‘Wavy Turbulence’ and Transport in Elastic Systems: A Look at Some VERY Simple Examples

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N.B.:

• For background material on ‘wave turbulence’, see postings.

• More advanced topics:
  “Nonlinear Resonance Analysis”
  Elena Kartashova
  CUP
Recent Collaboration:
• Xiang Fan, Luis Chacon

Past Collaboration and Discussion:
• D. W. Hughes, Steve Tobias, E. Kim, D. R. Nelson, F. Cattaneo, M. R. E. Proctor, A. Gruzinov, M. Vergassola, R. Pandit...
Outline

**Models**
-- What is an Elastic Fluid? (Pedagogic)
  - Oldroyd-B ‘family’, origins
  - MHD connection and Deborah number -> Waves enter!
  - Other systems, esp: Spinodal Decomposition in binary mixture

**(Linked) Single Eddy**
  - Flux Expulsion – 2D MHD
    - Kinematics – two views
    - Dynamics – vortex disruption
  - Cahn-Hilliard Flows and Target Patterns
Outline

Turbulence

• 2D MHD – Quick Review
  o Dual cascade
  o A closer at $\langle \tilde{A}^2 \rangle$

• Cahn-Hilliard Navier-Stokes (CHNS)
  o Scales, ranges, trends
  o Cascades and power laws
  o Lessons
Outline

Active Scalar Transport

• 2D MHD – Flux Diffusion
  o Kinematics
  o Quenching: Alfvénization for vortex disruption
  o Thoughts on transport dynamics -> Transport Bifurcations and Barriers

• CHNS -- $\psi$ as the Active Scalar

Conclusions, of Sorts
Models
Elastic Fluid -> Oldroyd-B Family Models

\[ \gamma \left( \frac{d\vec{r}_{1,2}}{dt} - \vec{v}(\vec{r}_{1,2}, t) \right) = -\frac{\partial U}{\partial \vec{r}_{1,2}} + \tilde{\xi}, \text{ where } U = \frac{k}{2} (\vec{r}_1 - \vec{r}_2)^2 + \cdots \]

\[ \text{so } \frac{d\vec{R}}{dt} = \vec{v}(\vec{R}, t) + \tilde{\xi} / \gamma, \text{ and } \frac{d\tilde{q}}{dt} = \tilde{q} \cdot \nabla \vec{v}(\vec{R}, t) - \frac{2}{\gamma} \frac{\partial U}{\partial \tilde{q}} + \text{noise} \]
Seek \( f(\mathbf{q}, \mathbf{R}, t|\mathbf{v}, \ldots) \rightarrow \text{distribution} \)

\[
\begin{align*}
\partial_t f + \partial_{\mathbf{R}} \cdot [\mathbf{v}(\mathbf{R}, t)f] + \partial_{\mathbf{q}} \cdot [\mathbf{v} \cdot \nabla \mathbf{v}(\mathbf{R}, t)f - \frac{2}{\gamma} \frac{\partial U}{\partial \mathbf{q}} f] &= \partial_{\mathbf{R}} \cdot \mathbf{D}_0 \cdot \frac{\partial f}{\partial \mathbf{R}} + \partial_{\mathbf{q}} \cdot \mathbf{D}_q \cdot \frac{\partial f}{\partial \mathbf{q}} \\
\text{Is F.P. valid?!}
\end{align*}
\]

and moments:

\[
Q_{ij}(\mathbf{R}, t) = \int d^3 \mathbf{q} \; q_i q_j \, f(\mathbf{q}, \mathbf{R}, t) \rightarrow \text{elastic energy field (tensor)}
\]

so:

\[
\begin{align*}
\partial_t Q_{ij} + \mathbf{v} \cdot \nabla Q_{ij} &= Q_{i\gamma} \partial_{\gamma} v_j + Q_{j\gamma} \partial_{\gamma} v_i \\
-\omega_z Q_{ij} + D_0 \nabla^2 Q_{ij} + 4 \frac{k_B T}{\gamma} \delta_{ij} &= \text{and concentration equation}
\end{align*}
\]

- Defines Deborah number: \( \nabla \mathbf{v}/\omega_z \)
Reaction on Dynamics

\[ \rho [ \partial_t \mathbf{v}_i + \mathbf{v} \cdot \nabla \mathbf{v}_i ] = -\nabla P + \nabla \cdot [c_p k Q_{ij}] + \eta \nabla^2 \mathbf{v}_i + f_i \]

- Classic systems; Oldroyd-B (1950).
- Extend to nonlinear springs (FENE), rods, rods + springs, networks, director fields, etc...
- Supports elastic \textit{waves} and fluid dynamics, depending on Deborah number.
- Oldroyd-B \leftrightarrow \textit{active tensor} field
Constitutive Relations

- J. C. Maxwell:

\[
\text{(stress)} + \tau_R \frac{d(\text{stress})}{dt} = \eta \frac{d}{dt} (\text{strain})
\]

- If \( \tau_R/T = D \ll 1 \), stress \( = \eta \frac{d}{dt} (\text{strain}) \)

\[
\sigma = -\eta \nabla \dot{v}
\]

- If \( \tau_R/T = D \gg 1 \), stress \( \cong \frac{\eta}{\tau_R} (\text{strain}) \)

\[
\sim E (\text{strain})
\]

- Limit of “freezing-in”: \( D>1 \) is criterion.

\( T \equiv \text{dynamic time scale} \)
Relation to MHD?!

- Re-writing Oldroyd-B:
  \[
  \frac{\partial}{\partial t} \mathbf{T} + \mathbf{v} \cdot \nabla \mathbf{T} - \mathbf{T} \cdot \nabla \mathbf{v} - (\nabla \mathbf{v})^T \cdot \mathbf{T} = \frac{1}{\tau} (\mathbf{T} - \frac{\mu}{\tau} \mathbf{I})
  \]

- MHD: \( \mathbf{T}_m = \frac{\mathbf{B} \mathbf{B}}{4\pi} \)
  \[
  \partial_t \mathbf{B} + \mathbf{v} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{v} + \eta \nabla^2 \mathbf{B}
  \]

- So
  \[
  \frac{\partial}{\partial t} \mathbf{T}_m + \mathbf{v} \cdot \nabla \mathbf{T}_m - \mathbf{T}_m \cdot \nabla \mathbf{v} - (\nabla \mathbf{v})^T \cdot \mathbf{T}_m = \eta \left[ \mathbf{B} \nabla^2 \mathbf{B} + (\nabla^2 \mathbf{B}) \mathbf{B} \right]
  \]

- \( \lim_{D \to \infty} \) (Oldroyd-B) \( \iff \lim_{R_m \to \infty} \) (MHD)

  c.f. Ogilvie and Proctor
Elastic Media -- What Is the CHNS System?

- Elastic media – Fluid with internal DoFs → “springiness”
- The Cahn-Hilliard Navier-Stokes (CHNS) system describes **phase separation** for binary fluid (i.e. **Spinodal Decomposition**)

![Miscible phase → Immiscible phase](image)

- [Fan *et al.* Phys. Rev. Fluids 2016]
- [Kim *et al.* 2012]
Elastic Media? -- What Is the CHNS System?

- How to describe the system: the concentration field

\[ \psi(\vec{r}, t) \overset{\text{def}}{=} \frac{[\rho_A(\vec{r}, t) - \rho_B(\vec{r}, t)]}{\rho} : \text{scalar field } \rightarrow \text{density contrast} \]

- \( \psi \in [-1,1] \)

- CHNS equations (2D):

\[
\begin{align*}
\partial_t \psi + \vec{v} \cdot \nabla \psi &= D \nabla^2 (-\psi + \psi^3 - \xi^2 \nabla^2 \psi) \\
\partial_t \omega + \vec{v} \cdot \nabla \omega &= \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega
\end{align*}
\]
Why Should a Plasma Physicist Care?

- Useful to examine familiar themes in plasma turbulence from new vantage point
- Some key issues in plasma turbulence:

1. Electromagnetics Turbulence
   - CHNS vs 2D MHD: analogous, with interesting differences.
   - Both CHNS and 2D MHD are elastic systems
   - Most systems = 2D/Reduced MHD + many linear effects
     - Physics of dual cascades and constrained relaxation → relative importance, selective decay...
     - Physics of wave-eddy interaction effects on nonlinear transfer (i.e. Alfven effect ↔ Kraichnan)
Why Care?

2. Zonal flow formation $\rightarrow$ negative viscosity phenomena
   - ZF can be viewed as a “spinodal decomposition” of momentum.
   - What determines scale?

Spinodal Decomposition

Arrows: $\psi$ for CHNS; flow for ZF.

Why Care?

3. “Blobby Turbulence”
   - CHNS is a naturally blobby system of turbulence.
   - What is the role of structure in interaction?
   - How to understand blob coalescence and relation to cascades?
   - How to understand multiple cascades of blobs and energy?

   • CHNS exhibits all of the above, with many new twists
A Brief Derivation of the CHNS Model

- Second order phase transition $\rightarrow$ Landau Theory.
- **Order parameter**: $\psi(\mathbf{r}, t) \overset{\text{def}}{=} \frac{[\rho_A(\mathbf{r}, t) - \rho_B(\mathbf{r}, t)]}{\rho}$
- **Free energy**: 

$$F(\psi) = \int d\mathbf{r} \left( \frac{1}{2} C_1 \psi^2 + \frac{1}{4} C_2 \psi^4 + \frac{\xi^2}{2} |\nabla \psi|^2 \right)$$

- $C_1(T), C_2(T)$.
- Isothermal $T < T_C$. Set $C_2 = -C_1 = 1$:

$$F(\psi) = \int d\mathbf{r} \left( -\frac{1}{2} \psi^2 + \frac{1}{4} \psi^4 + \frac{\xi^2}{2} |\nabla \psi|^2 \right)$$
A Brief Derivation of the CHNS Model

- Continuity equation: \( \frac{d\psi}{dt} + \nabla \cdot \vec{J} = 0 \). Fick’s Law: \( \vec{J} = -D\nabla \mu \).

- Chemical potential: \( \mu = \frac{\delta F(\psi)}{\delta \psi} = -\psi + \psi^3 - \xi^2 \nabla^2 \psi \).

- Combining above \( \rightarrow \) Cahn-Hilliard equation:

\[
\frac{d\psi}{dt} = D\nabla^2 \mu = D\nabla^2 (-\psi + \psi^3 - \xi^2 \nabla^2 \psi)
\]

- \( d_t = \partial_t + \vec{v} \cdot \nabla \). Surface tension: force in Navier-Stokes equation:

\[
\partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v} = -\frac{\nabla p}{\rho} - \psi \nabla \mu + \nu \nabla^2 \vec{v}
\]

- For incompressible fluid, \( \nabla \cdot \vec{v} = 0 \).
2D CHNS and 2D MHD

2D CHNS Equations:

\[
\partial_t \psi + \mathbf{v} \cdot \nabla \psi = D \nabla^2 (-\psi + \psi^3 - \xi^2 \nabla^2 \psi)
\]

\[
\partial_t \omega + \mathbf{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \mathbf{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega
\]

With \( \mathbf{v} = \mathbf{z} \times \nabla \phi \), \( \omega = \nabla^2 \phi \), \( \mathbf{B}_\psi = \mathbf{z} \times \nabla \psi \), \( j_\psi = \xi^2 \nabla^2 \psi \).

2D MHD Equations:

\[
\partial_t A + \mathbf{v} \cdot \nabla A = \eta \nabla^2 A
\]

\[
\partial_t \omega + \mathbf{v} \cdot \nabla \omega = \frac{1}{\mu_0 \rho} \mathbf{B} \cdot \nabla \nabla^2 A + \nu \nabla^2 \omega
\]

With \( \mathbf{v} = \mathbf{z} \times \nabla \phi \), \( \omega = \nabla^2 \phi \), \( \mathbf{B} = \mathbf{z} \times \nabla A \), \( j = \frac{1}{\mu_0} \nabla^2 A \).

\(-\psi\): Negative diffusion term
\(\psi^3\): Self nonlinear term
\(-\xi^2 \nabla^2 \psi\): Hyper-diffusion term

\(A\): Simple diffusion term

<table>
<thead>
<tr>
<th>2D MHD</th>
<th>2D CHNS</th>
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<tbody>
<tr>
<td>Magnetic Potential</td>
<td>( A )</td>
</tr>
<tr>
<td>Magnetic Field</td>
<td>( \mathbf{B} )</td>
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<tr>
<td>Current</td>
<td>( j )</td>
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<tr>
<td>Diffusivity</td>
<td>( \eta )</td>
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<tr>
<td>Interaction strength</td>
<td>( \frac{1}{\mu_0} )</td>
</tr>
<tr>
<td>( \xi^2 )</td>
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Linear Wave

- CHNS supports linear “elastic” wave:

\[ \omega(k) = \pm \sqrt{\frac{\xi^2}{\rho}} |\vec{k} \times \vec{B}_0| - \frac{1}{2} i(CD + \nu)k^2 \]

Where \( C \equiv [-1 - 6\psi_0 \nabla^2 \psi_0/k^2 - 6(\nabla \psi_0)^2/k^2 - 6\psi_0 \nabla \psi_0 \cdot i\kappa/k^2 + 3\psi_0^2 + \xi^2 k^2] \)

- Akin to capillary wave at phase interface. Propagates **only** along the interface of the two fluids, where \( |\vec{B}_0| = |\nabla \psi| \neq 0 \).

- Analogue of Alfven wave.

- Important differences:
  - \( \vec{B}_0 \) in CHNS is large only in the interfacial regions.
  - Elastic wave activity does not fill space.
(Linked) Single Eddy
Flux Expulsion

- Simplest dynamical problem in MHD (Weiss ‘66, et. seq.)
- Closely related to “PV Homogenization”

- Field wound-up, “expelled” from eddy
- For large $Rm$, field concentrated in boundary layer of eddy
- Ultimately, back-reaction asserts itself for sufficient $B_0$

$Rm \sim vL/\eta \gg 1$
How to Describe?

- Flux conservation: $B_0 L \sim b l$
- Wind up: $b = n B_0$ (field stretched)
- Rate balance: wind-up $\sim$ dissipation

$$\frac{\nu}{L} B_0 \sim \frac{\eta}{l^2} b \cdot \tau_{\text{expulsion}} \sim \left( \frac{L}{v_0} \right) Rm^{1/3}.$$  

$$l \sim \delta_{BL} \sim \frac{L}{Rm^{1/3}} \cdot \ b \sim Rm^{1/3} B_0.$$  

after $n$ turns:  
$n l = L$

N.B. differs from Sweet-Parker!
What’s the Physics?

- Shear dispersion! (Moffatt, Kamkar ‘82)

\[ \partial_t A + \vec{v} \cdot \nabla A = \eta \nabla^2 A \]  
(Shearing coordinates)

\[ \nu_y = \nu_y(x) = \nu_{y0} + xv_y' + \cdots \]

\[ \frac{dk_x}{dt} = -k_y v_y', \frac{dk_y}{dt} = 0 \]

\[ \partial_t A + xv_y' \partial_y A - \eta (\partial_x^2 + \partial_y^2) A = 0 \]

\[ A = A(t) \exp i(\vec{k}(t) \cdot \vec{x}) \]

(Shear enhanced dissipation annihilates interior field)

- So \( \tau_{mix} \cong \tau_{shear} Rm^{1/3} = (\nu_y'^{-1}) Rm^{1/3} \)
Single Eddy Mixing -- Cahn-Hilliard

- Structures are the key $\rightarrow$ need understand how a single eddy interacts with $\psi$ field
- Mixing of $\nabla \psi$ by a single eddy $\rightarrow$ characteristic time scales?
- Evolution of structure?
- Analogous to flux expulsion in MHD (Weiss, ‘66)

$\nabla \psi \leftrightarrow B$

Transport / Relaxation
3 stages: (A) the "jelly roll" stage, (B) the topological evolution stage, and (C) the target pattern stage.

ψ ultimately homogenized in slow time scale, but metastable target patterns formed and merge.

Additional mixing time emerges.

Note coarsening!

Single Eddy Mixing

- The bands merge on a time scale long relative to eddy turnover time.
- The 3 stages are reflected in the elastic energy plot.
- The target bands mergers are related to the dips in the target pattern stage.
- The band merger process is similar to the step merger in drift-ZF staircases.

Episodic relaxation-coarsening Cahn-Hilliard dynamics

[Ashourvan et al. 2016]
Back Reaction – Vortex Disruption

- (MHD only) (A. Gilbert et.al. ‘16; J. Mak et.al. ‘17)
- Demise of kinematic expulsion?
  - Magnetic tension grows to react on vorticity evolution!
- Recall: $b \sim B_0 (Rm^{1/3})$
  - B.L. field stretched!
- and $\vec{B} \cdot \nabla \vec{B} = -\frac{|B|^2}{r_c} \hat{n} + \frac{d}{ds} \left( \frac{|B|^2}{2} \right) \hat{t}$
- $|\vec{B} \cdot \nabla \vec{B}| \cong \frac{b^2}{L_0}$
  - vortex scale
Back Reaction – Vortex Disruption

So \( \rho \frac{d\omega}{dt} = \hat{z} \cdot [\nabla \times (\vec{B} \cdot \nabla \vec{B})] \)

\( \rightarrow \rho u \cdot \nabla \omega \sim \frac{b^2}{lL_0} \)

Feedback \( \rightarrow 1 \) for: \( Rm \left( \frac{v_{A0}}{u} \right)^2 \sim 1 \)

Critical value to disrupt vortex, end kinematics

Related Alfven wave emission.

Note for \( Rm \gg 1 \) \( \rightarrow \) strong field not required

Will re-appear...

\( \nu_{A0}^2 = \frac{B_0^2}{4\pi \rho} \)
Turbulence
MHD Turbulence – Quick Primer

- (Weak magnetization / 2D)
- Enstrophy conservation broken
- Alfvenic in $B_{rms}$ field – “magneto-elastic” (E. Fermi ‘49)

$$\epsilon = \frac{\langle \tilde{v}^2 \rangle^2}{l^2} \frac{l}{B_{rms}} \implies E(k) = (\epsilon B_{rms})^{1/2} k^{-3/2}$$ (I-K)

- Dual cascade:
  - Forward in energy
  - Inverse in $\langle A^2 \rangle \sim k^{-7/3}$

- What is dominant (A. Pouquet)?
  - conventional wisdom focuses on energy
  - yet $\langle A^2 \rangle$ conservation – freezing-in law!? 
Ideal Quadratic Conserved Quantities

• 2D MHD
  1. Energy
     \[ E = E^K + E^B = \int \left( \frac{v^2}{2} + \frac{B^2}{2\mu_0} \right) d^2x \]
  2. Mean Square Magnetic Potential
     \[ H^A = \int A^2 d^2x \]
  3. Cross Helicity
     \[ H^C = \int \vec{v} \cdot \vec{B} d^2x \]

• 2D CHNS
  1. Energy
     \[ E = E^K + E^B = \int \left( \frac{\nu^2}{2} + \frac{\xi^2 B^2}{2} \right) d^2x \]
  2. Mean Square Concentration
     \[ H^\psi = \int \psi^2 d^2x \]
  3. Cross Helicity
     \[ H^C = \int \vec{v} \cdot \vec{B}_\psi d^2x \]

Dual cascade expected!
Scales, Ranges, Trends

Fluid forcing → Fluid straining vs Blob coalescence

Straining vs coalescence is fundamental struggle of CHNS turbulence

Scale where turbulent straining ~ elastic restoring force (due surface tension): 

\[ L_H \sim \left( \frac{\rho}{\zeta} \right)^{-1/3} \varepsilon^{-2/9} \]

How big is a raindrop?
• Turbulent straining vs capillarity.
• \( \rho v^2 \) vs \( \sigma/l \).
[Hinze 1955]
Scales, Ranges, Trends

- $L_H/L_d \sim \left(\frac{\rho}{\xi}\right)^{-1/3} \nu^{-1/2} \epsilon^{-1/18}_\Omega \rightarrow$ Extent of the elastic range
- $L_H \gg L_d$ required for large elastic range $\rightarrow$ case of interest

[Diagram showing $H^\psi$ Spectrum with $H^\psi_k = \langle \psi^2 \rangle_k$]
Scales, Ranges, Trends

- Key elastic range physics: **Blob coalescence**
- Unforced case: \( L(t) \sim t^{2/3} \).
  
  (Derivation: \( \hat{\nu} \cdot \nabla \hat{\nu} \sim \frac{\xi^2}{\rho} \nabla^2 \psi \nabla \psi \Rightarrow \frac{L^2}{L} \sim \frac{1}{\rho} \frac{L^2}{L} \))

- Forced case: blob coalescence arrested at Hinze scale \( L_H \).

- \( L(t) \sim t^{2/3} \) recovered
- Blob growth arrest observed
- Blob growth saturation scale tracks Hinze scale (dashed lines)

- Blob coalescence suggests inverse cascade is fundamental here.
Cascades: Comparing the Systems

- Blob coalescence in the elastic range of CHNS is analogous to flux coalescence in MHD.
- Suggests **inverse cascade** of $\langle \psi^2 \rangle$ in CHNS.
- Supported by statistical mechanics studies (absolute equilibrium distributions).
- Arrested by straining.
Cascades

So, dual cascade:

• *Inverse* cascade of $\langle \psi^2 \rangle$
• *Forward* cascade of $E$

Inverse cascade of $\langle \psi^2 \rangle$ is formal expression of blob coalescence process $\rightarrow$ generate larger scale structures till limited by straining

Forward cascade of $E$ as usual, as elastic force breaks enstrophy conservation

Forward cascade of energy is analogous to counterpart in 2D MHD
Cascades

- Spectral flux of $\langle A^2 \rangle$:
  \[
  \Pi_{HA}(k) = \sum_{k' < k} T_{HA}(k') \quad \text{where} \quad T_{HA}(k) = \langle A^2 \rangle \nabla \cdot \nabla A_k
  \]

- Spectral flux of $\langle \psi^2 \rangle$:
  \[
  \Pi_{H\psi}(k) = \sum_{k' < k} T_{H\psi}(k') \quad \text{where} \quad T_{H\psi}(k) = \langle \psi^2 \rangle \nabla \cdot \nabla \psi_k
  \]

- MHD: weak small scale forcing on $A$ drives inverse cascade
- CHNS: $\psi$ is unforced $\rightarrow$ aggregates naturally $\iff$ structure of free energy
- Both fluxes negative $\rightarrow$ inverse cascades
Power Laws

- $\langle A^2 \rangle$ spectrum:

- $\langle \psi^2 \rangle$ spectrum:

- Both systems exhibit $k^{-7/3}$ spectra.

- Inverse cascade of $\langle \psi^2 \rangle$ exhibits same power law scaling, so long as $L_H \gg L_\alpha$, maintaining elastic range: Robust process.
Power Laws

Derivation of -7/3 power law:

For MHD, key assumptions:

• Alfvenic equipartition \((\rho\langle v^2 \rangle \sim \frac{1}{\mu_0}\langle B^2 \rangle)\)

• Constant mean square magnetic potential dissipation rate \(\epsilon_{HA}\), so
\[
\epsilon_{HA} \sim \frac{H_A^3}{\tau} \sim (H_k^A)^\frac{7}{2}k^\frac{7}{2}.
\]

Similarly, assume the following for CHNS:

• Elastic equipartition \((\rho\langle v^2 \rangle \sim \xi^2\langle B_{\psi}^2 \rangle)\)

• Constant mean square magnetic potential dissipation rate \(\epsilon_{H\psi}\), so
\[
\epsilon_{H\psi} \sim \frac{H^\psi}{\tau} \sim (H_k^\psi)^\frac{7}{2}k^\frac{7}{2}.
\]
More Power Laws

- Kinetic energy spectrum (Surprise!):
  - 2D CHNS: $E^K_k \sim k^{-3}$;
  - 2D MHD: $E^K_k \sim k^{-3/2}$.

- The -3 power law:
  - Closer to enstrophy cascade range scaling, in 2D Hydro turbulence.
  - Remarkable departure from expected -3/2 for MHD. Why?

- Why does CHNS $\leftrightarrow$ MHD correspondence hold well for $\langle \psi^2 \rangle_k \sim \langle A^2 \rangle_k \sim k^{-7/3}$, yet break down drastically for energy??
- **What physics** underpins this surprise??
Need to understand **differences**, as well as similarities, between CHNS and MHD problems.

**2D MHD:**
- Fields pervade system.

**2D CHNS:**
- Elastic back-reaction is limited to regions of density contrast i.e. $|\mathbf{B}_\psi| = |\nabla \psi| \neq 0$.
- As blobs coalesce, interfacial region diminished. ‘Active region’ of elasticity decays.
Interface Packing Matters!

- Define the **interface packing fraction** $P$:
  
  $$P = \frac{\text{# of grid points where } |\vec{B}_\psi| > B_{\psi}^{rms}}{\text{# of total grid points}}$$

- $P$ for CHNS decays;
- $P$ for MHD stationary!

- $\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega$: small $P \rightarrow$ local back reaction is weak.

- Weak back reaction $\rightarrow$ reduce to 2D hydro $\rightarrow$ k-spectra
- Blob coalescence coarsens interface network
What Are the Lessons?

- Avoid power law tunnel vision!
- **Real space** realization of the flow is necessary to understand key dynamics. Track interfaces and packing fraction $P$.
- One player in dual cascade (i.e. $\langle \psi^2 \rangle$) can modify or constrain the dynamics of the other (i.e. $E$).
- Against conventional wisdom, $\langle \psi^2 \rangle$ inverse cascade due to blob coalescence is the robust nonlinear transfer process in CHNS turbulence.
- Begs more attention to magnetic helicity in 3D MHD.
Transport
Active Scalar Transport

- Magnetic diffusion, $\psi$ transport are cases of active scalar transport
- (Focus: 2D MHD) (Cattaneo, Vainshtein ’92, Gruzinov, P. D. ’94, ’95)

\[
\partial_t A + \nabla \phi \times \hat{z} \cdot \nabla A = \eta \nabla^2 A
\]
\[
\partial_t \nabla^2 \phi + \nabla \phi \times \hat{z} \cdot \nabla \nabla^2 \phi = \nabla A \times \hat{z} \cdot \nabla \nabla^2 A + \nu \nabla^2 \nabla^2 \phi
\]

- Seek $\langle \nu_x A \rangle = -D_T \frac{\partial \langle A \rangle}{\partial x} - \eta \frac{\partial \langle A \rangle}{\partial x}$
- Point: $D_T \neq \sum_k |\nu_k|^2 \tau_k^E$, often substantially less
- Why: Memory! $\leftrightarrow$ Freezing-in
Origin of Memory?

➤ (a) flux advection vs flux coalescence
   • intrinsic to 2D MHD (and CHNS)
   • rooted in inverse cascade of $\langle A^2 \rangle$

➤ (b) tendency of (even weak) mean magnetic field to “Alfvenize” turbulence [cf: vortex disruption feedback threshold!]

➤ Re (a): Basic physics of 2D MHD

Forward transfer: fluid eddies chop up scalar $A$. 
Memory Cont’d

v.s.

Obvious analogy: straining vs coalescence; CHNS

Upshot: closure calculation yields:

\[ \Gamma_A = - \sum_{\mathbf{k}} \left[ \tau_c^\phi \langle \nu^2 \rangle_{\mathbf{k}} - \tau_c^A \langle B^2 \rangle_{\mathbf{k}} \right] \frac{\partial \langle A \rangle}{\partial x} + \cdots \]

flux of potential competition
scalar advection vs. coalescence (“negative resistivity”)
(+)(-)
Zeldovich and Alfvenization

- Re (b): Competition winner? → Alfvenization!
- Alfvenization is a natural consequence of stronger $\langle B \rangle$, ala’ vortex disruption
- fluid stretches $\langle B \rangle$, ala’ $B_0 \rightarrow b$ in flux expulsion
- How to quantify: Zeldovich Theorem

\[
H_A = \int d^2 x \ H_A = \int d^2 x \langle A^2 \rangle \\
\frac{1}{2} \frac{\partial H_A}{\partial t} = -\Gamma_A \frac{\partial \langle A \rangle}{\partial x} - \eta \langle B^2 \rangle
\]

production  dissipation
Zeldovich and Alfvenization, Cont’d

So \( \langle B^2 \rangle \approx -\frac{\Gamma_A}{\eta} \frac{\partial \langle A \rangle}{\partial x} \approx \frac{D_T}{\eta} \left( \frac{\partial \langle A \rangle}{\partial x} \right)^2 \) (meta-stationary state)

\[ \langle B^2 \rangle \approx \frac{D_T}{\eta} \langle B \rangle^2 \]

O(Rm)

- Strong RMS field generated from modest \( \langle B \rangle \)
- Reflects the effect of small scale B-field amplification (i.e. \( B_0 \rightarrow b \))
- Ultimately, \( \eta \) asserts itself (Cowling)
- Best think \( \langle B^2 \rangle \leftrightarrow T_m \) (elastic energy)
Eliminate $\langle B^2 \rangle$ in $\Gamma_A$ using Zeldovich. So:

$$D_T = D_K / \left[ 1 + Rm \frac{v_A^2}{\langle v^2 \rangle} \right]$$

where:

- $D_K$ is usual kinematic diffusivity
- $Rm \frac{v_A^2}{\langle v^2 \rangle} \sim 1$ identical to vortex disruption threshold
- Weak $B$ “quenches” flux diffusion for large $Rm$

Physics is memory enforced by strong, small scale field.

Implications for $\alpha$, dynamo, etc. (Well-established numerically)
Active scalar transport bifurcation!

\[ \Gamma_A = - \frac{D_K \frac{\partial \langle A \rangle}{\partial x}}{\left[ 1 + \frac{Rm}{\rho \langle v^2 \rangle} \left( \frac{\partial \langle A \rangle}{\partial x} \right)^2 \right]} - \eta \frac{\partial \langle A \rangle}{\partial x} \]  

(Standard form)

i.e.

Spatio-temporal dynamics largely unexplored
- bi-stable system
- fronts, barriers, domains

Expect analogue in CHNS, modulo density gradient
Something Old: Quenching

- $M^2 = \langle \tilde{v}^2 \rangle / v_{A0}^2$
- Higher $v_{A0}^2 / \langle \tilde{v}^2 \rangle \rightarrow$ lower $D_T \rightarrow$ longer $E_m$ persistence
- Ultimately $\eta$ asserts itself

- Blue: $\langle B \rangle$ sufficient for suppression
- Yellow: Ohmic decay phase

[Cattaneo and Vainshtein ’91]
Spatial Structure (Preliminary)

- Initial condition: $\cos(x)$ for $A$
- Shorter time (suppression phase)
  - Domains, and domain boundaries evident, resembles CHNS
  - A transport barriers?!
- Longer time (Ohmic decay phase)
  - Well mixed
  - No evidence nontrivial structure
Something New, Cont’d

- For analysis: pdf of A
- Suppression phase:
  - quenched diffusion
  - bi-modal distribution
    - quenching prevents fill-in
    - consequence i.c.
- Ohmic decay phase:
  - uni-modal distribution returns
Higher Pm (Lower $\eta_T$)

- Bi-modal pdf of A structure persists longer
- Barrier resists Ohmic decay

- A field exhibits strikingly sharp domain structure
- Transition layer (barrier) evident
- Clear example of decoupling of transport, intensity.
What of CHNS?

- So far much the same, without Ohmic decay phase

- CH structure feeds elastic energy $\leftrightarrow$ resembles forcing in B-field in MHD

- Ongoing $\rightarrow$ Layering, staircases?!
Conclusion
Conclusion, of Sorts

- Elastic fluids ubiquitous, interestingly similar and different. Comparison/contrast is useful approach.
- Simple problems, like flux expulsion (50+ years), reveal a lot about basic feedback dynamics.
- CHNS is interesting example of elastic turbulence where energy cascade is not fundamental or dominant.
- Spatio-temporal dynamics of (bi-stable) active scalar transport is a promising direction. Pattern formation in this system is terra novo.
- Revisiting polymer drag reduction would be interesting.