From Scatter-Free to Diffusive Propagation of Energetic Particles

Exact Solution of Fokker-Planck equation

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Abstract / Objectives

Abstract Propagation of energetic particles through magnetized turbulent media is reconsidered using the exact solution of Fokker-Planck equation [1]. It shows that the cosmic ray (CR) transport in weakly scattering media is nondiffusive. Poor understanding of the CR transport obscures their sources and acceleration mechanisms.

We present a simplified approximate version [2] of the exact solution of Fokker-Planck equation that accurately describes a ballistic, diffusive and transdiffusive (intermediate between the first two) propagation regimes. The transdiffusive phase lasts for a (surprisingly) long time, $\sim 5t_c$ (five collision times), while starting as early as at $\sim 0.5t_c$. Since the scattering rate is energy dependent ($t_c = t_c(E)$), a large part of the energy spectrum propagates neither diffusively nor ballistically. Its treatment should rely on the exact solution. Significant parts of the spectra affected by the heliospheric modulation, for example, falls into this category.

We present a new approximation of an exact Fokker-Planck propagator. It conveniently unifies the ballistic and Gaussian propagators, currently used (separately) in major Solar modulation and other CR transport models. The maximum deviation of the new propagator from the exact solution (at $t \approx t_c$) is less than a few percent. The work on the further improvement is ongoing.

Questions to Answer At times much shorter than the collision time, $t \ll t_c$, most particles propagate with their initial velocities or their projections on the magnetic field direction, if present. This regime is called the ballistic, or rectilinear propagation. The question then is what happens next, namely at $t \sim t_c$ but before the onset of diffusion at $t \gg t_c$? What exactly is the value of $t > t_c$, when it is safe to switch to the simple diffusive description?

Exact solution of Fokker-Planck equation $\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} = \frac{\partial}{\partial \mu}\left(1-\mu^2\right)\frac{\partial f}{\partial \mu}$

The Fokker-Planck equation

$$\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} = \frac{\partial}{\partial \mu}\left(1-\mu^2\right)\frac{\partial f}{\partial \mu} \qquad (1)$$

is written in units with the particle velocity, v, and scattering frequency, $D = 1/t_c$, are set to unity: $v = D(E) = 1$ (i.e., obtained via $D(E)\,t \to t$, $\quad Dx/v \to x$). Restriction at $t = 0$, $x^n f(x) \to 0$ for $|x| \to \infty$ and $n \geq 0$ is imposed (existence of all moments)

$$M_{ij}(t) = \left\langle \mu^i x^j \right\rangle = \int_{-\infty}^{\infty} dx \int_{-1}^{1} \mu^i x^j f d\mu/2 \qquad (2)$$

for integer $i, j \geq 0$. Multiplying eq.(1) by $\mu^i x^j$ and integrating by parts, we obtain an infinite hierarchy of matrix equations. It can be resolved inductively, by combining the triads of elements (red) and progressing along each matrix anti-diagonal:

$$M = \begin{pmatrix} 1 & \langle x \rangle & \langle x^2 \rangle & \langle x^3 \rangle \\ \langle \mu \rangle & \langle \mu x \rangle & \langle \mu x^2 \rangle & \nearrow \\ \langle \mu^2 \rangle & \langle \mu^2 x \rangle & \nearrow & \\ \langle \mu^3 \rangle & \nearrow & & \\ \nearrow & & & \end{pmatrix}$$

$$M_{ij}(t) = M_{ij}(0)\, e^{-i(i+1)t} + \int_0^t e^{i(i+1)(t'-t)} \times \left[jM_{i+1,j-1}(t') + i(i-1)\, M_{i-2,j}(t') \right] dt' \qquad (3)$$

Setting $M_{00} = 1$, the moment-generating function

$$f_\lambda(t) = \int_{-\infty}^{\infty} f_0(x,t)\, e^{\lambda x} dx = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} M_{0,2n}(t) \qquad (4)$$

relates to the Green function $f_0 = \int f(x,\mu,t) d\mu/2$, with $f_0(x,0) = \delta(x)$. This can be converted into a Fourier transform by setting $\lambda = -ik$. The Green function $f_0(x,t)$

$$f_0(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \sum_{n=0}^{\infty} (-1)^n \frac{k^{2n}}{(2n)!} M_{0,2n}(t) \qquad (5)$$

For $t \gg 1$, after summing up the series, one obtains the conventional diffusive result:

$$f_0(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx - k^2 t/6} = \sqrt{\frac{3}{2\pi t}} e^{-3x^2/2t} \qquad (6)$$

Simplified Universal FP Propagator $f_0(x,t) \approx \frac{1}{4y}\left[\mathrm{erf}\left(\frac{x+y}{\Delta}\right) - \mathrm{erf}\left(\frac{x-y}{\Delta}\right)\right]$ for arbitrary x and t

- for arbitrary $t \sim 1$, the series in eq.(5) need to sum for arbitrary λt (to capture sharp fronts). First, for $t < 1$

$$f_\lambda(t) = \frac{1}{\lambda t'} \sinh(\lambda t') + \frac{t^2}{45}\left[2\cosh(\lambda t) + \left(\lambda t - \frac{2}{\lambda t}\right)\sinh(\lambda t)\right] + \ldots$$

$$+ \ldots \approx f_\lambda \approx \frac{1}{\lambda t'} \sinh(\lambda t')\, e^{\lambda^2 \Delta^2/4}$$

where $t' = t - t^2/3 + \ldots$, and $\Delta(t) \approx 2t^2/3\sqrt{5}$

- for $t > 1$ - similar result, can be unified with $t < 1$ expansion
  - after taking inverse Fourier transform ($\lambda = -ik$)

$$f_0(x,t) = \frac{1}{2\pi} \int e^{ikx} f_{-ik}(t)\, dk$$

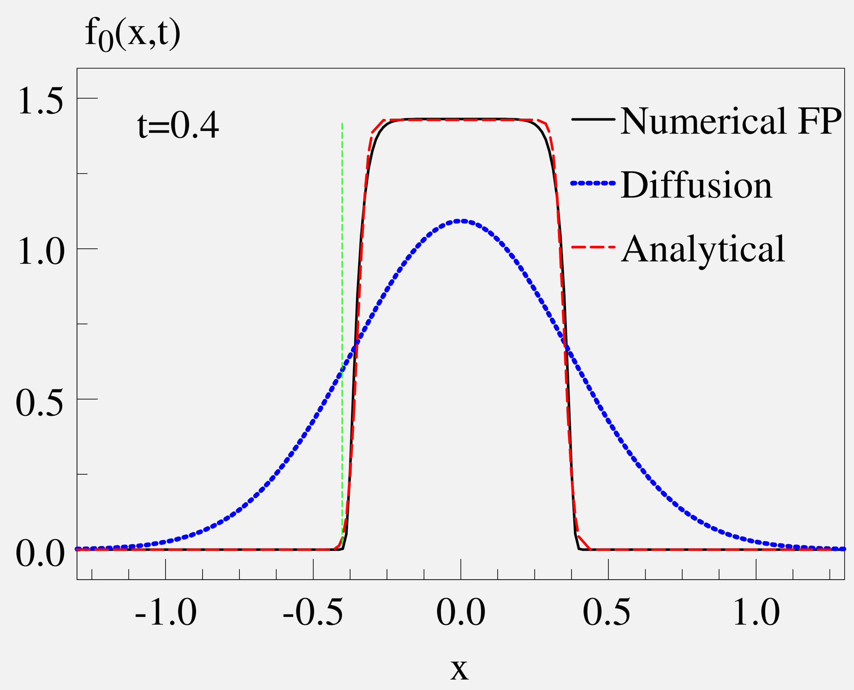
$$f_0(x,t) \approx \frac{1}{4y}\left[\mathrm{erf}\left(\frac{x+y}{\Delta}\right) - \mathrm{erf}\left(\frac{x-y}{\Delta}\right)\right] \qquad (7)$$

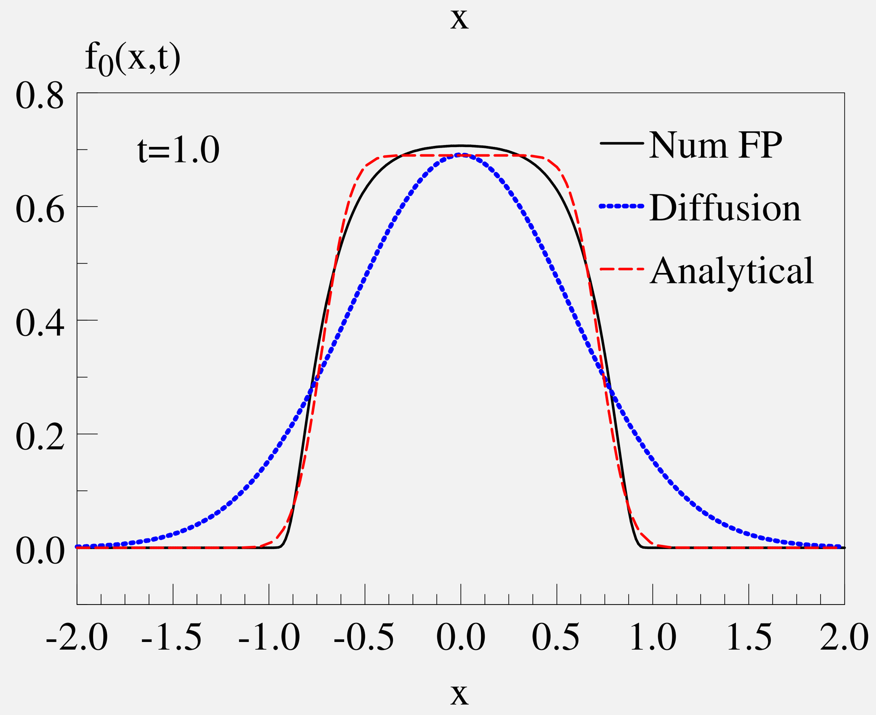
- $t \ll 1$, fronts at $\pm y$, $y \approx t$, thickness $\Delta \approx 2t^2/3\sqrt{5} \ll 1$.
- $t \sim 1$ -transdiffusive phase,
- $t \gg 1$-diffusion: $y \approx (11t/6)^{1/4}$, $\Delta \approx (2t)^{1/2}/3\sqrt{3}$
- propagator in eq.(7) is valid for all $0 < t < \infty$
- the only difference in $y(t)$, and $\Delta(t)$ for $t \ll 1$ and $t \gg 1$
- suggests determination of y and Δ from *exact* relations [2]:

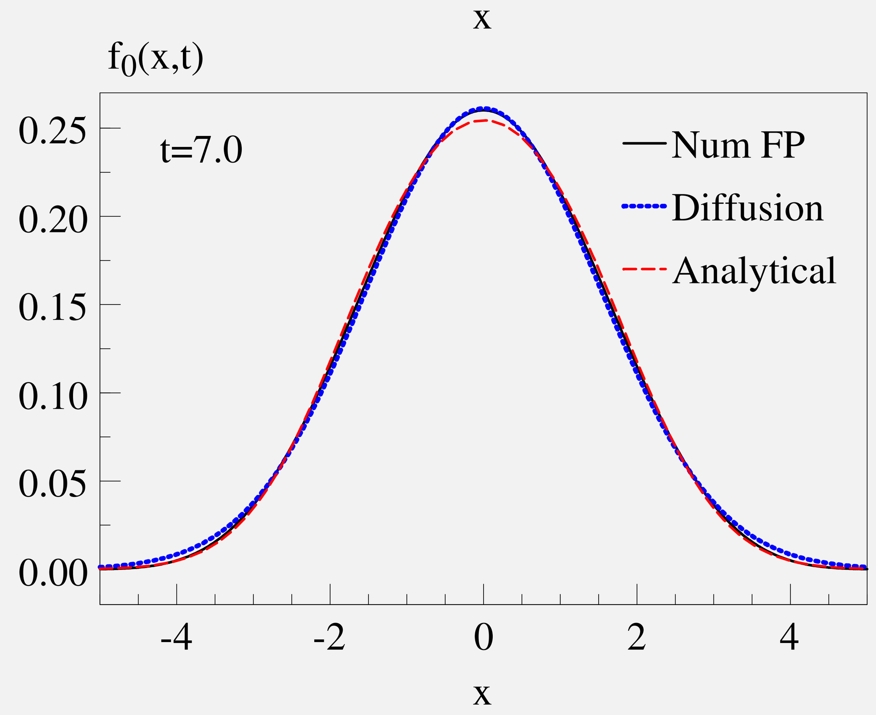
$$M_2 \equiv \int x^2 f_0(x,t)\, dx, \quad M_4 \equiv \int x^4 f_0(x,t)\, dx$$

$$y = \left[\frac{45}{2}\left(M_2^2 - \frac{1}{3}M_4\right)\right]^{1/4}, \quad \Delta = \sqrt{2M_2 - \sqrt{10}\sqrt{M_2^2 - \frac{1}{3}M_4}}$$

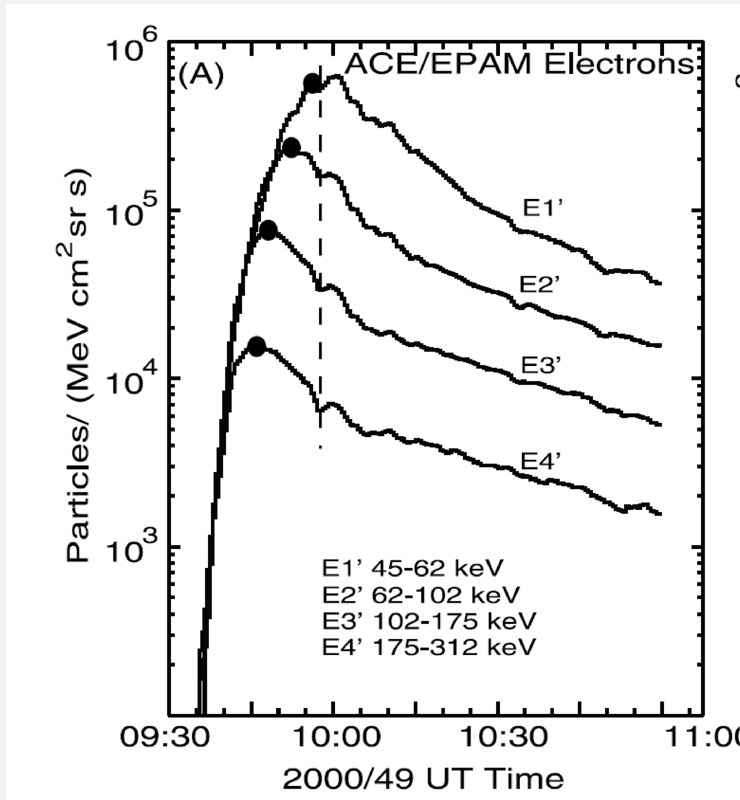
$$M_2 = \frac{t}{3} - \frac{1}{6}\left(1 - e^{-2t}\right), \quad M_4 = \frac{1}{270}e^{-6t} - \frac{t+2}{5}e^{-2t} + \frac{t^2}{3} - \frac{26}{45}t + \frac{107}{270}$$



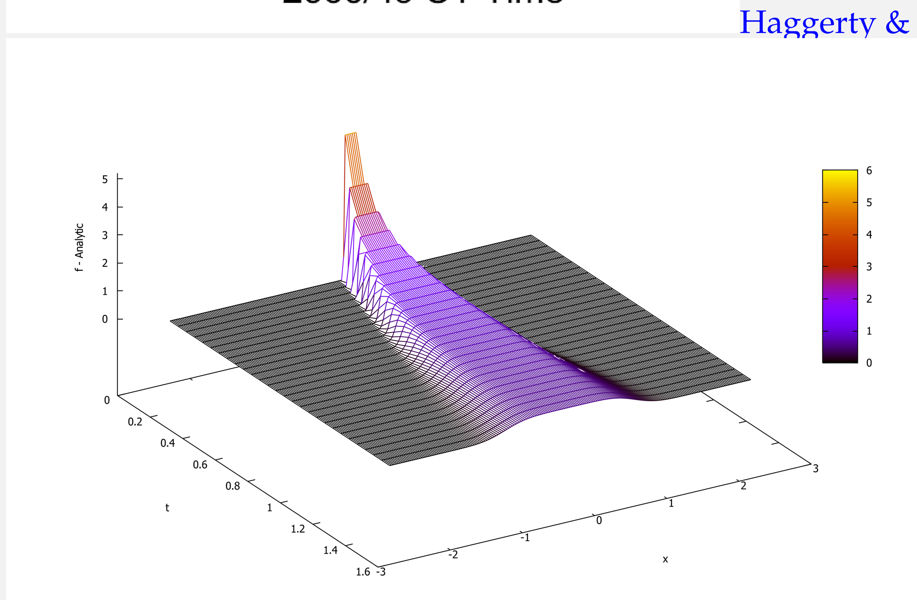




- only for $t \gtrsim 5$, the solution becomes diffusive, eq.(6).
- **comparison with observations**: rapid commencement of SEP event requires ballistic/transdiffusive regimes, way too early for diffusion



Haggerty & Roelof 2002



References

[1] M. A. Malkov. Exact solution of the Fokker-Planck equations for isotropic scattering. *Physical Review D*, 95(2):023007, 2017.

[2] M. A. Malkov. Propagating Cosmic Rays with exact Solution of Fokker-Planck Equation, arXiv:1703.02554. *ArXiv e-prints*, March 2017.