Turbulence and Transport in Elastic Systems: A Look at Some VERY Simple Examples

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Outline

Models

-- What is an Elastic Fluid? (Pedagogic)
  • Oldroyd-B ‘family’, origins
  • MHD connection and Deborah number
  • Other systems, esp: Spinodal Decomposition in binary mixture

(Linked) Single Eddy

• Flux Expulsion – 2D MHD
  o Kinematics – two views
  o Dynamics – vortex disruption
• Cahn-Hilliard Flows and Target Patterns
Outline

➢ Turbulence

• 2D MHD – Quick Review
  o Dual cascade
  o A closer at $\langle A^2 \rangle$

• Cahn-Hilliard Navier-Stokes (CHNS)
  o Scales, ranges, trends
  o Cascades and power laws
  o Lessons
Outline

Active Scalar Transport

- 2D MHD – Flux Diffusion
  - Kinematics
  - Quenching: Alfvénization for vortex disruption
  - Thoughts on transport dynamics
- CHNS -- $\psi$ as the Active Scalar

Conclusions, of Sorts
Models
Elastic Fluid -> Oldroyd-B Family Models

\[ \gamma \left( \frac{d\vec{r}_{1,2}}{dt} - \vec{v}(\vec{r}_{1,2}, t) \right) = -\frac{\partial U}{\partial \vec{r}_{1,2}} + \vec{\xi}, \text{ where } U = \frac{k}{2} (\vec{r}_1 - \vec{r}_2)^2 + \cdots \]

so
\[ \frac{d\vec{R}}{dt} = \vec{v}(\vec{R}, t) + \vec{\xi}/\gamma, \text{ and } \frac{d\vec{q}}{dt} = \vec{q} \cdot \nabla \vec{v}(\vec{R}, t) - \frac{2}{\gamma} \frac{\partial U}{\partial \vec{q}} + \text{noise} \]
Seek \( f(\hat{q}, \hat{R}, t|\hat{v}, ...) \rightarrow \) distribution

\[ \partial_t f + \partial_{\hat{R}} \cdot [\hat{v}(\hat{R}, t)f] + \partial_{\hat{q}} \cdot [\hat{q} \cdot \nabla \hat{v}(\hat{R}, t)f - \frac{2}{\gamma} \frac{\partial U}{\partial \hat{q}} f] = \partial_{\hat{R}} \cdot D_0 \frac{\partial f}{\partial \hat{R}} + \partial_{\hat{q}} \cdot D_q \frac{\partial f}{\partial \hat{q}} \]

Is F.P. valid?

and moments:

\[ Q_{ij}(\hat{R}, t) = \int d^3 q \ q_i q_j f(\hat{q}, \hat{R}, t) \rightarrow \text{electric energy field (tensor)} \]

so:

\[ \partial_t Q_{ij} + \hat{v} \cdot \nabla Q_{ij} = Q_{i\gamma} \partial_\gamma v_j + Q_{j\gamma} \partial_\gamma v_i - \omega_z Q_{ij} + D_0 \nabla^2 Q_{ij} + 4 \frac{k_B T}{\gamma} \delta_{ij} \]

and concentration equation

Defines Deborah number: \( \nabla \hat{v} / \omega_z \)
Reaction on Dynamics

\[ \rho [ \partial_t v_i + \hat{v} \cdot \nabla v_i ] = -\nabla_i P + \nabla_i \cdot [ c_p k Q_{ij} ] + \eta \nabla^2 v_i + f_i \]

- Classic systems; Oldroyd-B (1950).
- Extend to nonlinear springs (FENE), rods, rods + springs, networks, director fields, etc...
- Supports elastic waves and fluid dynamics, depending on Deborah number.
- Oldroyd-B \( \leftrightarrow \) active tensor field
Constitutive Relations

- J. C. Maxwell:

\[
(\text{stress}) + \tau_R \frac{d(\text{stress})}{dt} = \eta \frac{d}{dt} (\text{strain})
\]

- If \( \tau_R / T = D \ll 1 \), stress = \( \eta \frac{d}{dt} (\text{strain}) \)
  \[
  J = - \eta \nabla \dot{\nu}
  \]

- If \( \tau_R / T = D \gg 1 \), stress \( \approx \frac{\eta}{\tau_R} (\text{strain}) \)
  \[
  \sim E (\text{strain})
  \]

- Limit of “freezing-in”: \( D > 1 \) is criterion.

\( T \equiv \text{dynamic time scale} \)
Relation to MHD?!

- Re-writing Oldroyd-B:
  \[
  \frac{\partial}{\partial t} \mathbf{T} + \mathbf{\hat{v}} \cdot \nabla \mathbf{T} - \mathbf{T} \cdot \nabla \mathbf{\hat{v}} - (\nabla \mathbf{\hat{v}})^T \cdot \mathbf{T} = \frac{1}{\tau} \left( \mathbf{T} - \frac{\mu}{\tau} \mathbf{I} \right)
  \]
- MHD: \( \mathbf{T}_m = \frac{\mathbf{B} \cdot \mathbf{B}}{4\pi} \)
  \[
  \partial_t \mathbf{B} + \mathbf{\hat{v}} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{\hat{v}} + \eta \nabla^2 \mathbf{B}
  \]
- So
  \[
  \frac{\partial}{\partial t} \mathbf{T}_m + \mathbf{\hat{v}} \cdot \nabla \mathbf{T}_m - \mathbf{T}_m \cdot \nabla \mathbf{\hat{v}} - (\nabla \mathbf{\hat{v}})^T \cdot \mathbf{T}_m = \eta \left[ \mathbf{B} \nabla^2 \mathbf{B} + (\nabla^2 \mathbf{B}) \mathbf{B} \right]
  \]
- \( \lim_{D \to \infty} \) (Oldroyd-B) \( \iff \) \( \lim_{R_m \to \infty} \) (MHD)
Elastic Media -- What Is the CHNS System

- Elastic media – Fluid with internal DoFs $\rightarrow$ “springiness”
- The Cahn-Hilliard Navier-Stokes (CHNS) system describes phase separation for binary fluid (i.e. Spinodal Decomposition)

Miscible phase $\rightarrow$ Immiscible phase

$[Fan \textit{et.al.} \text{Phys. Rev. Fluids 2016}]$

$[Kim \textit{et.al.} 2012]$
Elastic Media? -- What Is the CHNS System?

- How to describe the system: the concentration field
- \( \psi(\vec{r}, t) \overset{\text{def}}{=} \left[ \rho_A(\vec{r}, t) - \rho_B(\vec{r}, t) \right] / \rho \) : scalar field \( \rightarrow \) density contrast
- \( \psi \in [-1,1] \)
- CHNS equations (2D):

\[
\begin{align*}
\partial_t \psi + \vec{v} \cdot \nabla \psi &= D \nabla^2 \left( -\psi + \psi^3 - \xi^2 \nabla^2 \psi \right) \\
\partial_t \omega + \vec{v} \cdot \nabla \omega &= \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega
\end{align*}
\]
Why Should a Plasma Physicist Care?

- Useful to examine familiar themes in plasma turbulence from new vantage point
- Some key issues in plasma turbulence:

1. Electromagnetics Turbulence
   - CHNS vs 2D MHD: analogous, with interesting differences.
   - Both CHNS and 2D MHD are elastic systems
   - Most systems = 2D/Reduced MHD + many linear effects
     - Physics of dual cascades and constrained relaxation → relative importance, selective decay...
     - Physics of wave-eddy interaction effects on nonlinear transfer (i.e. Alfven effect ↔ Kraichnan)
Why Care?

2. Zonal flow formation → negative viscosity phenomena
   • ZF can be viewed as a “spinodal decomposition” of momentum.
   • What determines scale?


Spinodal Decomposition

Arrows:
ψ for CHNS; flow for ZF.

Zonal Flow

Porter 1981
Why Care?

3. “Blobby Turbulence”
   - CHNS is a naturally blobby system of turbulence.
   - What is the role of structure in interaction?
   - How to understand blob coalescence and relation to cascades?
   - How to understand multiple cascades of blobs and energy?

   • CHNS exhibits all of the above, with many new twists
A Brief Derivation of the CHNS Model

- Second order phase transition $\Rightarrow$ Landau Theory.
- Order parameter: $\psi(\vec{r}, t) \equiv [\rho_A(\vec{r}, t) - \rho_B(\vec{r}, t)]/\rho$
- Free energy:

$$F(\psi) = \int d\vec{r}\left(\frac{1}{2}C_1\psi^2 + \frac{1}{4}C_2\psi^4 + \frac{\xi^2}{2}(|\nabla\psi|^2)\right)$$

- $C_1(T), C_2(T)$.
- Isothermal $T < T_C$. Set $C_2 = -C_1 = 1$:

$$F(\psi) = \int d\vec{r}\left(-\frac{1}{2}\psi^2 + \frac{1}{4}\psi^4 + \frac{\xi^2}{2}(|\nabla\psi|^2)\right)$$
A Brief Derivation of the CHNS Model

- Continuity equation: \( \frac{d\psi}{dt} + \nabla \cdot \mathbf{j} = 0 \). Fick’s Law: \( \mathbf{j} = -D \nabla \mu \).

- Chemical potential: \( \mu = \frac{\delta F(\psi)}{\delta \psi} = -\psi + \psi^3 - \xi^2 \nabla^2 \psi \).

- Combining above \( \rightarrow \) Cahn Hilliard equation:
  \[
  \frac{d\psi}{dt} = D \nabla^2 \mu = D \nabla^2 (-\psi + \psi^3 - \xi^2 \nabla^2 \psi)
  \]

- \( d_t = \partial_t + \mathbf{v} \cdot \nabla \). Surface tension: force in Navier-Stokes equation:
  \[
  \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} - \psi \nabla \mu + \nu \nabla^2 \mathbf{v}
  \]

- For incompressible fluid, \( \nabla \cdot \mathbf{v} = 0 \).
2D CHNS and 2D MHD

2D CHNS Equations:

\[ \partial_t \psi + \mathbf{v} \cdot \nabla \psi = D \nabla^2 (-\psi + \psi^3 - \xi^2 \nabla^2 \psi) \]

\[ \partial_t \omega + \mathbf{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \mathbf{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega \]

With \( \mathbf{v} = \hat{z} \times \nabla \phi \), \( \omega = \nabla^2 \phi \), \( \mathbf{B}_\psi = \hat{z} \times \nabla \psi \), \( j_\psi = \xi^2 \nabla^2 \psi \).

2D MHD Equations:

\[ \partial_t A + \mathbf{v} \cdot \nabla A = \eta \nabla^2 A \]

\[ \partial_t \omega + \mathbf{v} \cdot \nabla \omega = \frac{1}{\mu_0 \rho} \mathbf{B} \cdot \nabla \nabla^2 A + \nu \nabla^2 \omega \]

With \( \mathbf{v} = \hat{z} \times \nabla \phi \), \( \omega = \nabla^2 \phi \), \( \mathbf{B} = \hat{z} \times \nabla A \), \( j = \frac{1}{\mu_0} \nabla^2 A \).
CHNS supports linear “elastic” wave:

$$\omega(k) = \pm \sqrt{\frac{\xi^2}{\rho} |\hat{k} \times \hat{B}_\psi|} - \frac{1}{2}i(\nu k^2 + \nu k^2)$$

Where $C \equiv [-1 - 6\psi_0 \nabla^2 \psi_0 / k^2 - 6(\nabla \psi_0)^2 / k^2 - 6\psi_0 \nabla \psi_0 \cdot i k / k^2 + 3\psi_0^2 + \xi^2 k^2]$

Akin to capillary wave at phase interface. Propagates **only** along the interface of the two fluids, where $|\hat{B}_\psi| = |\nabla \psi| \neq 0$.

Analogue of Alfven wave.

Important differences:

- $\hat{B}_\psi$ in CHNS is large only in the interfacial regions.
- Elastic wave activity does not fill space.
Flux Expulsion

- Simplest dynamical problem in MHD (Weiss ‘66, et. seq.)
- Closely related to “PV Homogenization”

Field wound-up, “expelled” from eddy
For large Rm, field concentrated in boundary layer of eddy
Ultimately, back-reaction asserts itself for sufficient $B_0$

$Rm \sim vL/\eta \gg 1$
Flux conservation: \( B_0 L \sim b l \)  
Wind up: \( b = n B_0 \) (field stretched)

Rate balance: wind-up \( \sim \) dissipation

\[
\frac{v}{L} B_0 \sim \frac{\eta}{l^2} b \cdot \tau_{\text{expulsion}} \sim \left( \frac{L}{v_0} \right) Rm^{1/3}.
\]

\[
l \sim \delta_{BL} \sim L/Rm^{1/3} \cdot \quad b \sim Rm^{1/3} B_0.
\]

after n turns: \( nl = L \)

N.B. differs from Sweet-Parker!
What’s the Physics?

Shear dispersion! (Moffatt, Kamkar ‘82)

\[ \partial_t A + \vec{v} \cdot \nabla A = \eta \nabla^2 A \quad \text{(Shearing coordinates)} \]

\[ v_y = v_y(x) = v_{y0} + xv_y' + \cdots \]

\[ \frac{dk_x}{dt} = -k_y v_y', \quad \frac{dk_y}{dt} = 0 \]

\[ \partial_t A + xv_y' \partial_y A - \eta \left( \partial_x^2 + \partial_y^2 \right) A = 0 \]

\[ A = A(t) \exp i(\vec{k}(t) \cdot \vec{x}) \]

(Shear enhanced dissipation annihilates interior field)

So \( \tau_{mix} \approx \tau_{shear} \text{Rm}^{1/3} = (v_{y}'^{-1}) \text{Rm}^{1/3} \)
Single Eddy Mixing -- Cahn-Hilliard

- Structures are the key → need understand how a single eddy interacts with $\psi$ field
- Mixing of $\nabla \psi$ by a single eddy → characteristic time scales?
- Evolution of structure?
- Analogous to flux expulsion in MHD (Weiss, ‘66)

$\nabla \psi \leftrightarrow \vec{B}$

Transport / Relaxation
3 stages: (A) the "jelly roll" stage, (B) the topological evolution stage, and (C) the target pattern stage.

ψ ultimately homogenized in slow time scale, but metastable target patterns formed and merge.

A: Jelly roll

B: reconnection

C: Target

Additional mixing time emerges.

Note coarsening!

Single Eddy Mixing

- The bands merge on a time scale long relative to eddy turnover time.
- The 3 stages are reflected in the elastic energy plot.
- The target bands mergers are related to the dips in the target pattern stage.
- The band merger process is similar to the step merger in drift-ZF staircases.

Episodic relaxation-coarsening Cahn-Hilliard dynamics

[Ashourvan et al. 2016]
Back Reaction – Vortex Disruption

➢ (MHD only) (A. Gilbert et.al. ‘16; J. Mak et.al. ‘17)
➢ Demise of kinematic expulsion?
  • Magnetic **tension** grows to react on vorticity evolution!
➢ Recall: \( b \sim B_0 (Rm^{1/3}) \)
  • B.L. field stretched!
➢ and \( \vec{B} \cdot \nabla \vec{B} = -\frac{|B|^2}{r_c} \hat{n} + \frac{d}{ds} \left( \frac{|B|^2}{2} \right) \hat{t} \)

\[
|\vec{B} \cdot \nabla \vec{B}| \cong \frac{b^2}{L_0}
\]

\[
\begin{align*}
  r_c & \sim L_0 \\
  \frac{d}{ds} & \sim L_0^{-1}
\end{align*}
\]
Back Reaction – Vortex Disruption

➢ So \( \rho \frac{d \omega}{dt} = \hat{z} \cdot [\nabla \times (\vec{B} \cdot \nabla \vec{B})] \)
→ \( \rho u \cdot \nabla \omega \sim b^2 / lL_0 \)

small BL scale enters

\( v_{A0}^2 = B_0^2 / 4\pi \rho \)

➢ Feedback → 1 for: \( Rm \left( \frac{v_{A0}}{u} \right)^2 \sim 1 \)

\( Rm \gg 1 \rightarrow \text{strong field not required} \)

Remember this!

➢ Critical value to disrupt vortex, end kinematics
➢ Related Alfvén wave emission.
➢ Note for \( Rm \gg 1 \rightarrow \text{strong field not required} \)
➢ Will re-appear...

4/4/18  WIN 2018
Turbulence
MHD Turbulence – Quick Primer

- (Weak magnetization / 2D)
- Enstrophy conservation broken
- Alfvenic in $B_{rms}$ field – “magneto-elastic” (E. Fermi ‘49)

$$\epsilon = \frac{\langle \hat{v}^2 \rangle^2}{l^2} \frac{l}{B_{rms}} \implies E(k) = (\epsilon B_{rms})^{1/2} k^{-3/2} \ (l-K)$$

- Dual cascade:
  - Forward in energy
  - Inverse in $\langle A^2 \rangle \sim k^{-7/3}$

- What is dominant (A. Pouquet)?
  - conventional wisdom focuses on energy
  - yet $\langle A^2 \rangle$ conservation – freezing-in law!?
Ideal Quadratic Conserved Quantities

**2D MHD**

1. Energy
\[ E = E^K + E^B = \int \left( \frac{v^2}{2} + \frac{B^2}{2\mu_0} \right) d^2 x \]

2. Mean Square Magnetic Potential
\[ H^A = \int A^2 d^2 x \]

3. Cross Helicity
\[ H^C = \int \vec{v} \cdot \vec{B} d^2 x \]

**2D CHNS**

1. Energy
\[ E = E^K + E^B = \int \left( \frac{v^2}{2} + \frac{\xi^2 B^2}{2} \right) d^2 x \]

2. Mean Square Concentration
\[ H^\psi = \int \psi^2 d^2 x \]

3. Cross Helicity
\[ H^C = \int \vec{v} \cdot \vec{B}_\psi d^2 x \]

Dual cascade expected!
Scales, Ranges, Trends

Fluid forcing → Fluid straining vs Blob coalescence

Straining vs coalescence is fundamental struggle of CHNS turbulence

Scale where turbulent straining ~ elastic restoring force (due surface tension):

Hinze Scale

\[ L_H \sim \left( \frac{\rho}{\xi} \right)^{-1/3} \epsilon^{-2/9} \]

How big is a raindrop?
• Turbulent straining vs capillarity.
• \( \rho v^2 \) vs \( \sigma / l \).
[Hinze 1955]
Scales, Ranges, Trends

- Elastic range: \( L_H < l < L_d \): where elastic effects matter.
- \( L_H / L_d \sim (\frac{\rho}{\xi})^{-1/3} \nu^{-1/2} \epsilon^{-1/18}_\Omega \) \( \Rightarrow \) Extent of the elastic range
- \( L_H \gg L_d \) required for large elastic range \( \Rightarrow \) case of interest
Scales, Ranges, Trends

• Key elastic range physics: **Blob coalescence**

• Unforced case: \( L(t) \sim t^{2/3} \).

  (Derivation: \( \vec{v} \cdot \nabla \vec{v} \sim \frac{\xi^2}{\rho} \nabla^2 \psi \nabla \psi \Rightarrow \frac{L^2}{L} \sim \frac{1}{\rho L^2} \))

• Forced case: blob coalescence arrested at Hinze scale \( L_H \).

• \( L(t) \sim t^{2/3} \) recovered

• Blob growth arrest observed

• Blob growth saturation scale tracks Hinze scale (dashed lines)

• Blob coalescence suggests inverse cascade is fundamental here.
Cascades: Comparing the Systems

- Blob coalescence in the elastic range of CHNS is analogous to flux coalescence in MHD.
- Suggests inverse cascade of $\langle \psi^2 \rangle$ in CHNS.
- Supported by statistical mechanics studies (absolute equilibrium distributions).
- Arrested by straining.
Cascades

➢ So, *dual cascade*:
  • *Inverse* cascade of $\langle \psi^2 \rangle$
  • *Forward* cascade of $E$

➢ Inverse cascade of $\langle \psi^2 \rangle$ is formal expression of blob coalescence process → generate larger scale structures till limited by straining

➢ Forward cascade of $E$ as usual, as elastic force breaks enstrophy conservation

➢ Forward cascade of energy is analogous to counterpart in 2D MHD
Cascades

- Spectral flux of $\langle A^2 \rangle$:
  $$\Pi_{HA}(k) = \sum_{k' \leq k} T_{HA}(k'), \text{ where } T_{HA}(k) = \langle A^*_k (v \cdot \nabla)A \rangle$$

- Spectral flux of $\langle \psi^2 \rangle$:
  $$\Pi_{H\psi}(k) = \sum_{k' \leq k} T_{H\psi}(k'), \text{ where } T_{H\psi}(k) = \langle \psi^*_k (v \cdot \nabla)\psi \rangle$$

- MHD: weak small scale forcing on $A$ drives inverse cascade
- CHNS: $\psi$ is unforced $\Rightarrow$ aggregates naturally $\iff$ structure of free energy
- Both fluxes negative $\Rightarrow$ inverse cascades
Power Laws

- \( \langle A^2 \rangle \) spectrum:

- \( \langle \psi^2 \rangle \) spectrum:

- Both systems exhibit \( k^{-7/3} \) spectra.

- Inverse cascade of \( \langle \psi^2 \rangle \) exhibits same power law scaling, so long as \( L_H \gg L_\alpha \), maintaining elastic range: Robust process.
Power Laws

Derivation of -7/3 power law:

For MHD, key assumptions:

- Alfvenic equipartition \( \rho \langle v^2 \rangle \sim \frac{1}{\mu_0} \langle B^2 \rangle \)
- Constant mean square magnetic potential dissipation rate \( \epsilon_{HA} \), so \( \epsilon_{HA} \sim \frac{H^A}{\tau} \sim (H^A_k)^2 k^{-2} \).

Similarly, assume the following for CHNS:

- Elastic equipartition \( \rho \langle v^2 \rangle \sim \xi^2 \langle B^2_{\psi} \rangle \)
- Constant mean square magnetic potential dissipation rate \( \epsilon_{H\psi} \), so \( \epsilon_{H\psi} \sim \frac{H^\psi}{\tau} \sim (H^\psi_k)^2 k^{-2} \).
More Power Laws

- Kinetic energy spectrum (Surprise!):
  - 2D CHNS: $E_k^K \sim k^{-3}$;
  - 2D MHD: $E_k^K \sim k^{-3/2}$.
- The -3 power law:
  - Closer to enstrophy cascade range scaling, in 2D Hydro turbulence.
  - Remarkable departure from expected $-3/2$ for MHD. **Why?**
- Why does CHNS $\longleftrightarrow$ MHD correspondence hold well for $\langle \psi^2 \rangle_k \sim \langle A^2 \rangle_k \sim k^{-7/3}$, yet break down drastically for energy???
- **What physics** underpins this surprise??
Interface Packing Matters! – Pattern!

- Need to understand **differences**, as well as similarities, between CHNS and MHD problems.

**2D MHD:**
- Fields pervade system.

**2D CHNS:**
- Elastic back-reaction is limited to regions of density contrast i.e. $|\mathbf{B}_\psi| = |\nabla \psi| \neq 0$.
- As blobs coalesce, interfacial region diminished. ‘Active region’ of elasticity decays.
Interface Packing Matters!

- Define the **interface packing fraction** $P$:
  
  $$P = \frac{\text{# of grid points where } |\vec{B}_\psi| > B_{\psi}^{rms}}{\text{# of total grid points}}$$

- $P$ for CHNS decays;
- $P$ for MHD stationary!

- $\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega$: small $P \rightarrow$ local back reaction is weak.

- Weak back reaction $\rightarrow$ reduce to 2D hydro $\rightarrow$ k-spectra
- Blob coalescence coarsens interface network
What Are the Lessons?

- Avoid power law tunnel vision!
- **Real space** realization of the flow is necessary to understand key dynamics. Track interfaces and packing fraction $P$.
- One player in dual cascade (i.e. $\langle \psi^2 \rangle$) can modify or constrain the dynamics of the other (i.e. $E$).
- Against conventional wisdom, $\langle \psi^2 \rangle$ inverse cascade due to blob coalescence is the robust nonlinear transfer process in CHNS turbulence.
- Begs more attention to magnetic helicity in 3D MHD.
Transport
Active Scalar Transport

Magnetic diffusion, $\psi$ transport are cases of active scalar transport

(Focus: 2D MHD) (Cattaneo, Vainshtein ’92, Gruzinov, P. D. ’94, ’95)

Scalar mixing – the usual

$$\partial_t A + \nabla \phi \times \hat{z} \cdot \nabla A = \eta \nabla^2 A$$

$$\partial_t \nabla^2 \phi + \nabla \phi \times \hat{z} \cdot \nabla \nabla^2 \phi = \nabla A \times \hat{z} \cdot \nabla \nabla^2 A + \nu \nabla^2 \nabla^2 \phi$$

Turbulent resistivity

Seek $\langle v_x A \rangle = -D_T \frac{\partial \langle A \rangle}{\partial x} - \eta \frac{\partial \langle A \rangle}{\partial x}$

Point: $D_T \neq \sum_k |v_k|^2 \frac{E_k}{\tau_k}$, often substantially less

Why: Memory! $\iff$ Freezing-in
Origin of Memory?

- (a) flux advection vs flux coalescence
  - intrinsic to 2D MHD (and CHNS)
  - rooted in inverse cascade of \( \langle A^2 \rangle \)
- (b) tendency of (even weak) mean magnetic field to “Alfvenize” turbulence [cf: vortex disruption feedback threshold!]
- Re (a): Basic physics of 2D MHD

Forward transfer: fluid eddies chop up scalar \( A \).
Memory Cont’d

v.s.

Inverse transfer: current filaments and A-blobs attract and coagulate.

Obvious analogy: straining vs coalescence; CHNS

Upshot: closure calculation yields:

\[ \Gamma_A = - \sum_{k'} \left[ \tau_c^\phi \langle v^2 \rangle_{k'} - \tau_c^A \langle B^2 \rangle_{k'} \right] \frac{\partial \langle A \rangle}{\partial x} + \cdots \]

flux of potential competition

scalar advection vs. coalescence ("negative resistivity")

(+)

(-)
Zeldovich and Alfvenization

- Re (b): Competition winner? → Alfvenization!
- Alfvenization is a natural consequence of stronger $\langle B \rangle$, ala' vortex disruption
- fluid stretches $\langle B \rangle$, ala' $B_0 \rightarrow b$ in flux expulsion
- How to quantify: Zeldovich Theorem

$$H_A = \int d^2 x \ H_A = \int d^2 x \langle A^2 \rangle$$

$$\frac{1}{2} \frac{\partial H_A}{\partial t} = -\Gamma_A \frac{\partial \langle A \rangle}{\partial x} - \eta \langle B^2 \rangle$$

production dissipation
Zeldovich and Alfvenization, Cont’d

So \( \langle B^2 \rangle \approx -\frac{\Gamma A}{\eta} \frac{\partial \langle A \rangle}{\partial x} \approx \frac{D_T}{\eta} \left( \frac{\partial \langle A \rangle}{\partial x} \right)^2 \)  

\( \langle B^2 \rangle \approx \frac{D_T}{\eta} \langle B \rangle^2 \)  

(meta-stationary state)

Strong RMS field generated from modest \( \langle B \rangle \)

Reflects the effect of small scale B-field amplification (i.e. \( B_0 \rightarrow b \))

Ultimately, \( \eta \) asserts itself (Cowling)

Best think \( \langle B^2 \rangle \leftrightarrow T_m \) (elastic energy)
Eliminate $\langle B^2 \rangle$ in $\Gamma_A$ using Zeldovich

So: $D_T = D_K / \left[ 1 + Rm \frac{v_A^2}{\langle v^2 \rangle} \right]$

where:

- $D_K$ is usual kinematic diffusivity
- $Rm \frac{v_A^2}{\langle v^2 \rangle} \sim 1$ identical to vortex disruption threshold
- Weak $\langle B \rangle$ “quenches” flux diffusion for large Rm

Physics is memory enforced by strong, small scale field.

[Implications for $\alpha$, dynamo, etc.]
(Well-established numerically)
Active scalar transport bifurcation!

\[ \Gamma_A = -\frac{D_K \frac{\partial \langle A \rangle}{\partial x}}{\left[ 1 + \frac{Rm}{\rho \langle v^2 \rangle} \left( \frac{\partial \langle A \rangle}{\partial x} \right)^2 \right]} - \eta \frac{\partial \langle A \rangle}{\partial x} \]

(Standard form)

i.e.

Spatio-temporal dynamics largely unexplored
- bi-stable system
- fronts, barriers, domains

Expect analogue in CHNS, modulo density gradient
Something Old: Quenching

- $M^2 = \langle \tilde{\nu}^2 \rangle / \nu_{A0}^2$
- Higher $\nu_{A0}^2 / \langle \tilde{\nu}^2 \rangle \rightarrow$ lower $D_T \rightarrow$ longer $E_m$ persistence
- Ultimately $\eta$ asserts itself

- Blue: $\langle B \rangle$ sufficient for suppression
- Yellow: Ohmic decay phase

[Cattaneo and Vainshtein '91]
Spatial Structure (Preliminary)

- **Initial condition:** $\cos(x)$ for $A$
- **Shorter time (suppression phase)**
  - Domains, and domain boundaries evident, resembles CHNS
  - A transport barriers?!
- **Longer time (Ohmic decay phase)**
  - Well mixed
  - No evidence nontrivial structure
Something New, Cont’d

- For analysis: pdf of A

- Suppression phase:
  - quenched diffusion
  - bi-modal distribution
    - quenching prevents fill-in
    - consequence i.c.

- Ohmic decay phase:
  - uni-modal distribution returns
Higher Pm (Lower $\eta_T$)

- Bi-modal pdf of A structure persists longer
- Barrier resists Ohmic decay

- A field exhibits strikingly sharp domain structure
- Transition layer (barrier) evident
- Clear example of decoupling of transport, intensity.
What of CHNS?

- So far much the same, without Ohmic decay phase
- CH structure feeds elastic energy $\leftrightarrow$ resembles forcing in B-field in MHD
- Ongoing
Conclusion
Conclusion, of Sorts

- Elastic fluids ubiquitous, interestingly similar and different. Comparison/contrast is useful approach.
- Simple problems, like flux expulsion (50+ years), reveal a lot about basic feedback dynamics.
- CHNS is interesting example of elastic turbulence where energy cascade is *not* fundamental or dominant.
- Spatio-temporal dynamics of (bi-stable) active scalar transport is a promising direction. Pattern formation in this system is terra novo.
- Revisiting polymer drag reduction would be interesting.