mathbf{x}_2$, $\omega^i_*(1) \equiv (k_\theta c T_i\langle f_i(1)\rangle')/(qB\langle f_i(1)\rangle) = k_\theta v^i_*(1 + \eta_i(\bar{E} - 3/2))$, $v^i_* \equiv -(\rho_i/|L_n|)v_{thi}$ and $\langle\ldots\rangle_{k\omega}$ is the Fourier spectrum of a correlation function, i.e., $\langle f(\mathbf{x}_1,t_1)g(\mathbf{x}_2,t_2)\rangle = \sum_{k\omega}\langle fg\rangle_{k\omega}e^{i\mathbf{k}\cdot\mathbf{x}_- - i\omega t_-}$. As $\delta h = \delta h^c + \widetilde{\delta h}$, the production term consists of two parts, namely coherent and incoherent production[10–12] (Fig. 6). The coherent part originates from a response of phase space density fluctuation to potential fluctuation $\tilde{\phi}$, namely $\delta h^c_{k\omega} \sim R(k,\omega)\tilde{\phi}_{k\omega}$. The incoherent part originates from granulation $\widetilde{\delta h}$. While moving through plasmas, granulations, or macro-particles in phase space can emit wave wakes via Cerenkov processes (Fig. 2). These wakes are in turn absorbed by plasma. This process effectively produces a macro-particle wake and induces dynamical friction on ion phase space density granulations. This in turn produces incoherent production, $\tilde{P} \propto \mathrm{Im}\epsilon$. We calculate both coherent and incoherent production in the following (Fig. 6).

To calculate the production term due to the coherent response, we need $\delta h^c_{k\omega}$, the part of δh phase-coherent with fluctuation potential. $\delta h^c_{k\omega}$ can be calculated as a response to $\tilde{\phi}_{k\omega}$ from Eq. (10a) as

$$\delta h^c_{k\omega} = g_{k\omega}(-i\omega + i\omega^i_*)\left(\frac{q\tilde{\phi}}{T_i}\right)_{k\omega}\langle f_i\rangle, \tag{17}$$

where $g_{k\omega} \equiv (-i\omega + i\bar{\omega}_D\bar{E} + ik_\theta\langle v_E\rangle' x + 1/\tau_c)^{-1}$ is a propagator, and $1/\tau_c$ comes from resonance broadening[33] due to the nonlinear $\mathbf{E}\times\mathbf{B}$ scattering, so $\tau_c^{-1} \sim k_\perp^2 D_\perp$ and $D_\perp = \sum_{k\omega}\mathrm{Re}g_{k\omega}(c/B)^2\langle\tilde{\phi}^2\rangle_{k\omega}$. Note that in the weak turbulence limit of $1/\tau_c < \omega$, $\mathrm{Re}g_{k\omega} \to \pi\delta(\omega - \bar{\omega}_D\bar{E} - k_\theta\langle v_E\rangle' x)$, which is the expression we use later to obtain the net ion phasetrophy production, Eq. (28). This is essentially the linear response. Then using the expression for $\delta h^c_{k\omega}$ gives

$$\begin{array}{lll} & P^c(1,2) & \\ P(1,2) \nearrow\!\searrow & & \tilde{P}_i \propto \mathrm{Im}\epsilon_i \\ & \tilde{P}(1,2) \propto \mathrm{Im}\epsilon \;\to & \tilde{P}_e \propto \mathrm{Im}\epsilon_e \\ & & \tilde{P}_{pol} \propto \mathrm{Im}\epsilon_{pol} \end{array}$$

FIG. 6. List of terms to be calculated.

$$P^c(1,2) \equiv \mathrm{Re}\sum_{k\omega}(-i\omega + i\omega_*^i(1))\left\langle \frac{q\tilde{\phi}(1)}{T_i}\delta h^{c*}(2)\right\rangle_{k\omega} \times \langle f_i(1)\rangle e^{i\mathbf{k}\cdot\mathbf{x}_-} + (1 \leftrightarrow 2), \tag{18a}$$

$$= \sum_{k\omega}(\omega - \omega_*^i(1))(\omega - \omega_*^i(2))\mathrm{Re}g_{k\omega}(2) \times \left\langle \frac{q\tilde{\phi}(1)}{T_i}\frac{q\tilde{\phi}^*(2)}{T_i}\right\rangle_{k\omega}\langle f_i(1)\rangle\langle f_i(2)\rangle e^{i\mathbf{k}\cdot\mathbf{x}_-} + (1 \leftrightarrow 2), \tag{18b}$$

$$\rightarrow 2\sum_{k\omega}(\omega - \omega_*^i(E))^2\langle f_i\rangle^2 \mathrm{Re}g_{k\omega}\lim_{1\rightarrow 2}\left\langle \frac{q\tilde{\phi}(1)}{T_i}\frac{q\tilde{\phi}^*(2)}{T_i}\right\rangle_{k\omega}, \tag{18c}$$

where in the last line we took the limit of $1 \rightarrow 2$. This term corresponds to the production due to diffusive flux of phase space density by $\tilde{v}_{E\times B}$ scattering. This can be checked, for $\omega - \omega_*^i \sim k_\theta \rho_i v_{thi}\langle f\rangle'/\langle f\rangle$,

$$P^c \sim 2\sum_{k\omega} v_{thi}^2 \mathrm{Re}g_{k\omega}k_\theta^2\rho_i^2\left\langle \left(\frac{q\tilde{\phi}}{T_i}\right)^2\right\rangle_{k\omega}\langle f_i\rangle'^2 = 2D\langle f_i\rangle'^2. \tag{18d}$$

Here $D \equiv \sum_{k\omega} v_{thi}^2 \mathrm{Re}g_{k\omega}k_\theta^2\rho_i^2\left\langle (q\tilde{\phi}/T_i)^2\right\rangle_{k\omega}$ is the diffusion coefficient due to $E\times B$ scattering. Eq. (18d) is the familiar quasi-linear result.

Note that the spectrum $\langle\tilde{\phi}^2\rangle_{k\omega}$ is not arbitrary here; $\langle\tilde{\phi}^2\rangle_{k\omega}$ is produced by granulation $\langle\widetilde{\delta h}^2\rangle_{k\omega}$ via Cerenkov emission. The potential fluctuation is self-consistently calculated[9] by solving the quasi-neutrality condition via the GK Poisson equation,

$$\hat{\epsilon}(k,\omega)\frac{q\tilde{\phi}_{k\omega}}{T_i} = \left(\frac{\widetilde{\delta n_i}}{n_0}\right)_{k\omega}, \tag{19}$$

where $(\widetilde{\delta n_i}/n_0)_{k\omega} = \int d^3v\widetilde{\delta h}_{k\omega}$ may be thought of as the emission by incoherent granulation. The quasi-neutrality condition can be solved with the help of Green's function defined by

$$\hat{\epsilon}(x)G(x,x') = \delta(x - x'), \tag{20}$$

which yields

$$\frac{q\tilde{\phi}_{k\omega}(x)}{T_i} = \int dx' G(x,x')\frac{\widetilde{\delta n}_{k\omega}(x')}{n_0} = \int dx' d^3v G(x,x')\widetilde{\delta h}_{k\omega}(\mathbf{v},x'). \tag{21}$$

Note that Eq. (21) is only valid for nearly steady state with saturated waves, since in that case the homogeneous or eigenvalue solution of the quasi-neutrality condition, i.e., $\epsilon(k,\omega_k) = 0$ with $\tilde{\phi}_k \sim e^{-i\omega_k t}$, is damped so only the inhomogeneous solution due to the incoherent emission remains in Eq. (21). Given that caveat, the self-consistent spectrum is obtained as

$$\left\langle \frac{q\tilde{\phi}(1)}{T_i}\frac{q\tilde{\phi}^*(2)}{T_i}\right\rangle_{k\omega} = \int dx_1' dx_2' d^3v_1 d^3v_2 G(x_1,x_1')G^*(x_2,x_2') \times \langle\widetilde{\delta h}(x_1',\mathbf{v}_1)\widetilde{\delta h}^*(x_2',\mathbf{v}_2)\rangle_{k\omega}.$$

Equation (22) suggests that the self-consistent spectrum at x depends on the granulation fluctuations at different locations x_1', x_2'. Note that in the local limit $G(x,x') = \epsilon^{-1}(k,\omega)\delta(x - x')$ and Eq. (22) reduces to a familiar form $\langle\tilde{\phi}^2\rangle_{k\omega} \sim \langle\widetilde{\delta n}^2\rangle_{k\omega}/|\epsilon(k,\omega)|^2$.

With the self-consistent spectrum, the coherent production can be rewritten as

$$\lim_{1\rightarrow 2}P^c(1,2) = 2\sum_{k\omega}(\omega - \omega_*^i(E))^2\langle f_i\rangle^2\mathrm{Re}g_{k\omega} \times \int dx_1' dx_2' d^3v_1 d^3v_2 G(x_1,x_1')G^*(x_2,x_2') \times \langle\widetilde{\delta h}(x_1',\mathbf{v}_1)\widetilde{\delta h}^*(x_2',\mathbf{v}_2)\rangle_{k\omega}. \tag{22}$$

A form which is more useful for the later calculation is obtained by relating k,ω spectrum to k spectrum via the orbit propagator[9,12,16]

$$\langle\widetilde{\delta h}(1)\widetilde{\delta h}^*(2)\rangle_{k\omega} \cong 2\pi\delta(\omega - \bar{\omega}_D\bar{E}_2 - k_\theta\langle v_E\rangle x_2)\langle\widetilde{\delta h}(1)\widetilde{\delta h}^*(2)\rangle_k. \tag{23}$$

Upon integrating over energy, we obtain

$$\lim_{1\rightarrow 2}P^c(1,2) = 2\sum_{k\omega}(\omega - \omega_*^i(E))^2\langle f_i\rangle\mathrm{Re}g_{k\omega} \times \sqrt{2\epsilon_0}\frac{2}{\sqrt{\pi}}\frac{2\pi}{|\bar{\omega}_D|}e^{-\bar{E}}\int dx_1' dx_2' G(x,x_1')G^*(x,x_2') \times \sqrt{\bar{E}_{res}(x_2')}\left\langle \frac{\widetilde{\delta n}(x_1')}{n_0}\widetilde{\delta h}^*(x_2',\bar{E}_{res}(x_2'))\right\rangle_k. \tag{24}$$

Note that Eq. (24) is not as simple as its local limit,

$$\lim_{1\rightarrow 2}P^c(1,2) = 2\sum_{k\omega}(\omega - \omega_*^i(E))^2\langle f_i\rangle\mathrm{Re}g_{k\omega}\sqrt{2\epsilon_0}\frac{2}{\sqrt{\pi}}\frac{2\pi}{|\bar{\omega}_D|} \times e^{-\bar{E}}\frac{\sqrt{\bar{E}_{res}}}{|\epsilon(k,\omega)|^2}\left\langle \frac{\widetilde{\delta n}}{n_0}\widetilde{\delta h}^*(\bar{E}_{res})\right\rangle_k. \tag{25}$$

The difference arises from zonal flow coupling. In the presence of zonal flows, the plasma dielectric becomes an operator via envelope coupling $\mathrm{Im}\epsilon_{pol} \propto \partial_r$ and the resonance is altered from $\delta(\omega - \bar{\omega}_D\bar{E})$ to $\delta(\omega - \bar{\omega}_D\bar{E} - k_\theta\langle v_E\rangle' x)$. The modified resonance functions introduce a spatial integral (via inversion of the operator using a Green's function) and space dependent velocity integral Jacobian $\sqrt{\bar{E}_{res}(x)}$. These, then, form a "non-local" influence kernel.

Now we turn to the calculation of the production term due to incoherent, granular fluctuations,

$$\tilde{P}(1,2) \equiv \mathrm{Re}\sum_{k\omega}(-i\omega + i\omega_*^i(1))\left\langle \frac{q\tilde{\phi}(1)}{T_i}\widetilde{\delta h}^*(2)\right\rangle_{k\omega} \times \langle f_i(1)\rangle e^{i\mathbf{k}\cdot\mathbf{x}_-} + (1 \leftrightarrow 2). \tag{26a}$$

Expressing the potential fluctuation in terms of the incoherent fluctuation and inserting a unit operator $\hat{\epsilon}^*(x)\int dx' G^*(x,x') = 1$ gives

$$\tilde{P}(1,2) = \mathrm{Re}\sum_{k\omega}(-i\omega + i\omega_*^i(1))\langle f_i(1)\rangle e^{i\mathbf{k}\cdot\mathbf{x}_-}$$
$$\times\int dx_1' dx_2' G(x_1,x_1')\hat{\epsilon}^*(x_2)G^*(x_2,x_2')$$
$$\times\left\langle\frac{\widetilde{\delta n}(x_1')}{n_0}\widetilde{\delta h}^*(x_2)\right\rangle_{k\omega} + (1\leftrightarrow 2)$$
$$\to -2\sum_{k\omega}(\omega - \omega_*^i(E))\langle f_i(E)\rangle\int dx_1' dx_2' G(x,x_1')$$
$$\times\,\mathrm{Im}\hat{\epsilon}(x)G^*(x,x_2')\left\langle\frac{\widetilde{\delta n}(x_1')}{n_0}\widetilde{\delta h}^*(x)\right\rangle_{k\omega}. \qquad (26b)$$

As noted above, $\tilde{P}(1,2)$ is directly proportional to $\mathrm{Im}\epsilon$. As $\mathrm{Im}\epsilon = \mathrm{Im}\epsilon_i + \mathrm{Im}\epsilon_e + \mathrm{Im}\epsilon_{pol}$, $\tilde{P}(1,2)$ consists of pieces from ions, electrons, and polarization charges. We calculate each piece in the following (Fig. 6).

We start by calculating ion induced incoherent production. This term is related to drag on phase space density exerted by ions—i.e., granulations emit waves via Cerenkov emission, while waves are in turn absorbed by ions, $\propto \mathrm{Im}\epsilon_i$. This leads to a drag on ion phase space density granulations, and incoherent production. Using the expression for $\mathrm{Im}\epsilon_i$ and noting,[16]

$$\left\langle\frac{\widetilde{\delta n}(1)}{n_0}\widetilde{\delta h}^*(2)\right\rangle_{k\omega} = \left\langle\frac{\widetilde{\delta n}(1)}{n_0}\widetilde{\delta h}^*(2)\right\rangle_{k}$$
$$\times\, 2\pi\delta(\omega - \bar{\omega}_D\bar{E}_2 - k_\theta\langle v_E\rangle' x_2) \quad (27a)$$

gives

$$\tilde{P}_i = -2\sum_{k\omega}(\omega - \omega_*^i(E))^2\langle f_i(E)\rangle\sqrt{2\epsilon_0}\frac{2}{\sqrt{\pi}}\sqrt{\bar{E}_{res}}\frac{\pi}{|\omega_D|}$$
$$\times\, e^{-\bar{E}_{res}}2\pi\delta(\omega - \bar{\omega}_D\bar{E} - k_\theta\langle v_E\rangle' x)$$
$$\times\int dx_1' dx_2' G(x,x_1')G^*(x,x_2')\left\langle\frac{\widetilde{\delta n}(x_1')}{n_0}\widetilde{\delta h}^*(x)\right\rangle_k. \qquad (27b)$$

In the local limit we have

$$\tilde{P}_i = -2\sum_{k\omega}(\omega - \omega_*^i(E))^2\langle f_i(E)\rangle\sqrt{2\epsilon_0}\frac{2}{\sqrt{\pi}}\sqrt{\bar{E}_{res}}\frac{\pi}{|\omega_D|}$$
$$\times\, e^{-\bar{E}_{res}}2\pi\delta(\omega - \bar{\omega}_D\bar{E})\frac{1}{|\epsilon(k,\omega)|^2}\left\langle\frac{\widetilde{\delta n}}{n_0}\widetilde{\delta h}^*\right\rangle_k. \qquad (27c)$$

The incoherent production by ions Eq. (27b) and the coherent production Eq. (24) adds to give an effective coherent production $P_{i,i} \equiv P^c + \tilde{P}_i$,

$$P_{i,i} = 2\sum_{k\omega}(\omega - \omega_*^i(E))^2\langle f_i(E)\rangle^2\mathrm{Re}g_{k\omega}S_{k\omega}, \qquad (28)$$

where

$$S_{k\omega} \equiv \sqrt{2\epsilon_0}\frac{2}{\sqrt{\pi}}\frac{2\pi}{|\bar{\omega}_D|}\left(\frac{2\pi T_i}{m_i}\right)^{3/2}\int dx_1' dx_2' G(x,x_1')G^*(x,x_2')$$
$$\times\left\{\sqrt{\bar{E}_{res}(x_2')}\left\langle\frac{\widetilde{\delta n}(x_1')}{n_0}\widetilde{\delta h}^*(x_2',\bar{E}_{res}(x_2'))\right\rangle_k - \sqrt{\bar{E}_{res}(x)}\left\langle\frac{\widetilde{\delta n}(x_1')}{n_0}\widetilde{\delta h}^*(x)\right\rangle_k\right\}. \qquad (29)$$

Here $S_{k\omega}$ is the effective fluctuation spectrum, shifted by the incoherent production contribution, as can be seen in the subtraction in the curly bracket. In the absence of the incoherent production contribution, $S_{k\omega} \to \langle(q\tilde{\phi}/T_i)^2\rangle_{k\omega}$. The net production Eq. (28) takes the form of coherent production - the incoherent contribution is rescaled into the coherent part. This may be viewed as a renormalization of coherent production due to self-feedback from *ion* incoherent production (Fig. 7), i.e., "bare" coherent production P^c produces $\widetilde{\delta h}^2$ which acts back through the incoherent ion production term $\tilde{P}_i$. Physically speaking, we may understand this as a self-field due to phase space density; while a test ion phase space density is scattered by $\tilde{v}_{E\times B}$, it also produces a self-field $\tilde{\phi}_{self} \sim \epsilon^{-1}(k,\omega)\int d^3v\widetilde{\delta h}_{test}$. The self-field, in turn, is coupled to other ions, which leads to absorption $\propto \mathrm{Im}\epsilon_i$. Through the coupling, the test ion phase space density feels the effect of the other ions as dynamical friction due to Cerenkov emission, thus leading to the renormalization of the $\tilde{v}_{E\times B}$ scattering of test phase space density. This can be expressed as a net "renormalized" production $P_{i,i} \sim 2\bar{D}\langle f\rangle'^2$ where

$$\bar{D} \equiv \sum_{k\omega}v_{thi}^2\mathrm{Re}g_{k\omega}k_\theta^2\rho_i^2 S_{k\omega} \qquad (30)$$

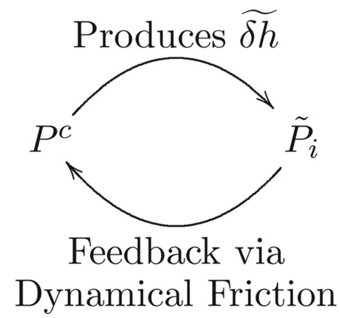

FIG. 7. Renormalization of coherent production.

is a "renormalized" diffusion coefficient. Of course, if we turn off the self-feedback from the incoherent production, $\bar{D}$ reduces to a "bare" diffusion coefficient $\bar{D} \to \sum_{k\omega} v_{thi}^2 \mathrm{Re} g_{k\omega} k_\theta^2 \rho_i^2 \left\langle (q\tilde{\phi}/T_i)^2 \right\rangle_{k\omega}$.

Note that we do *not* have the cancelation between the coherent and incoherent parts as found in analyses with effectively 1D resonance dynamics,[16] $\delta(\omega - \bar{\omega}_D \bar{E})$. Here, the resonance function is $\delta(\omega - \bar{\omega}_D \bar{E} - k_\theta \langle v_E \rangle' x)$ and thus 2D with (E,x). In 2D, E and x can change their value while $\bar{\omega}_D \bar{E} + k_\theta \langle v_E \rangle' x$ unchanged, as in a $(E,x) \to (E', x')$ scattering event. In contrast, in 1D, $\bar{E}$ cannot change its value while $\bar{\omega}_D \bar{E}$ is unchanged. This leaves initial state = final state, so relaxation is impossible and thus the like-species production must vanish. Indeed, by turning off zonal flow in the resonance dynamics in Eq. (28), we can recover the 1D result,

$$P^c + \tilde{P}_i \propto \delta(\omega - \bar{\omega}_D \bar{E} - k_\theta \langle v_E \rangle' x) S_{k\omega} \to \delta(\bar{E}_{res} - \bar{E}) \left\{ e^{-\bar{E}} \left\langle \widetilde{\frac{\delta n}{n_0}} \widetilde{\delta h^*}(\bar{E}_{res}) \right\rangle_k - e^{-\bar{E}_{res}} \left\langle \widetilde{\frac{\delta n}{n_0}} \widetilde{\delta h^*}(E) \right\rangle_k \right\} \to 0. \tag{31}$$

Thus the cancellation is a special case and an artifact of the 1D resonance dynamics.

Now we consider the calculation of other components in the incoherent production term (Fig. 6). The incoherent production due to electrons arises from the coupling of ion phase space density to electrons via $\mathrm{Im}\epsilon_e$ (i.e., drag). Here, ion phase space granulation emits waves via Cerenkov emission, while waves are, in turn, collisionally dissipated by electrons, $\mathrm{Im}\epsilon_e \propto \nu_e^{-1}$. This leads to drag on phase space density and incoherent production by electrons,

$$\tilde{P}_e = -2 \sum_{k\omega} (\omega - \omega_*^i(E)) \langle f_i(E) \rangle \mathrm{Im}\epsilon_e \times \int dx_1' dx_2' G(x, x_1') G^*(x, x_2') \left\langle \frac{\widetilde{\delta n}(x_1')}{n_0} \widetilde{\delta h^*}(x) \right\rangle_{k\omega}. \tag{32a}$$

This term reduces to the result which was derived earlier[16] by going to the local limit $G(x - x') = \epsilon^{-1}(k, \omega)\delta(x - x')$, relating the k, ω spectrum to the k spectrum via the orbit propagator, and utilizing the frequency ordering $\omega_*^i > \omega$,

$$\tilde{P}_e = 4\pi \sum_k \omega_*^i \left(\frac{\eta_i}{\eta_{i,cr}(\bar{E})} - 1 \right) \langle f_i(E) \rangle \frac{\mathrm{Im}\epsilon_e}{|\epsilon(k, \bar{\omega}_D \bar{E})|^2} \left\langle \widetilde{\frac{\delta n}{n_0}} \widetilde{\delta h^*} \right\rangle_k, \tag{32b}$$

where $\eta_{i,cr} \equiv (3/2 - \bar{E})^{-1}$ is an energy dependent threshold for the onset of the ion transport driven by granulations. This term was utilized to calculate anomalous transport of ion heat and particles.[16]

Incoherent production also arises from polarization charge. While moving through phase space, an ion granulation leaves a wake with a spatial extent (Fig. 2). The resulting spatial envelope of the fluctuation spectrum necessarily is coupled to the polarization charge via $\mathrm{Im}\epsilon_{pol} \propto \partial_r$. This leads to a wake drag on the phase space macro-particle and thus incoherent production by the polarization charge coupling,

$$\tilde{P}_{pol} = -2 \sum_{k\omega} (\omega - \omega_*^i(E)) \langle f_i(E) \rangle \int dx_1' dx_2' G(x, x_1') \times (-2\rho_i^2 k_r \partial_r) G^*(x, x_2') \left\langle \frac{\widetilde{\delta n}(x_1')}{n_0} \widetilde{\delta h^*}(x) \right\rangle_{k\omega}. \tag{33a}$$

Noting that polarization charges correspond to fluid vorticity and introduces zonal flow coupling in the *single* structure growth, we expect that $\tilde{P}_{pol}$ induces zonal flow coupling in the *multi*-structure case. To see the connection to zonal flow, we go to the local limit,

$$\tilde{P}_{pol} \simeq -2 \sum_{k\omega} (\omega - \omega_*^i(E)) \langle f_i(E) \rangle (-2\rho_i^2 k_r) \times \frac{1}{|\epsilon(k, \omega)|^2} \left\langle \widetilde{\frac{\delta n}{n_0}} \partial_r \widetilde{\delta h^*} \right\rangle_{k\omega}. \tag{33b}$$

Since it contains $k_\theta k_r$ weighed by spectrum via $\omega_*^i \simeq k_\theta v_*^i$, Eq. (33b) resembles the Reynolds stress. To show the Reynolds stress connection explicitly, and for the sake of simplicity, we take $\omega - \omega_*^i(E) \simeq -k_\theta v_*^i$, divide $\tilde{P}_{pol}$ by $\langle f \rangle$, and integrate $\tilde{P}_{pol}$ over velocity space, which yields

$$\int d^3v \frac{\tilde{P}_{pol}}{2\langle f_i \rangle} \simeq -\sum_{k\omega} \frac{v_*^i k_\theta \rho_i^2 k_r}{|\epsilon(k, \omega)|^2} \partial_r \left\langle \widetilde{\frac{\delta n}{n_0}} \int d^3v \widetilde{\delta h^*} \right\rangle_{k\omega} = -\sum_{k\omega} v_*^i k_\theta \rho_i^2 k_r \partial_r \left\langle \frac{q\tilde{\phi}}{T_i} \frac{q\tilde{\phi}^*}{T_i} \right\rangle_{k\omega} = \frac{v_*^i}{v_{thi}^2} \partial_r \langle \tilde{v}_r \tilde{v}_\theta \rangle. \tag{33c}$$

Hence, we see that the incoherent production via polarization charge induces zonal flow coupling to the granulation dynamics, and clearly links production to the Reynolds force.

Relative magnitude of $\tilde{P}_{pol}$, for example to $\tilde{P}_e$ (the latter leads to anomalous ion heat and particle transport[16]), can be evaluated as

$$\frac{\tilde{P}_{pol}}{\tilde{P}_e} \sim \frac{\overline{k_r k_\theta}}{\overline{k_\theta^2}} \frac{\eta_e}{1 + 3\eta_e/2} \frac{\nu_e}{\epsilon_0^{3/2} \omega_{c,i}} \frac{L_{T_e}}{L_{env}}. \tag{34}$$

Here the bar denotes spectral average, i.e., $\overline{(\ldots)} \equiv \sum_{k\omega} (\ldots) \langle \tilde{\phi}^2 \rangle_{k\omega} / \sum_{k\omega} \langle \tilde{\phi}^2 \rangle_{k\omega}$, L_{env} is the scale length of envelope variation. Setting $L_{env} \sim \sqrt{\rho_i L_{T_e}}$, typical of mesoscales, we have

$$\frac{\tilde{P}_{pol}}{\tilde{P}_e} \sim \frac{\overline{k_r k_\theta}}{\overline{k_\theta^2}} \frac{\eta_e}{1 + 3\eta_e/2} \frac{\nu_e/\epsilon_0}{v_{thi}/L_{T_e}} \frac{1}{\epsilon_0^{1/2}} \sqrt{\frac{\rho_i}{L_{T_e}}}. \tag{35}$$

Here typically $\overline{k_r k_\theta}/\overline{k_\theta^2} \sim \eta_e/(1 + 3\eta_e/2) \sim O(1)$. While $\sqrt{(\rho_i/L_{T_e})}$ is small, it is multiplied by $\epsilon_0^{-1/2} > 1$ and $(\nu_e/\epsilon_0)/(v_{thi}/L_{T_e}) > 1$ (by the frequency ordering for electron collisions). Then we can see that $\tilde{P}_{pol}/\tilde{P}_e$ can easily be order unity and thus $\tilde{P}_{pol}$ should be included in the analysis. This is especially true around the ion barrier region where the gradients are steep.

In summary, we obtained the evolution equation for phasetrophy as

$$\left(\frac{\partial}{\partial t}+\tau_L^{-1}\right)\lim_{1\to 2}\langle\delta h(1)\delta h(2)\rangle=\lim_{1\to 2}P(1,2), \quad (36)$$

where τ_L^{-1} is

$$\tau_L^{-1}\lim_{1\to 2}\langle\delta h(1)\delta h(2)\rangle=\lim_{1\to 2}\left(v_{rel}\frac{\partial}{\partial y_-}\langle\delta h(1)\delta h(2)\rangle + T(1,2)+C(1,2)\right), \quad (37)$$

where $v_{rel}=\bar{v}_D\bar{E}_- + \langle v_E\rangle' x_-$, T(1,2) is the triplet term defined by Eq. (13b) and C(1,2) is the collision term defined by Eq. (13c). The key difference between τ_L^{-1} and P(1,2) is their small scale behavior. As $1\to 2$, τ_L^{-1} approaches to a small value which is determined by collisions, $\tau_L^{-1}\to 2\nu_{eff}\langle\delta h^2\rangle$ where a Krook operator $C(1,2)=\nu_{eff}\langle\delta h(1)\delta h(2)\rangle$ is used for the purposes of estimation. As $1\to 2$, P(1,2) remains finite and the total production is

$$\lim_{1\to 2}P(1,2)\equiv P_{i,i}+P_{i,e}+P_{i,pol}, \quad (38a)$$

$$P_{i,i}=2\sum_{k\omega}(\omega-\omega_*^i(E))^2\langle f_i\rangle^2\mathrm{Re}g_{k\omega}S_{k\omega}, \quad (38b)$$

$$P_{i,e}=-2\sum_{k\omega}(\omega-\omega_*^i(E))\langle f_i\rangle\frac{\mathrm{Im}\epsilon_e}{|\epsilon(k,\omega)|^2}\left\langle\frac{\widetilde{\delta n}}{n_0}\widetilde{\delta h^*}\right\rangle_{k\omega}, \quad (38c)$$

$$P_{i,pol}=-2\sum_{k\omega}\langle f_i\rangle\frac{2v_*^i\rho_i^2k_rk_\theta}{|\epsilon(k,\omega)|^2}\left\langle\frac{\widetilde{\delta n}}{n_0}\partial_r\widetilde{\delta h^*}\right\rangle_{k\omega}, \quad (38d)$$

where $S_{k\omega}$ is the spectrum defined by Eq. (29). $P_{i,i}$ is renormalized coherent production. $P_{i,e}$ is due to the coupling of the ion wake to electron dissipation. $P_{i,pol}$ is from the coupling of the ion wake to polarization charge. This term introduces a novel zonal flow effect into granulation dynamics. This new effect is different from the conventional zonal flow effects such as shearing suppression of turbulence or cross-phase modification.[34] Indeed, this effect is akin to a reduction of the production term by scattering of momentum (and energy) to the zonal flow.

C. Phase space density granulation and zonal flows: Connection to the momentum theorem in quasigeostrophic system and its consequences

The phasetrophy evolution derived in the above can be put in a form where zonal flow coupling is more apparent. In doing so, we divide phasetrophy evolution by $\langle f\rangle$ and integrate over velocity space to obtain

$$\left(\frac{\partial}{\partial t}+\tau_L^{-1}\right)\int d^3v\frac{\langle\delta h^2\rangle}{2\langle f\rangle}=\int d^3v\frac{P_{i,i}+P_{i,e}+P_{i,pol}}{2\langle f\rangle}, \quad (39)$$

where $\langle\delta h^2\rangle\equiv\lim_{1\to 2}\langle\delta h(1)\delta h(2)\rangle$. $P_{i,pol}$ term introduces zonal flow coupling as discussed above. Utilizing the expression for $P_{i,pol}$ gives

$$\frac{\partial}{\partial t}\int d^3v\frac{\langle\delta h^2\rangle}{2\langle f\rangle}=\int d^3v\frac{P_{i,i}+P_{i,e}}{2\langle f\rangle}+\frac{v_*^i}{v_{thi}^2}\partial_r\langle\tilde{v}_r\tilde{v}_\theta\rangle - \tau_L^{-1}\int d^3v\frac{\langle\delta h^2\rangle}{2\langle f\rangle}. \quad (40)$$

Thus, as in the single structure limit, phase space density granulation evolution is also dynamically coupled to zonal flows.

Equation (40) has the same structure as the fundamental momentum constraint in QG system, namely the C-D momentum theorem. For comparison, below we re-write the C-D theorem for the Hasegawa-Wakatani system in a similar form,

$$\frac{\partial}{\partial t}\frac{\langle\delta q^2\rangle}{2\langle q\rangle'}=-\langle\tilde{v}_r\tilde{n}_e\rangle+\partial_r\langle\tilde{v}_r\tilde{v}_\theta\rangle - \frac{1}{\langle q\rangle'}\left(\frac{\partial}{\partial r}\left\langle\tilde{v}_r\frac{\delta q^2}{2}\right\rangle+D_0\langle(\nabla\delta q)^2\rangle\right), \quad (41)$$

$$\frac{\partial}{\partial t}\int d^3v\frac{\langle\delta h^2\rangle}{2\langle f\rangle}=\int d^3v\frac{P_{i,i}+P_{i,e}}{2\langle f\rangle}+\frac{v_*^i}{v_{thi}^2}\partial_r\langle\tilde{v}_r\tilde{v}_\theta\rangle - \tau_L^{-1}\int d^3v\frac{\langle\delta h^2\rangle}{2\langle f\rangle}. \quad (42)$$

Each term in Eq. (41) has a clear counterpart in Eq. (42). The phasetrophy $\langle\delta h^2\rangle$ is the counterpart of potential enstrophy $\langle\delta q^2\rangle$. $P_{i,i}$ and $P_{i,e}$ represent the effect of relaxation, thus leading to flux of both particles and heat. This clearly corresponds to the particle flux in the momentum theorem for the Hasegawa-Wakatani system. The lifetime of phasetrophy τ_L is analogous to the lifetime of enstrophy via turbulence spreading[35] and viscous dissipation of $\langle\delta q^2\rangle$.

As a consequence, a similar statement as the Charney-Drazin non-acceleration theorem[25] follows: *in the absence of production and dissipation of phase space density granulation, stationary granulation cannot accelerate flow against frictional drag*. This in turn implies that if we have any production or dissipation of phase space density granulation, we must have a corresponding adjustment of the flow, and vice versa.

The coupled system of phase space density granulation and zonal flow shows a self-regulating behavior, as Eq. (40) and the momentum balance equation for zonal flow form a type of predator-prey system,

$$\frac{\partial}{\partial t}\int d^3v\frac{\langle\delta h^2\rangle}{2\langle f\rangle}=\int d^3v\frac{P_{i,i}+P_{i,e}}{2\langle f\rangle}+\frac{v_*^i}{v_{thi}^2}\partial_r\langle\tilde{v}_r\tilde{v}_\theta\rangle - \tau_L^{-1}\int d^3v\frac{\langle\delta h^2\rangle}{2\langle f\rangle}, \quad (43a)$$

$$\frac{\partial}{\partial t}\langle v_\theta\rangle = -\partial_r\langle\tilde{v}_r\tilde{v}_\theta\rangle - \nu\langle v_\theta\rangle, \tag{43b}$$

which clearly has the same structure as the familiar predator-prey model,[29,36]

$$\partial_t\varepsilon = \gamma_L\varepsilon - \alpha V'^2\varepsilon - \Delta\omega(\varepsilon)\varepsilon, \tag{44a}$$

$$\partial_t V'^2 = \alpha V'^2\varepsilon - \nu_{col}V'^2, \tag{44b}$$

where ε is the turbulence intensity, V'^2 is flow shear, γ_L is linear growth rate of a mode, α represents a coupling between flow and fluctuations, $\Delta\omega$ is a decorrelation rate, ν_{col} is the collisional drag on the flow. Then, by comparison, Eq. (43a) can be viewed as the equation for prey, which here is the phasetrophy. The prey are produced by mean field relaxation due to $P_{i,i}$ and $P_{i,e}$. Death of prey occurs due to the granulation dispersion τ_L^{-1} and due to coupling to the predator, namely the zonal flow $\partial_r\langle\tilde{v}_r\tilde{v}_\theta\rangle$. Eq. (43b) is the equation for the predator. The predator is pumped by consuming the prey, such as phase space density granulations which drive the Reynolds stress. *The predator-prey system here may be compared to the kinetic predator-prey system derived based on entropy balance.*[37] *Ultimately, the both systems are derived from the dynamics of the same quantity, namely phase space density correlation δf^2, which is the fundamental quantity.*

The coupled system of granulations and zonal flows can lead to non-trivial, finite intensity state with zonal flow coupling, namely the zero production state. In a weakly collisional system, zonal flows allow a stationary state with zero total production,

$$0 \cong \int d^3v\frac{P_{i,i}+P_{i,e}}{2\langle f\rangle} + \frac{v_*^i}{v_{thi}^2}\partial_r\langle\tilde{v}_r\tilde{v}_\theta\rangle. \tag{45}$$

The zero production state is of practical interest, since mean field evolution is also vanishes, $\partial_t\langle f\rangle \sim P_{tot} \simeq 0$. Access to the zero production state requires the balance of the relaxation drive $P_{i,i}+P_{i,e}$ with the zonal flow drive $P_{i,pol} \sim \partial_r\langle\tilde{v}_r\tilde{v}_\theta\rangle$. In turn, in the zero net production state a stationary zonal flow can be sustained against collisional drag,

$$\begin{aligned}\langle v_\theta\rangle &= \frac{1}{\nu}\frac{v_{thi}^2}{v_*^i}\int d^3v\frac{P_{i,i}+P_{i,e}}{2\langle f\rangle}\\ &\simeq -\frac{\omega_{c,i}}{\nu}\frac{\langle\tilde{v}_r\tilde{n}_e\rangle}{n_0} + \frac{1}{\nu}\frac{v_{thi}^2}{v_*^i}\int d^3v\frac{P_{i,i}}{2\langle f\rangle},\end{aligned} \tag{46}$$

where we used

$$\begin{aligned}\int d^3v\frac{P_{i,e}}{2\langle f\rangle} &\simeq \int d^3v\sum_{k\omega}k_\theta v_*^i\frac{\mathrm{Im}\epsilon_e}{|\epsilon(k,\omega)|^2}\left\langle\widetilde{\frac{\delta n}{n_0}}\widetilde{\delta h^*}\right\rangle_{k\omega}\\ &= -\frac{v_*^i}{v_{thi}}\frac{1}{\rho_i}\frac{\langle\tilde{v}_r\tilde{n}_e\rangle}{n_0}.\end{aligned} \tag{47}$$

Equation (46) can be compared to the stationary zonal flows in the Hasegawa-Wakatani system and GK system in the single structure limit. The close correspondence is evident. In each system, electron flux can support stationary zonal flow against collisional drag.

D. Transport

Since $2P = \partial_t\langle\delta f^2\rangle \simeq -\partial_t\langle f\rangle^2$, the production term is related to the mean field evolution and transport. The transport flux can be extracted from the phasetrophy production term, $\lim_{1\to2}P(1,2) \sim -\langle\tilde{v}_r\delta f\rangle\langle f\rangle'$, as

$$J(r) \equiv \langle\tilde{v}_r\delta f\rangle = J_{i,i}+J_{i,e}+J_{i,pol}, \tag{48a}$$

$$J_{i,i} = \sum_{k\omega}(\omega-\omega_*^i(E))\langle f_i(E)\rangle k_\theta\rho_i v_{thi}\mathrm{Re}g_{k\omega}S_{k\omega}, \tag{48b}$$

$$J_{i,e} = -\sum_{k\omega}k_\theta\rho_i v_{thi}\frac{\mathrm{Im}\epsilon_e}{|\epsilon(k,\omega)|^2}\left\langle\widetilde{\frac{\delta n}{n_0}}\widetilde{\delta h^*}\right\rangle_{k\omega}, \tag{48c}$$

$$J_{i,pol} = -\sum_{k\omega}k_\theta\rho_i v_{thi}\frac{(-2\rho_i^2k_r)}{|\epsilon(k,\omega)|^2}\left\langle\widetilde{\frac{\delta n}{n_0}}\partial_r\widetilde{\delta h^*}\right\rangle_{k\omega}, \tag{48d}$$

where $S_{k\omega}$ is the spectrum defined by Eq. (29). $J_{i,i}$ is the flux which arises from the net ion production $P_{i,i}$. $J_{i,i}$ is the diffusive part of the total flux, as $J_{i,i} \simeq -\bar{D}\langle f_i\rangle'$ for $\omega < \omega_*$. Here $\bar{D} = \sum_{k\omega}v_{thi}^2\mathrm{Re}g_{k\omega}k_\theta^2\rho_i^2S_{k\omega}$ is the renormalized diffusivity. $J_{i,i}$ simplifies in certain limits. If we neglect the effect of incoherent fluctuations and retain the spectrum only due to eigenmodes, $J_{i,i}$ reduces to the quasilinear flux,

$$J_{i,i} \simeq -\sum_k\rho_s^2k_\theta^2c_s^2\mathrm{Re}g_k\left|\frac{q\tilde{\phi}}{T_e}\right|_k^2\langle f\rangle'. \tag{49}$$

If we go to the local limit, we have $J_{i,i}\to0$ corresponding to $P_{i,i}\to0$. The net $J_{i,i}\propto\bar{D}\propto P_{i,i}$ arises from the non-cancelation between the coherent and the ion incoherent productions.

$J_{i,e}$ is the dynamical friction which originates from the ion wake drag on the electrons. Here dissipative non-adiabatic electrons are assumed, $\mathrm{Im}\epsilon_e\propto\nu_e^{-1}$. $J_{i,e}$ is utilized to explain the anomalous transport of ion heat and particles due to ion clumps.[16]

$J_{i,pol}$ is the novel piece here, which originates from polarization charge and describes zonal flow coupling. $J_{i,pol}$ may be understood as a zonal flow induced collisionless friction[38] exerted on the ion phase space density. Note that $J_{i,pol}$ algebraically competes against other fluxes. The competition can lead to a saturated state with the zero total flux $J(r) = \langle\tilde{v}_r\delta f\rangle = J_{i,i}+J_{i,e}+J_{i,pol}\simeq0$, which corresponds to the zero net production state discussed above. Here, transport suppression is achieved by the competition between relaxation and Reynolds work, i.e., Eq. (45). This is different from the conventional view of transport suppression by zonal flow shearing. Moreover, retaining dynamical friction is *essential* to the recovery of this effect.

IV. CONCLUSIONS

In this paper, we present a theory for relaxation and transport in collisionless GK turbulence with zonal flow.

This theory treats the effects of both phase space structures and zonal flows. The principal results of the paper are

1. In the strongly resonant limit for $K \gg 1$, the growth of even a single localized structure in phase space is seen to be strongly coupled to zonal flows. $-\int \sqrt{E}dE\langle \delta f^2\rangle/\langle f\rangle|_0$ is identified as the zonal pseudomomentum carried by the phase space structure. The net invariance of total dipole moment was used to reveal the zonal flow coupling in the structure growth equation. The foundation of this is the conservation of zonal momentum between phase space fluid and zonal flow. The resultant expression, Eq. (8), is shown to be closely related and very similar to the Charney-Drazin theorem for the Hasegawa-Wakatani system, a fundamental momentum constraint in quasigeostrophic systems.
2. For $K \sim 1$, a statistical theory of granulation evolution and mean field ($\langle f\rangle$) evolution was formulated in the presence of zonal flows. In particular,

   (a) Zonal flow coupling in the granulation dynamics is introduced by the production due to polarization charge mixing. The production due to polarization charge arises due to envelope coupling, which can be introduced by the spatial variation intrinsic to the wake emitted by granulation. Other processes, such as mode propagation and absorption, etc., can contribute to the envelope structure. The production due to the polarization charge, necessarily coupled via the GK Poisson equation, is explicitly related to Reynolds force, Eq. (33c).
   (b) The coupled system of granulations and zonal flow form a type of self-regulating, kinetic predator-prey system, Eqs. (43a) and (43b). The coupling allows the system to achieve a finite amplitude state of vanishing production, by balancing granulation induced production due to ∇T relaxation with the Reynolds work which produces the zonal flow.
   (c) The mean field evolution is calculated and various contributions to the transport fluxes are given, including the diffusive flux as well as dynamical friction. Dynamical friction arises from the zonal flow, Eq. (48d). The dynamical friction competes against other fluxes algebraically, which is similar to the effect of zonal flow in the predator-prey system and is different from the conventionally invoked zonal flow effects on transport, namely cross phase modification and simple amplitude suppression.

Throughout the paper, the quantity which plays the central role is the phase space density correlation $\langle\delta f(1)\delta f(2)\rangle$. $\langle\delta f(1)\delta f(2)\rangle$ is the fundamental correlation, as we can easily translate or relate $\langle\delta f(1)\delta f(2)\rangle$ to other physical quantities, including the pseudomomentum of phase space turbulence, the fluctuation entropy, and the fluctuation phasetrophy $\langle\delta f^2\rangle$ which is similar to potential enstrophy, Fig. 8.

As $\langle\delta f^2\rangle$ is closely related to different quantities, its time evolution can also be interpreted in several ways. The evolution of $\langle\delta f^2\rangle$ can be related to the momentum conservation constraint for phase space turbulence, which is similar to the Charney-Drazin momentum theorem for QG turbulence. Alternatively, the evolution of $\langle\delta f^2\rangle$ can be related to the entropy balance equation with production from relaxation drive, destruction from Reynolds work, and $\langle\delta f^2\rangle$ coupling to small scale dissipation. See Table III. In either case, GK turbulence and zonal flows are coupled via dynamical friction due to polarization charge and so form a self-regulating system.

Phase space density correlation

$\langle\delta f(1)\delta f(2)\rangle$

Kinetic wave activity density

$\int \sqrt{E}dE\langle\delta f^2\rangle/\langle f\rangle'|_0$

$\rightarrow$ pseudomomentum

Fluctuation entropy

$\int \sqrt{E}dE\langle\delta f^2\rangle/\langle f\rangle$

FIG. 8. Relation of "phasetrophy" $\langle\delta f(1)\delta f(2)\rangle$ to other relevant physical quantities.

The coupled system derived in this paper for describing GK turbulence and zonal flow conserves energy and zonal momentum. The momentum conservation is via dynamical friction due to zonal flow and sets a fundamental constraint on the modeling of GK turbulence and zonal flow generation. Thus, any GK model which includes zonal flow generation *must* also include dynamical friction, otherwise momentum and energy are not conserved between fluctuation and flows. Then, for example, we see that the usual quasilinear description of GK turbulence production and transport is not compatible with a proper description of zonal flow generation.

A similar behavior to the predator-prey type system described here is observed in the recent computational work[39] on the entropy transfer between ITG/ETG and zonal flow. The *non-local* transfer of entropy between drift wave and zonal flow described in the work can be then understood as a simple variant of the well known zonal flow shearing feedback in the predator-prey system. Indeed, it has *long* been known that large scale shears produces non-local potential enstrophy transfer to small scales in QG systems.[28] Then, it is no surprise that large scale shear produces non-local entropy (closely related to phasetrophy, akin to potential enstrophy) transfer to small scales in GK systems.

In the related vein, nonlocality in *physical space* or *transport* is an important unresolved issue. The most clear physical process which underpins nonlocality is avalanching, process akin to coupled topplings of neighboring cites in a sandpile, which has also been observed in GK simulations. This paper does not treat avalanching. We note, however, that the response to phase space density granulations is intrinsically non-local, i.e., Eq. (21), and so can serve as a seed in granulation formation.

This paper sets forth the basic theory of relaxation in a system with granulations which also couple to zonal flows. The next step in this program is to solve the coupled phase space density correlation and zonal flow equations, i.e., Eqs.

TABLE III. Comparison of phase space density correlation evolution to other drift wave-zonal flow system.

	Fluctuation intensity in fluid DWT models, ϵ	Potential enstrophy in QG system, $\langle \delta q^2 \rangle$	Entropy $\int d^3v \langle \delta f^2 \rangle / \langle f \rangle$	Phase space density correlation $\langle \delta h(1) \delta h(2) \rangle$
Drive	Linear instability $\gamma_L |\hat{\phi}|^2$	Forcing,…	Production by heattransport $-\langle \tilde{v}_r \tilde{T}_i \rangle \langle T_i \rangle'$	Production by relaxation $P_{i,i} + P_{i,e}$
Zonal flow coupling	Modulational Instability$-\alpha |\hat{\phi}|^2 V'^2$	Vorticity flux $\langle \tilde{v}_r \nabla^2 \varphi \rangle = \partial_r \langle \tilde{v}_r \tilde{v}_\theta \rangle$	Entropy destruction by flow organization $-\langle \tilde{v}_r \tilde{v}_\theta \rangle \langle v_\theta \rangle' \rightarrow -\gamma_{ZF} \langle v_\theta \rangle'^2$	Dynamical friction by polarization charge $P_{i,pol} \rightarrow v_* \partial_r \langle \tilde{v}_r \tilde{v}_\theta \rangle$
Coupled equation for turbulence and flow	Predator-prey system, ϵ and V'	Charney-Drazinor predator-prey	Kinetic predator-prey $\int d^3v \langle \delta f^2 \rangle / \langle f \rangle$ and $\langle v_\theta \rangle$	Kinetic predator-prey or kinetic Charney-Drazin

(36) and (43b). This forthcoming work will examine possible wave and/or granulation driven relaxation processes, and the zonal flow effects on each of these. Special attention is focused on subcritical processes.

Finally, we point out that the paradigm considered here, namely relaxation and transport in the presence of phase space structures and zonal flows, is not only applicable to collisionless ITG turbulence but is also of interest in the context of energetic particle mode (EPM). Indeed, formation of structures in EPM is likely.[40] A key physical point here is that EPM excitation is due to precession resonance, which is rather coherent.[41] As a consequence, the mode localizes where the drive is strongest. Thus, a description in terms of screened macro-particles seems quite natural. Also, zonal flow generation in energetic particle induced Alfven turbulence has also been reported.[42] Thus, the framework presented here should be applicable to a self-consistent description of transport in EPM turbulence as well. This will be pursued in the near future.

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APPENDIX A: DERIVATION OF PLASMA DIELECTRIC

Here we derive the plasma dielectric for the trapped ion mode model used in Sec. III. The derivation is standard; we linearize the kinetic equation and require a quasi-neutrality. The frequency ordering of interest here is

$$\omega_{ti}, \quad \omega_{bi} > \omega_*^i > \omega \sim \omega_{Di} > v_{eff}^i,$$
$$\omega_{te}, \quad \omega_{be} > v_{eff}^e > \omega_*^e > \omega,$$

where ω_t is the transit frequency, ω_b is the bounce frequency, ω_* is the diamagnetic frequency, ω_D is a frequency due to magnetic drift, $v_{eff} = v_c/\epsilon_0$ is effective collision frequency, and ϵ_0 is the inverse aspect ratio. In the above frequency ordering, the linear calculation yields density perturbation as

$$\left(\frac{\delta n_i}{n_0}\right)_{k\omega} = -\frac{q\tilde{\phi}_{k\omega}}{T_i} + \int d^3v \frac{\omega - \omega_*^i(E)}{\omega - \bar{\omega}_D \bar{E} + k_\theta \langle v_E \rangle(r)} \frac{q\tilde{\phi}_{k\omega}}{T_i} \langle f_i \rangle + \left(\frac{\widetilde{\delta n_i}}{n_0}\right)_{k\omega}, \tag{A1}$$

$$\left(\frac{\delta n_e}{n_0}\right)_{k\omega} = \frac{q\tilde{\phi}_{k\omega}}{T_e} + i \int d^3v \frac{\omega - \omega_*^e(E)}{v_e/\epsilon_0} \bar{E}^{3/2} \frac{q\tilde{\phi}_{k\omega}}{T_e} \langle f_e \rangle, \tag{A2}$$

where $\omega_*^\sigma(E) \equiv (k_\theta c T_\sigma \langle f_\sigma(E) \rangle')/(q_\sigma B \langle f_\sigma(E) \rangle) = k_\theta v_*^\sigma (1 + \eta_\sigma (\bar{E} - 3/2))$, $v_*^i \equiv -(\rho_i/|L_n|) v_{thi}$, and $v_*^e = (\rho_s/|L_n|) c_s$. The ion density perturbation consists of two parts, as phase space density δh consists of two pieces, $\delta h = \delta h^c + \widetilde{\delta h}$. δh^c is a phase-coherent response to fluctuation potential, ϕ. This term, upon velocity integral, gives the ion density perturbation which is proportional to fluctuation potential. $\widetilde{\delta h}$ is an *in*coherent part which describes granulation effect. This leads to the last term in the ion density perturbation, $(\widetilde{\delta n_i}/n_0)_{k\omega} \equiv \int d^3v \widetilde{\delta h}_{k\omega}$. For electrons, the non-adiabatic response is retained from a phase shift due to collisions. On substituting the density perturbations into the Gyrokinetic Poisson equation, we have

$$\hat{\epsilon}(k,\omega) \frac{q\tilde{\phi}_{k\omega}}{T_i} = \left(\frac{\widetilde{\delta n_i}}{n_0}\right)_{k\omega}, \tag{A3}$$

where

$$\hat{\epsilon}(k,\omega) = \frac{T_i}{T_e} + \rho_i^2 k_\perp^2 + 1 - P \int d^3v \frac{\omega - \omega_*^i(E)}{\omega - \bar{\omega}_D \bar{E} - k_\theta \langle v_E \rangle(r)} \langle f_i \rangle + i\mathrm{Im}\epsilon_i + i\mathrm{Im}\epsilon_e + i\mathrm{Im}\epsilon_{pol}. \tag{A4}$$

P denotes the principle part of the integral. The imaginary part of the dielectric is defined as

$$\mathrm{Im}\epsilon_i \equiv \int d^3v (\omega - \omega_*^i(E)) \pi \delta(\omega - \bar{\omega}_D \bar{E} - k_\theta \langle v_E \rangle(r)) \langle f_i \rangle = \sqrt{2\epsilon_0} \frac{2}{\sqrt{\pi}} \sqrt{\bar{E}_{res}} \pi \frac{\omega - \omega_*^i(\bar{E}_{res})}{|\bar{\omega}_D|} e^{-\bar{E}_{res}}, \tag{A5}$$

$$\mathrm{Im}\epsilon_e \equiv \int d^3v \frac{\omega - \omega_*^e(E)}{v_e/\epsilon_0} \bar{E}^{3/2} \frac{T_i}{T_e} \langle f_e \rangle = \frac{4}{\sqrt{\pi}} \frac{\sqrt{2\epsilon_0}}{v_e/\epsilon_0} \frac{T_i}{T_e} \left(\omega - \omega_*^e \left(1 + \frac{3}{2}\eta_e\right)\right), \tag{A6}$$

$$\mathrm{Im}\epsilon_{pol} \equiv -2\rho_i^2 k_r \partial_r, \tag{A7}$$

where $\bar{E}_{res} \equiv (\omega - k_\theta \langle v_E \rangle (x))/\bar{\omega}_D$.

The dispersion relation is obtained by setting $\mathrm{Re}\epsilon(k,\omega) = 0$,

$$\frac{T_i}{T_e} + \rho_i^2 k_\perp^2 + 1 - P\int d^3v \frac{\omega - \omega_*^i(E)}{\omega - \bar{\omega}_D \bar{E} - k_\theta \langle v_E \rangle (r)} \langle f_i \rangle = 0. \tag{A8}$$

For $\omega > \bar{\omega}_D \bar{E}$, we have

$$\frac{T_i}{T_e} + \rho_i^2 k_\perp^2 + 1 - \sqrt{2\epsilon_0}\left(1 - \frac{\omega_*^i}{\omega}\right) = 0 \tag{A9}$$

and thus

$$\omega = \omega_k = -\frac{\sqrt{2\epsilon_0}\omega_*^i}{1 + \rho_i^2 k_\perp^2 + T_i/T_e - \sqrt{2\epsilon_0}}. \tag{A10}$$

APPENDIX B: DERIVATION OF ZONAL FLOW EVOLUTION EQUATION

The mean vorticity evolution is obtained by taking time derivative of mean quasi-neutrality (here $\langle \ldots \rangle$ is the zonal average),

$$\frac{\partial}{\partial t}\left(\left\langle \rho_s^2 \nabla_\perp^2 \frac{e\phi}{T_e} \right\rangle\right) = \frac{\partial}{\partial t}\left(\frac{\langle n_e \rangle}{n_0} - \frac{\langle n_i \rangle}{n_0}\right) \tag{B1}$$

$$= \int d^3v (\partial_t \langle f_e \rangle - \partial_t \langle f_i \rangle). \tag{B2}$$

The evolution of mean f_σ is

$$\partial_t \langle f_\sigma \rangle = -\partial_r \langle \tilde{v}_r \delta f_\sigma \rangle. \tag{B3}$$

Combining these, we obtain

$$\frac{\partial}{\partial t}\left(\left\langle \rho_s^2 \partial_r^2 \frac{e\phi}{T_e} \right\rangle\right) = -\frac{\partial}{\partial r}\left\langle \tilde{v}_r \left(\frac{\delta n_e}{n_0} - \frac{\delta n_i}{n_0}\right)\right\rangle. \tag{B4}$$

Using quasi-neutrality, we have

$$\frac{\partial}{\partial t}\left(\left\langle \rho_s^2 \partial_r^2 \frac{e\phi}{T_e} \right\rangle\right) = -\frac{\partial}{\partial r}\left\langle \tilde{v}_r \rho_s^2 \nabla_\perp^2 \frac{e\tilde{\phi}}{T_e} \right\rangle. \tag{B5}$$

The right-hand side contains the flux of vorticity, which is Reynolds forcing via the identity[17,18] $\langle \tilde{v}_r \phi \nabla_\perp^2 \tilde{\phi} \rangle = \partial_r \langle \tilde{v}_r \tilde{v}_\theta \rangle$. By integrating radially once, we obtain

$$\partial_t \langle v_\theta \rangle = -\partial_r \langle \tilde{v}_r \tilde{v}_\theta \rangle. \tag{B6}$$

As shown by Hinton and Rosenbluth,[43] zonal flow is damped by collisions in the time scale $\tau \sim \epsilon_0 \tau_{ii}$ where $\tau_{ii} \sim 1/v_{ii}$. We model the effect by adding a collisional drag on the flow,

$$\partial_t \langle v_\theta \rangle = -\partial_r \langle \tilde{v}_r \tilde{v}_\theta \rangle - v\langle v_\theta \rangle, \tag{B7}$$

where $v = v_{ii}/\epsilon_0$. We note that v is different from the collisional damping of fluctuation used in the text, namely $C(\delta h_i) = - v_i \delta h_i$ and $C(\delta h_e) = -(v_e/\epsilon_0)\bar{E}^{-3/2}\delta h_e$.

We also note that here the collisional drag was added in an ad hoc manner in the zonal flow evolution Eq. (B7). This effect may be recovered systematically by retaining a bounce averaged collision operator in Eq. (B3). Explicitly, by retaining the collision term in Eq. (B3), we see that the vorticity evolution equation (for zonal flow) becomes

$$\frac{\partial}{\partial t}\left(\left\langle \rho_s^2 \partial_r^2 \frac{e\phi}{T_e} \right\rangle\right) = -\frac{\partial}{\partial r}\left\langle \tilde{v}_r \rho_s^2 \nabla_\perp^2 \frac{e\tilde{\phi}}{T_e} \right\rangle + \int_{tr} d^3v \{\overline{C_e(\langle f_e \rangle)} - \overline{C_i(\langle f_i \rangle)}\}, \tag{B8}$$

where $\overline{(\ldots)}$ is the bounce average and the velocity integrals are limited to trapped electron and ions, respectively. Following the argument by Hinton,[43] we replace the ion collision integral by a collisional frictional damping $v = v_{ii}/\epsilon_0$ of the axisymmetric ($n = m = 0$) zonal potential,

$$\frac{\partial}{\partial t}\left(\left\langle \rho_s^2 \partial_r^2 \frac{e\phi}{T_e} \right\rangle\right) = -\frac{\partial}{\partial r}\left\langle \tilde{v}_r \rho_s^2 \nabla_\perp^2 \frac{e\tilde{\phi}}{T_e} \right\rangle - v\left\langle \rho_s^2 \partial_r^2 \frac{e\phi}{T_e} \right\rangle, \tag{B9}$$

which leads to Eq. (B7). Electron collisional effects are negligible, as that species is nearly Maxwellian for zonal modes.

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