Emergence of Zonal Shear Flow Pulses From Drift Wave Phase Dynamics

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Take Away Message

-Cross phase is critical to evolution of the Reynolds stress $< \widetilde{v_x}\widetilde{v_y} >= |\widetilde{v_x}||\widetilde{v_y}|\cos\theta_{\widetilde{v_x}\widetilde{v_y}}$ and thus is a key in the self-regulation between zonal flows and drift waves.

-Consider the nonlinear phase evolution directly beyond quasilinear theory from coupled phase, amplitude, and flow equations.

-Phase curvature can directly drive zonal flow evolution even in the absence of inhomogeneous turbulent intensity (i.e. no modulational instability).

-Nonlinear structures arise in the phase field

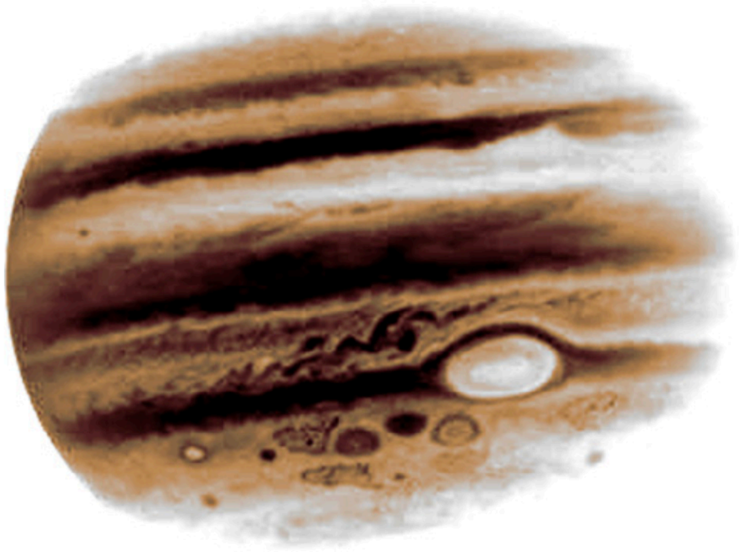
-Phase dynamics governed by phase gradient steepening vs. dispersion – leads to stable collisionless phase “shocks” and zonal shear flow pulses

-Exact result obtained for limiting case of phase field

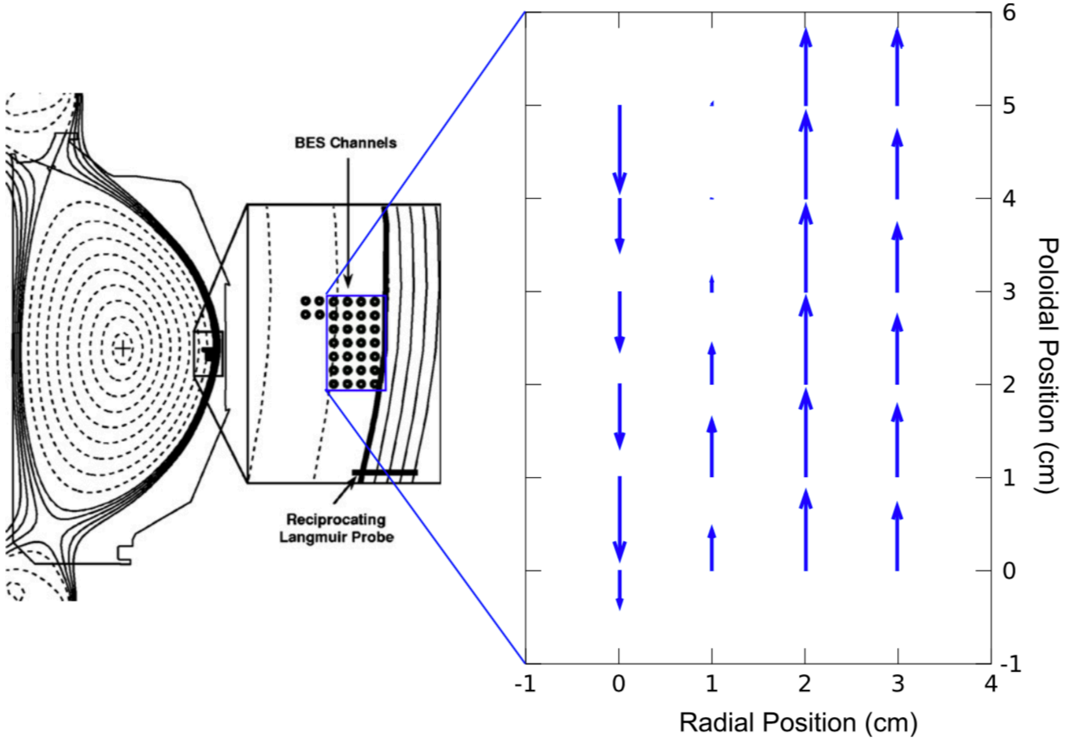
Motivation

- DW-ZF system: regulates transport in confined plasma

- Key element in the DW-ZF system is their self-regulation within predator-prey dynamics (exchange of energy)



Zonal flows in Jupiter (Gürcan, Diamond 2015)



Zonal flows in the D-IIID tokamak (McKee G R et al 2003 Phys. Plasmas)

Motivation

- Few models of ZF go into the **scale** of the flow and its nonlinear evolution.
- Critical feature in this self regulation is the Reynolds stress $< \tilde{v}_x \tilde{v}_y >$ which is related to the cross-phase of $< \partial_y \psi \, \partial_x \psi >$.

Thus, why not evolve the phase directly along with zonal flow evolution?

- Consider nonlinear evolution of the phase directly, beyond quasilinear theory
- ZBG, PD (PRL, 2016) begins to address this, but some issues:
  - Eikonal formulation — not systematic
  - Ad-hoc phase diffusion

Basic System — Hasegawa-Mima & Flow equation

- **Hasegawa-Mima equation** in the presence of mean flow:

$$\partial_t\left(\widetilde{\phi}-\rho_s^2\nabla^2\widetilde{\phi}\right)+v^*\partial_y\widetilde{\phi}+\overline{\mathbf{v}}_E\cdot\nabla\left(\widetilde{\phi}-\rho_s^2\nabla^2\widetilde{\phi}\right)+\widetilde{\mathbf{v}}_E\cdot\nabla\left(\overline{\phi}-\rho_s^2\nabla^2\overline{\phi}\right)=0$$

where $\mathbf{v}_E=\rho_s^2\Omega_i\,\widehat{\mathbf{z}}\times\nabla\phi$ is the **E** x **B** drift velocity, with **B** in the direction of $\widehat{\mathbf{z}}$.

The equation simplifies to

$$\frac{\partial}{\partial t}\widetilde{\phi}-\rho_s^2\frac{\partial}{\partial t}\nabla^2\widetilde{\phi}+2<v_y>\partial_y\widetilde{\phi}-\rho_s^2<v_y>\partial_y\nabla^2\widetilde{\phi}-\rho_s^2<v_y>''\,\partial_y\widetilde{\phi}+v^*\partial_y\widetilde{\phi}=0$$

- The 2D flow equation is

$$\partial_t\,\Delta\overline{\phi}+<\widetilde{\mathbf{v}_\mathbf{E}}\cdot\nabla\Delta\widetilde{\phi}>=0$$

We can reduce the second term on the left through the Taylor identity:

$$<\widetilde{v_x}\,\Delta\widetilde{\phi}>=\frac{1}{\rho_s^2\Omega_i}\,\partial_x<\widetilde{v_x}\widetilde{v_y}>=-\rho_s^2\Omega_i\,\partial_x<\partial_y\widetilde{\phi}\,\partial_x\widetilde{\phi}>$$

So, the flow equation is

$$\partial_t<v_y>=\rho_s^4\Omega_i^2\,\partial_x<\partial_y\widetilde{\phi}\,\partial_x\widetilde{\phi}>-\mu<v_y>$$

Coupled Phase & Amplitude Equations

Rewriting $\widetilde{\phi}$ as $\widetilde{\phi} = Ae^{i\psi}$ and separating the real and imaginary components, we get the **amplitude evolution equation**:

$$0 = \partial_t A + (2\nabla A \cdot \nabla\psi + A\nabla^2\psi)\,\hat{\chi}(\psi) + (\nabla\psi)^2\,\hat{\chi}(A) - \hat{\chi}(\nabla^2 A) + 2A\nabla\psi \cdot \hat{\chi}(\nabla\psi) + \\ 2 < v_y > \partial_y A - \rho_s^2 < v_y >'' \partial_y A + v^*\partial_y A$$

and the **phase evolution equation:**

$$0 = A\partial_t\psi - 2\nabla\psi \cdot \hat{\chi}(\nabla A) - 2\nabla A \cdot \hat{\chi}(\nabla\psi) - (\nabla^2 A - A(\nabla\psi)^2)\,\hat{\chi}(\psi) - \\ \nabla^2\psi\,\hat{\chi}(A) - A\,\hat{\chi}(\nabla^2\psi) - \rho_s^2 < v_y >'' A\partial_y\psi + 2 < v_y > A\partial_y\psi + v^* A\partial_y\psi$$

where the operator $\hat{\chi} \equiv \rho_s^2(\partial_t + < v_y > \partial_y)$.

Flow Equation

The flow equation is

$$\partial_t < v_y > = \rho_s^4 \Omega_i^2 \, \partial_x < \partial_y \tilde{\phi} \, \partial_x \tilde{\phi} > - \mu < v_y >$$

where the Reynolds term (1st term on r.h.s.) can be expanded to:

$$\begin{aligned}
\partial_x < \partial_y \tilde{\phi} \, \partial_x \tilde{\phi} > &= \partial_x [(\partial_y A + iA\partial_y \psi)(\partial_x A - iA\partial_x \psi)] \\
&= \partial_x [(\partial_x A)(\partial_y A) + iA[(\partial_x A)(\partial_y \psi) - (\partial_x \psi)(\partial_y A)] + A^2(\partial_x \psi)(\partial_y \psi)]
\end{aligned}$$

so, the flow equation becomes

$$\frac{1}{\rho_s^4 \Omega_i^2} \partial_t < v_y > = \partial_{xx} A(\partial_y A + iA\partial_y \psi) + \partial_{xy} A(\partial_x A - iA\partial_x \psi) + A[\partial_{xx} \psi(A\partial_y \psi - i\partial_y A) + \partial_{xy} \psi(A\partial_x \psi + i\partial_x A)] +$$
$$i\partial_x A[\partial_x A \partial_y \psi - 2iA\partial_x \psi \partial_y \psi - \partial_x \psi \partial_y A] - \frac{\mu}{\rho_s^4 \Omega_i^2} < v_y >$$

Phase & Amplitude Equations for homogeneous turbulence intensity

In the WKB limit, we drop all O(∇A) and higher in the phase & amplitude equations, so the amplitude equation now becomes:

$$0 = \nabla^2\psi\,\hat{\chi}(\psi) + 2\nabla\psi\,\hat{\chi}(\nabla\psi)$$

This effectively acts as a constraint on the phase flow system.

The phase equation becomes:

$$0 = \partial_t\psi + \underbrace{(\nabla\psi)^2}_{\text{phase steepening}}\hat{\chi}(\psi) - \hat{\chi}(\nabla^2\psi) - \rho_s^2 < v_y >'' \partial_y\psi + 2 < v_y > \partial_y\psi + v^*\partial_y\psi$$

which describes the evolution of the phase for homogeneous turbulent intensity.

Flow Equation for homogeneous turbulence intensity

The flow equation is

$$\frac{1}{\rho_s^4\Omega_i^2}\,\partial_t < v_y > = \partial_{xx}A(\partial_y A + iA\partial_y\psi) + \partial_{xy}A(\partial_x A - iA\partial_x\psi) + A[\partial_{xx}\psi(A\partial_y\psi - i\partial_y A) + \partial_{xy}\psi(A\partial_x\psi + i\partial_x A)] +$$
$$i\partial_x A[\partial_x A\partial_y\psi - 2iA\partial_x\psi\partial_y\psi - \partial_x\psi\partial_y A] - \frac{\mu}{\rho_s^4\Omega_i^2} < v_y >$$

The leading terms are then:

$$\partial_t < v_y > = \rho_s^4\Omega_i^2 A^2(\underbrace{\partial_{xx}\psi}_{\text{phase curvature}}\partial_y\psi + \partial_{xy}\psi\partial_x\psi) - \mu < v_y >$$

The second term on the right hand side goes away once averaged, as $\partial_{xy}\psi\partial_x\psi = \frac{1}{2}\,\partial_y(\partial_x\psi)^2$, which disappears once averaged due to poloidal periodicity.

Observation

$$\partial_t < v_y >= \rho_s^4 \Omega_i^2 A^2 (\partial_{xx}\psi \partial_y \psi + \partial_{xy}\psi \partial_x \psi) - \mu < v_y >$$

Note that if we average the above equation, the second term on the right disappears due to periodicity, and thus **the ZF can be directly driven by phase curvature even in the absence of modulational instability and / or for homogeneous intensity (∇I = 0).**

Specifically, since $\partial_x \psi \sim k_r$ and $\partial_y \psi \sim k_\theta$, the zonal flow evolution (without damping) goes like

$$\partial_t < v_y >= \rho_s^4 \Omega_i^2 \, \partial_x [A^2 k_r k_\theta]$$

<u>so for constant amplitude, the radial derivative affects the wavenumbers and thus, the flow is driven by the gradient of the radial wavenumber, k_r' , which is the phase curvature.</u>

Reduced Phase-Flow system (I)

-Let us eliminate the fast variance of the phase in $\hat{\mathbf{y}}$.

Specifically, let us consider the evolution of the phase from an initial simple plane wave, with fast dependence on $\hat{\mathbf{y}}$, such that

$$\psi = k_y y - \omega_{k_y} t + \Theta(x, y, t)$$

with an eigenfrequency that we take as constant for now, by considering the Doppler shift as constant:

$$\omega_{k_y} = \frac{v^* k_y}{1 + \rho_s^2 k_y^2} + k_y < v_y >$$

Reduced Phase-Flow system (II)

Since Θ is slowly varying, $k_y \gg \partial_y\Theta$ and $\omega_{k_y} \gg \partial_t\Theta$,

and because dynamics of Θ are predominantly in $\hat{\mathbf{x}}$ such that $\rho_s^2 < v_y > \partial_y \nabla^2\Theta \simeq 0$
and $\rho_s^2\, \partial_t \nabla^2\Theta \simeq \rho_s^2\, \partial_t\partial_{xx}\Theta$, the **phase equation** reduces to

$$\partial_t(\Theta - \rho_s^2\partial_{xx}\Theta) = -k_y < v_y > +\rho_s^2 \frac{v^* k_y}{1+\rho_s^2 k_y^2} (\partial_x\Theta)^2 + \rho_s^2 k_y < v_y >''$$

The first and last terms on the right hand side represent the frequency detuning by the zonal flow and its curvature, and the second term represents the quadratic self coupling.

The **flow equation**, once averaged, reduces to:

$$\partial_t < v_y >= \rho_s^4 \Omega_i^2\, A^2 k_y \partial_{xx}\Theta - \mu < v_y >$$

Nonlinear Phase Dynamics (I) — Nonlinear Phase-Flow Waves

-Let us consider weak damping. Then, our system described by the following set of equations:

$$\partial_t(\Theta - \rho_s^2\partial_{xx}\Theta) = -k_y < v_y > +\rho_s^2 \frac{v^* k_y}{1+\rho_s^2 k_y^2}(\partial_x\Theta)^2 + \rho_s^2 k_y < v_y >''$$

$$\partial_t < v_y >= \rho_s^4\Omega_i^2 A^2 k_y \partial_{xx}\Theta$$

We look for solutions in a moving frame, i.e. of the form $\Theta(x-ct)$. Thus, we can transform the temporal derivative as $\partial_t \to -c\partial_x$ where the spatial derivative is now taken to be in the moving frame. After simplification, we arrive at

$$< v_y >= -\frac{A^2 k_y}{c}\partial_x\Theta + \frac{\alpha}{c}$$

where the last term on the r.h.s. is an integration constant, and the scaling $\rho_s^2\Omega_i$ has been absorbed into A .

Nonlinear Phase Dynamics (II) — Nonlinear Phase-Flow Waves

Plugging the equation for flow back into the phase equation, multiplying both sides by Θ'', then integrating once, we get

$$-\frac{1}{2}(k_y^2 A^2 + c^2) f^2 + \frac{c^2 \rho_s^2}{2}(f')^2 = \frac{1}{3}\frac{\rho_s^2 v^* k_y c}{1+\rho_s^2 k_y^2} f^3 - \frac{\rho_s^2 k_y^2 A^2}{2}(f')^2 - k_y \alpha f + \beta$$

where $f \equiv \Theta'$ and β is an integration constant. Thus, the equation for the phase gradient is given as

$$x + x_0 = \int \frac{df}{Q(f,c)^{\frac{1}{2}}}$$

where

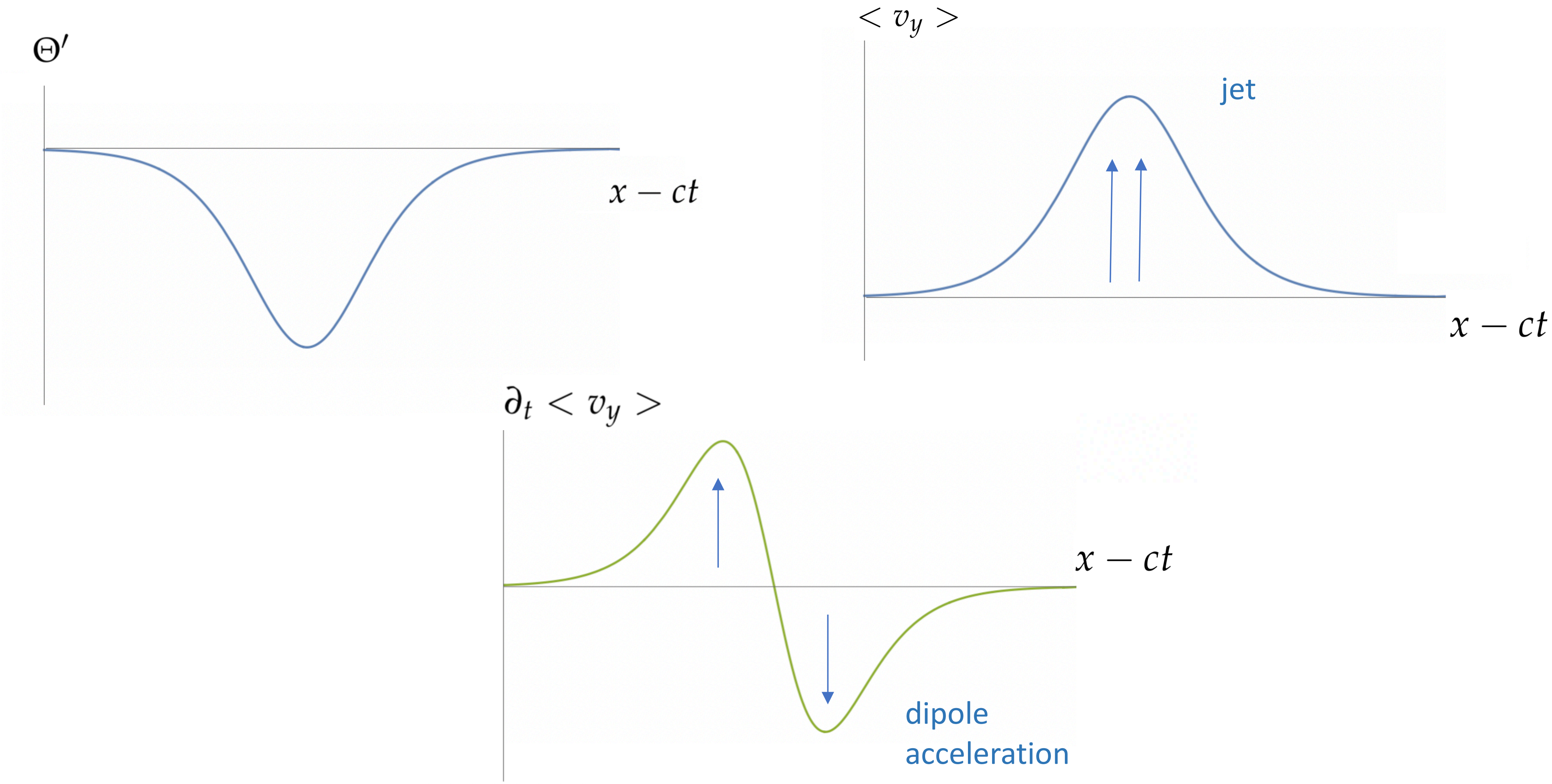
$$Q(f,c) \equiv (f')^2 = \frac{2}{\rho_s^2}\frac{1}{c^2 + k_y^2 A^2}\left[\frac{1}{3}\frac{\rho_s^2 v^* k_y c}{1+\rho_s^2 k_y^2} f^3 + \frac{1}{2}(c^2 + k_y^2 A^2) f^2 - k_y \alpha f + \beta\right]$$

Since A represents the amplitude of the potential perturbation, $k_y A$ is effectively the **E x B drift,** $v_{E\times B}$.

Nonlinear Phase Dynamics (III) — Zonal Shear Flow Pulses

If $\alpha, \beta = 0$, the equation admits the exact solution $\Theta' = \dfrac{3(c^2 + k_y^2 A^2)(1+\rho_s^2 k_y^2)}{2\, c\, \rho_s^2\, v^*\, k_y} \left[\tanh^2\left(\dfrac{x+x_0}{2\rho_s}\right) - 1 \right] + \Theta_0'$.

The analytic solution defines the scales of the zonal flow.



Nonlinear Phase Dynamics (IV) — Zonal Shear Flow Pulses

$$\Theta' = \frac{3(c^2 + k_y^2 A^2)(1 + \rho_s^2 k_y^2)}{2\, c\, \rho_s^2\, v^*\, k_y} \left[\tanh^2\left(\frac{x + x_0}{2\rho_s} \right) - 1 \right] + \Theta_0'$$

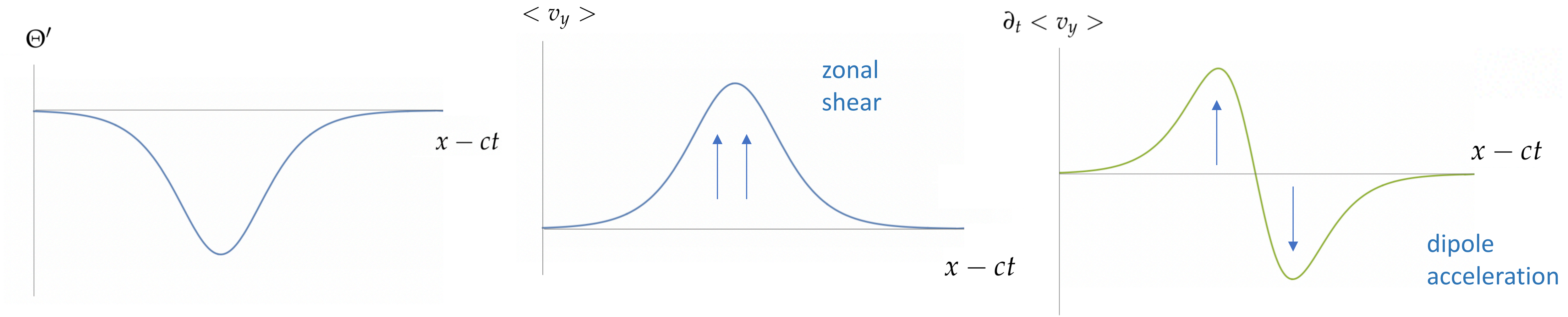
- The scale of the width of the pattern is set by $2\rho_s$.

- The magnitude of the zonal flows $< v_y >$ is set by $\frac{3A^2(c^2 + k_y^2 A^2)(1 + \rho_s^2 k_y^2)}{2\, c^2\, \rho_s^2\, v^*}$ for $\alpha = 0$.

By conducting perturbation theory in the moving frame for the zonal flow-based state i.e. A=A(x,t), we numerically show that our base states are ***stable*** solutions to our coupled system *(refer to appendix)*.

Message to Experimentalists

- Coherent nonlinear phase evolution can drive zonal flow production
- Cross phase dynamics must be treated on equal footing with intensity, $< v_y >$, etc.
- Jets emerge as stable solitons
- $< v_y >$, $\partial_t < v_y >$, length scale etc. set by phase pattern structure
- Phase evolution connects to new feedback loops
- More to the DW-ZF theory than predator-prey story



Conclusions & Future Work

- Cross phase is critical to evolution of the Reynolds stress and thus is a key in the self-regulation between zonal flows and drift waves, as it gives rise to the phase curvature term which can directly drive zonal flow evolution, even in the absence of inhomogeneous turbulent intensity, through k_r' .

- Must address phase **dynamics**

- Feedback loop between phase and flow can be driven through phase curvature

- Nonlinear phase dynamics governed by phase gradient steepening $(\partial_x \Theta)^2$ vs. dispersion $\partial_t \partial_{xx} \Theta$, which give rise to **stable** collisionless phase "shocks" and zonal shear flows.

- Future work:
    - Consider streamer-based state
    - Investigate phase and amplitude couplings in multi-mode nonlinear interactions
    - Look at dynamic density — coupling between ∇n and phase
    - Boundary effects: Fate of the pattern?

Appendix: Energetics (fluctuation energy)

Let us demonstrate the self regulation of drift waves and zonal flows through their exchange of energy under their predator prey dynamics.

The fluctuation energy is:

$$
\begin{aligned}
< \frac{d\varepsilon}{dt} > &= \frac{1}{2} < \partial_t(\tilde{\phi}^2 + \rho_s^2(\nabla\tilde{\phi})^2) > \\
&=< \tilde{\phi} \left[\partial_t\tilde{\phi} - \rho_s^2 \partial_t \nabla^2 \tilde{\phi} \right] >
\end{aligned}
$$

where we performed integration by parts to conclude to the second line. Plugging in our Hasegawa-Mima equation and eliminating terms due to their periodicity in the poloidal direction gives

$$
< \frac{d\varepsilon}{dt} > = - << v_y > A^2 \partial_x(\partial_x\psi\, \partial_y\psi) >
$$

Appendix: Energetics (flow energy)

The flow energy goes like

$$< \frac{dE_{ZF}}{dt} > = \frac{1}{\rho_s^4 \Omega_i^2} << v_y > \partial_t < v_y >>$$

Substituting in the flow equation

$$< \frac{dE_{ZF}}{dt} > =<< v_y > A^2\, \partial_{xx}\psi\, \partial_y\psi > - \frac{\mu}{\rho_s^4 \Omega_i^2} < v_y >^2$$

Thus,

$$\begin{aligned} < \frac{dE_{tot}}{dt} > &=< \frac{d\varepsilon}{dt} > + < \frac{dE_{ZF}}{dt} > \\ &= - << v_y > \partial_t < v_y >> + << v_y > A^2\, \partial_{xx}\psi\, \partial_y\psi > - \frac{\mu}{\rho_s^4 \Omega_i^2} < v_y >^2 \\ &= - \frac{\mu}{\rho_s^4 \Omega_i^2} < v_y >^2 \end{aligned}$$

confirming conservation of energy up to damping, as well as showing energy exchange through phase curvature.

Appendix: Dynamic Amplitude in the Zonal Flow-Based State (I)

We consider the zonal flow-based state. In this state, we restrict the amplitude to be only a function of **x** . Thus, along the equiamplitude surfaces of constant radius, the dynamics of the phase are dominated by **k_y y**.

Thus, by transforming $\nabla A \rightarrow \partial_x A$, our full coupled amplitude-phase-flow equations reduce to

$$\partial_t \left[\left(1 + \rho_s^2 [(\partial_x \Theta)^2 + k_y^2] \right) A - \rho_s^2 \partial_{xx} A \right] = \frac{v^* \rho_s^2 k_y}{1 + \rho_s^2 k_y^2} (2\,\partial_x \Theta\, \partial_x A + A \partial_{xx} \Theta)$$

$$A \partial_t \Theta - \rho_s^2 \partial_t \left[(A \partial_{xx} \Theta) + 2(\partial_x \Theta\, \partial_x A) \right] = \frac{v^* \rho_s^2 k_y}{1 + \rho_s^2 k_y^2} (A (\partial_x \Theta)^2 - \partial_{xx} A) + \\ + \rho_s^2 k_y A < v_y >'' - k_y A < v_y >$$

$$\frac{1}{\rho_s^4 \Omega_i^2} \partial_t < v_y > = k_y A^2 \partial_{xx} \Theta + 2 k_y A (\partial_x \Theta\, \partial_x A) - \frac{\mu}{\rho_s^4 \Omega_i^2} < v_y >$$

Appendix: Dynamic Amplitude (Zonal Flow-Based State) (II) — Perturbation Theory

We now consider a small perturbation of a homogeneous amplitude,

$$A = A_0 + \varepsilon A_1(x,t)$$

Thus, the flow and phase will be perturbed in the same order:

$$< v_y >=< v_y >_0 (x,t) + \varepsilon < v_y >_1 (x,t)$$
$$\Theta = \Theta_0(x,t) + \varepsilon\,\Theta_1(x,t)$$

Plugging this in to our reduced set of equations and separating the scales, our **zeroth order** equations are given by

$$\partial_t[(\Theta_0')^2] = \frac{v^* k_y}{1+\rho_s^2 k_y^2}\Theta_0''$$

$$\partial_t[\Theta_0 - \rho_s^2\Theta_0''] = \frac{v^* \rho_s^2 k_y}{1+\rho_s^2 k_y^2}(\Theta_0')^2 + \rho_s^2 k_y < v_y >_0'' - k_y < v_y >_0$$

$$\frac{1}{\rho_s^4\Omega_i^2}\partial_t < v_y >_0 = k_y A_0^2 \Theta_0'' - \frac{\mu}{\rho_s^4\Omega_i^2} < v_y >_0$$

which give the equations for the base state we have already calculated.

Appendix: Dynamic Amplitude (Zonal Flow-Based State) (III) — Perturbation Theory

The first order equations are given by

$$\partial_t\left(A_1+\rho_s^2\left[k_y^2A_1+2A_0\Theta_0'\Theta_1'-A_1''\right]\right)+\rho_s^2(\Theta_0'^2)\partial_tA_1=\frac{v^*\rho_s^2k_y}{1+\rho_s^2k_y^2}\left[2\Theta_0'A_1'+A_0\Theta_1''\right]$$

$$\begin{aligned}A_0\,\partial_t\Theta_1-\rho_s^2\left[\Theta_0''\,\partial_tA_1+\partial_t\left(A_0\Theta_1''+2\Theta_0'A_1'\right)\right]=&\frac{v^*\rho_s^2k_y}{1+\rho_s^2k_y^2}\left[2A_0\Theta_0'\Theta_1'-A_1''\right]\\&+\rho_s^2k_yA_0<v_y>_1''\\&-k_yA_0<v_y>_1\end{aligned}$$

$$\frac{1}{\rho_s^4\Omega_i^2}\partial_t<v_y>_1=k_yA_0^2\Theta_1''+\partial_x\left[2k_yA_0A_1\Theta_0'\right]-\frac{\mu}{\rho_s^4\Omega_i^2}<v_y>_1$$

Numerical simulation is employed to assess the stability of the perturbations. However, because our base solutions are in the moving frame, we transform the above set of equations in the moving frame through

$$\partial_x\rightarrow\partial_x$$

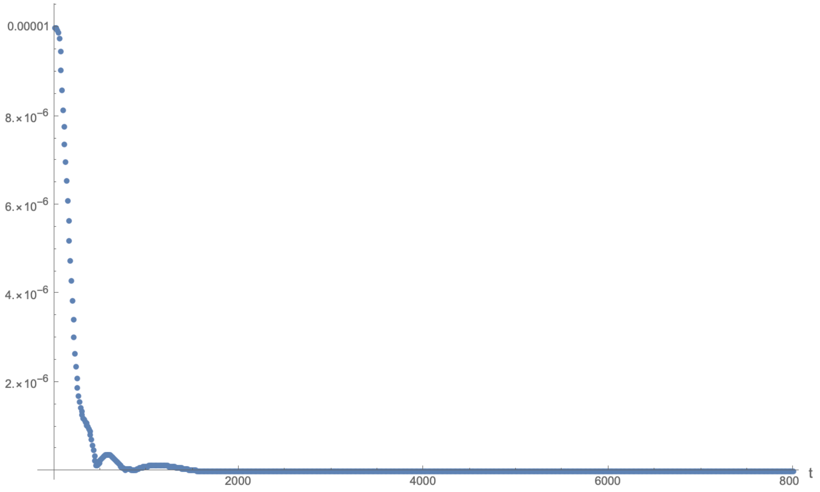
$$\partial_t\rightarrow-c\partial_x+\partial_t$$

before we simulate the above set of equations.

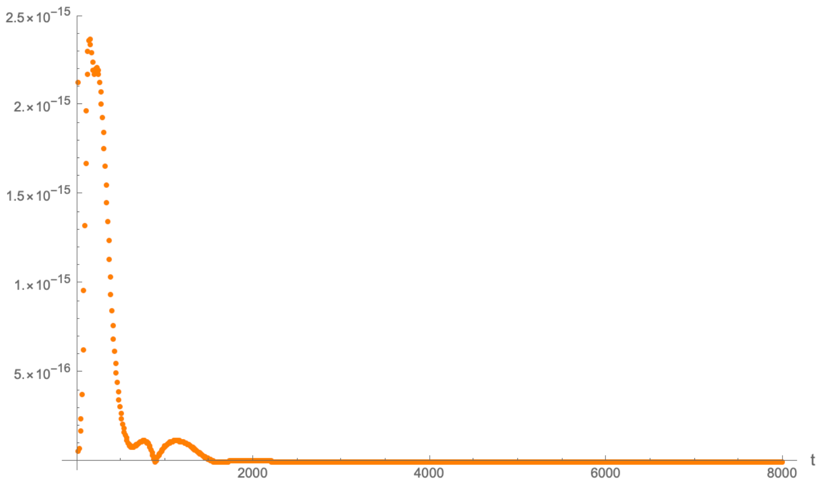
Appendix: Dynamic Amplitude (Zonal Flow-Based State) (IV) — Perturbation Theory

The results of the numerical simulation are shown below. Initial conditions were chosen such that the derivatives disappear at the edges, with an initial perturbation only to the amplitude, whose initial field is set by the boundary conditions of $A(x, 0) = 10^{-5}$, $A(x_f, t) = 0$, where x_f is the right edge of the tokamak, and c and k_y were chosen to be 4 and 20, respectively.

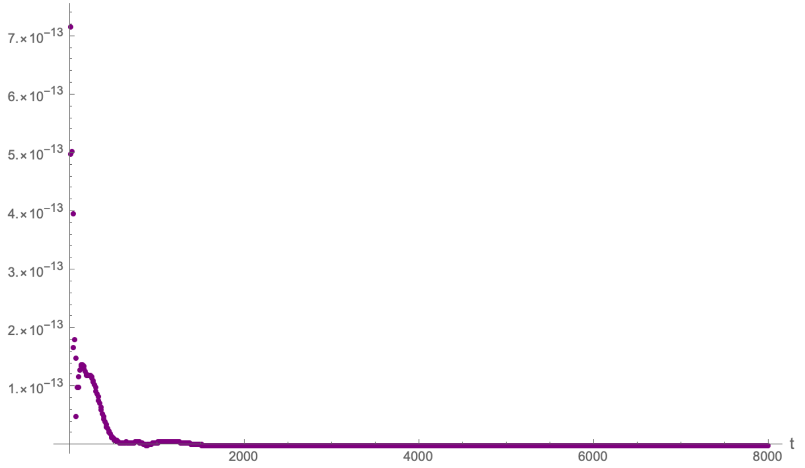
The following plots display the maximum magnitude of the perturbations across time.



Amplitude



Phase



Flow

Thus, our base states are **stable** solutions to our coupled system in the zonal flow-based state.