Subcritical turbulence spreading and avalanche birth

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Festival de Théorie 2019, Aix-en-Provence

Supported by the Department of Energy under Award Number DE-FG02-04ER54738
Introduction

- Turbulence spreading is an important nonlinear phenomenon in drift wave turbulence.
- Challenge the conventional wisdom on spreading and point out issues with the supercritical Fisher equation paradigm.
- Suggest a new model based on subcritical turbulence, which features avalanche-like spatiotemporal intermittency.
- We make testable predictions which distinguish it from Fisher.
- I might say the words ‘phase’ and ‘dynamics’ at some point, but probably not consecutively.
Outline

1. Background: turbulence spreading and avalanching
2. Bistable model
3. A teaser: machine learning model?
Background: turbulence spreading and avalanching
What the Fick?: turbulence spreading

- Spreading is important because it spells doom for local Fickian transport models
- Turbulence can radially self-propagate (even into linearly stable zones!) via nonlinear coupling

\[ \partial_t \varepsilon_k \sim - \sum_{k'} (\mathbf{k} \cdot \mathbf{k'} \times \mathbf{\hat{z}})^2 |\tilde{\phi}_{k'}|^2 R(\mathbf{k}, \mathbf{k'}) l_k \rightarrow \frac{\partial}{\partial x} D_x(l_k) \frac{\partial}{\partial x} l_k - \mathbf{k k} : \mathbf{D} l_k \]

\[ D_x = \sum_{k'} k_{y'}^2 |\phi_{k'}|^2 R(\mathbf{k}, \mathbf{k'}) \]

- Decouples flux-gradient relation: local turbulence intensity now depends on global properties of the profiles
Figure: Spatiotemporal evolution of flux-surface-averaged turbulence intensity in toroidal GK simulation. Linearly unstable region is $0.42 < r < 0.76$. From [Wang et al., 2006]
Bistable model

A teaser: machine learning model?

Avalanches

- Bursty, intermittent transport events associated with SOC. Account for a large percentage of total flux!
- Initially localized fluctuation cascades through neighboring regions via gradient coupling
- Closely related to spreading: both result in fast, mesoscopic turb front propagation. Unified model?

Figure: Heat flux spectrum from GK simulation showing $1/f$ scaling
Depiction of avalanching

Figure: Pressure (left) and potential (right) contours for simulations of resistive drift interchange turbulence [Carreras et al., 1996]
Conventional wisdom for spreading is Fisher-type equation for turbulence intensity:

$$\partial_t I = \gamma_0 I - \gamma_{nl} I^2 + \partial_x(D_0 I \partial_x I)$$

- **local lin. growth/decay**
- **local nonlin. coupling to dissipation**
- **nonlin. diffusion of turb. energy**

For $\gamma_0 > 0$, dynamics characterized by traveling fronts connecting unstable “laminar root” $I = 0$ and saturated “turbulent root” $I = \gamma_0 / \gamma_{nl}$ with speed $c = \sqrt{\frac{D_0 \gamma_0^2}{2 \gamma_{nl}}}$
Depiction of Fisher evolution

Figure: Evolution of traveling turbulence front in Fisher model. From [Gürcan and Diamond, 2006]
How does Fisher do?

- Propagation speed and characteristic front size $\ell \sim \sqrt{D/\gamma_0}$ in reasonable agreement with simulation
- Can be derived with some rigor from Fokker-Planck approach or renormalization of Hasegawa-Wakatani [Gürcan and Diamond, 2005, Gürcan and Diamond, 2006]
- But: weak spreading into stable zone. Dubiously consistent with experiment?

Figure: Experiment by Nazikian et al 2005 clearly showing fluctuations in stable zone
When does Fisher even make sense?

- Fisher model purports to describe spreading of a patch of turbulence in linearly unstable zone
- Begs the question: why didn’t noise already excite the whole system to turbulence?
- Only relevant if \( \gamma_0 \ll c/\Delta x \) i.e. \( \Delta x^2 \gamma_{nl} \ll D_0 \)
- Otherwise, physical fronts separating laminar/turbulent domains generally require bistability à la [Pomeau, 1986]
Heinonen and Diamond 2019: propose phenomenological model of form

$$\partial_t I = \gamma_1 I + \gamma_2 I^2 - \gamma_3 I^3 + \partial_x (D(I) \partial_x I)$$

take $D(I) = D_0 I$

New physics: nonlinear turbulence drive $\propto I^2$. Can sustain sufficiently large fluctuations even when linearly damped

*Bistable* in weak damping regime

Estimate $\gamma_1 \sim \epsilon \omega_*, \gamma_{2,3} \sim \omega_*, D_0 \sim \chi_{GB}$

But is MF plasma actually subcritically unstable?
Evidence for subcriticality

- [Inagaki et al., 2013]: experiments demonstrate hysteresis between fluctuation intensity and driving gradient (no TB present). Suggests bistable S-curve relation?

- Turbulence subcritical in presence of strong perpendicular flow shear [Carreras et al., 1992, Barnes et al., 2011, van Wyk et al., 2016] or in the presence of magnetic shear [Biskamp and Walter, 1985, Drake et al., 1995]

- Profile corrugations [Waltz, 1985, Waltz, 2010] and phase space structures [Lesur and Diamond, 2013] can drive nonlinear instability

Figure: Hysteresis between intensity and gradient, flux and gradient
Cousin models

- Compare to bistable models for subcritical transition to fluid turbulence [Barkley et al., 2015, Pomeau, 2015].
- Compare to [Gil and Sornette, 1996] model for sandpile avalanches

\[ \partial_t S = \gamma \left( |\partial_x h|/g_c - 1 \right) S + \beta S^2 - S^3 + \partial_x (D_S S \partial_x S) \]
\[ \partial_t h = \partial_x (D_h S \partial_x h). \]

- \( S \leftrightarrow I, \ h \leftrightarrow p \)
- Weak gradient coupling limit \( D_p \ll D_I \Rightarrow \) our model
- Strong gradient coupling limit: \( I \) slaved to \( p \). \( \partial_x p \propto I^{-1} \Rightarrow \)
  linear term is \( c - \gamma I \), where \( c \) is a constant which depends on BCs. Bistable again!
Model analysis I

\[ \partial_t I = \gamma_1 I + \gamma_2 I^2 - \gamma_3 I^3 + \partial_x (D(I) \partial_x I) \]

- Qualitatively similar to Fisher EXCEPT in weak damping case
  \( \gamma_1 < 0 \) and \( \gamma_2^2 > 4|\gamma_1|\gamma_3 \)
- Can then transform to Zel’dovich/Nagumo equation

\[ \partial_t I = f(I) + \partial_x (D I \partial_x I) \]
\[ f(I) \equiv \gamma I(I - \alpha)(1 - I) \]

where \( \alpha \equiv l_-/l_+ \), \( \gamma \equiv l_+^2 \gamma_3 \), \( D \equiv l_+ D_0 \), \( l_\pm \equiv (\gamma_2 \pm \sqrt{\gamma^2_2 - 4|\gamma_1|\gamma_3})/2\gamma_3 \)
Model analysis II

- Can write in variational form

\[ D(I) \partial_t I = -\frac{\delta F}{\delta I} \]

with free energy functional

\[ F = \int dx \left[ \frac{1}{2} (D(I) \partial_x I)^2 - \int_0^I dl' D(l') f(l') \right] \]

\[ \text{kine tic/flux} \quad \text{potential} \]

and \( dF/dt \leq 0 \)
Model analysis III

- $I = 0$ metastable for $\alpha < \alpha^* = 3/5$, abs. stable for $\alpha > \alpha^*$
- “Potential barrier” at $I = \alpha$: threshold for onset of nonlinear instability

Figure: “Potential” part of $\mathcal{F}$
Unlike Fisher, traveling fronts admitted in marginal/weak damping case!

- Propagation speed $c \sim \sqrt{D\gamma}$ (depends on $\alpha$), characteristic scale $\ell \sim \sqrt{D/\gamma}$
- “Maxwell construction” for speed

$$c \int_{-\infty}^{\infty} D(I(z))I'(z)^2 \, dz = \int_{0}^{1} D(I)f(I) \, dI$$

$z = x - ct$

- Thus turbulence spreads if $\alpha < \alpha^*$, recedes if $\alpha > \alpha^*$. Corresponds to (meta)stability of fixed points
Consider spreading of turbulence from lin. unstable to lin. stable zone

Simple model: $\gamma_1 = \gamma_g > 0$ for $x < 0$, $\gamma_1 = -\gamma_d < 0$ for $x > 0$

Allow turbulent front to form in lefthand region and propagate

In Fisher model, penetration is weak: forms stationary, exponentially-decaying profile with $\lambda \sim \sqrt{D_0/\gamma_{nl}} \sim \Delta_c$. Dubiously consistent with observation
Penetration into stable zone II

- However, in our model, a new front with reduced speed/amplitude forms in second region if weakly damped (i.e. $\gamma_d$ is small enough that $\alpha < \alpha^*$)
- Hence: can have ballistic propagation even in stable zone!
- More strongly delocalizing effect on the flux-gradient relation, compared to Fisher
Penetration into stable zone III

Figure: Spreading into stable zone in GK simulation with magnetic shear [Yi et al., 2014]. Ballistic propagation???
Local threshold behavior

- In contrast to Fisher, sufficiently large localized puff of turbulence will grow into front and spread. Suggestive of an avalanche triggered by initial seed
- How to determine threshold?

Two puffs differing only in spatial size are initialized; one grows and spreads, other collapses
Obviously puff amplitude must exceed $I_0 = \alpha$ or else $\gamma_{\text{eff}} = (I - \alpha)(1 - I) < 0$

Consider “cap” of puff (part exceeding $I = \alpha$)

Size threshold governed by competition between diffusion of turbulence out of cap and total nonlinear growth in cap (suggested by free energy functional)

Sets scale $\sqrt{D/\gamma}$
Avalanche threshold (details)

- Strategy: assume initial puff is symmetric, has single max $I_0$ and single lengthscale $L$
- Expand intensity curve about max to quadratic order, plug into dynamical equation, integrate over extent of cap
- Result: growth if

$$L > L_{\text{min}} = \sqrt{\frac{D(\alpha)I_0}{f(I_0) - \frac{1}{3}(I_0 - \alpha)f'(I_0)}} = \sqrt{\frac{3D\alpha I_0}{\gamma(I_0 - \alpha)((1 - 2\alpha)I_0 + \alpha)}}$$

- Power law $L_{\text{min}} \sim (I_0 - \alpha)^{-1/2}$
Avalanche threshold: analytical vs. simulation

Figure: Numerical result for threshold at $\alpha = 0.3$ for three types of initial condition (Gaussian ($I_1$), Lorentzian ($I_2$), parabola ($I_3$)), compared with analytical estimate.
Triggering an avalanche

- How might a puff of sufficient size form?
- Near linear marginality, threshold is weak:

$$I_- \sim \frac{|\gamma_1|}{\gamma_2} \ll 1, \quad L_{\text{min}} \sim \left(\frac{\chi_{GB}}{\omega_*}\right)^{1/2} \sim \Delta_c$$

- Suggests threshold can be triggered by noise
- Simulations of model with appropriate choice of noise (multiplicative + small additive background) show that front propagation events will be intermittently excited
Bistable model: conclusions

- Natural extension of Fisher model that allows for coexistence of laminar/turbulent domains
- Supported by substantial evidence for subcritical turbulence
- Provides simple framework for understanding avalanching: local exceedance of nonlinear instability by turbulent puffs
- Key testable predictions: ballistic spreading into weakly linearly damped regions, power-law threshold for spreading of puffs
References I

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A teaser: machine learning model?
Towards a complete model

- A realistic model should include coupling to zonal flow and pressure profile
- Start with Hasegawa-Wakatani:

\[ \partial_t n + \{ \phi, n \} = \alpha (\phi - n) + \text{diss.} \]
\[ \partial_t \nabla^2_\perp \phi + \{ \phi, \nabla^2_\perp \phi \} = \alpha (\phi - n) + \text{diss.} \]

with \( \alpha = -\eta \partial_z^2 \) the adiabatic operator representing parallel electron response
- Take zonal averages:

\[ \partial_t \langle n \rangle + \partial_x \langle \tilde{n} \tilde{v}_x \rangle = \text{diss.} \]
\[ \partial_t \langle \zeta \rangle + \partial_x \langle \tilde{\zeta} \tilde{v}_x \rangle = \text{diss.} \]
\[ \partial_t \langle \varepsilon \rangle + \langle (\tilde{n} - \tilde{\zeta}) \tilde{v}_x \rangle \partial_x \langle n - \zeta \rangle + \partial_x \langle \varepsilon \tilde{v}_x \rangle = \text{diss.} \]

where \( \zeta = \nabla^2_\perp \phi \), \( \varepsilon = \frac{1}{2}(\tilde{n} - \tilde{\zeta})^2 \)
Learning mean field theory

- How to proceed? Need model for turbulent fluxes $\Gamma_q = \langle \tilde{q}\tilde{v}_x \rangle$ but hard to calculate.
- Idea: use simulations to train machine learning model that maps mean profiles to local fluxes.
- Here ML is just a form of nonparametric regression: no need to impose a model.
- One approach: local model

$$\Gamma_q(x) = f(\partial_x n|_x, \partial_x^2 n|_x, \ldots, \zeta|_x, \partial_x \zeta|_x, \ldots, \varepsilon|_x, \partial_x \varepsilon|_x, \ldots)$$

- Challenges: feature selection, noise suppression. Also is local model even valid?
Preliminary results: particle flux

- Training on \( \sim 20 \) simulations of 2D Hasegawa-Wakatani at \( \alpha = 2 \) and constraining the model with symmetries of HW, a simple neural network learns a reasonable model for the particle flux.

Learned turbulent particle flux as function of density gradient at zero vorticity gradient (left) and vice versa (right.)
Preliminary results: particle flux

- Flux is approximately linear combination of terms prop. to $\partial_x \langle n \rangle$ and $\partial_x \langle \zeta \rangle$. First is obvious, latter less so!
- No clear dependence on shear itself

Figure: Dependence of particle flux on both density and vorticity gradients
Preliminary results: particle flux

- Results can be explained by simple quasilinear theory. However, must include effects of mean vorticity gradient on dispersion relation! Ignored in most studies

\[ \partial_t \tilde{n} + V(x) \partial_y \tilde{n} + \partial_x \langle n \rangle \partial_y \phi = \alpha (\tilde{\phi} - \tilde{n}) \]

\[ \partial_t \tilde{\zeta} + V(x) \partial_y \tilde{\zeta} - V''(x) \partial_y \phi = \alpha (\tilde{\phi} - \tilde{n}) \]

\[ \omega = \frac{k_y}{1 + k^2 \left( \partial_x \langle n \rangle + V'' \right) + k_y V} \]

for \( \alpha \gg 1 \)

- Real part of frequency proportional to PV gradient, not density gradient! This has lots of interesting consequences, just one of which is effect on particle flux
Preliminary results: particle flux

- Quasilinear flux in adiabatic limit can be computed as

\[ \Gamma_n \sim \frac{1}{\alpha} \sum_k -\frac{k_y^2}{1 + k^2} \left( k^2 \kappa - V'' \right) |\tilde{\phi}_k|^2 \]

- Good agreement with ML
- Vorticity gradient term can result in staircasing
- This project very much a work in progress (vorticity and enstrophy flux are harder), but this simple result shows its potential to elucidate new physics