Potential Vorticity Mixing in a Tangled Magnetic Field

— with applications to L-H transition

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Outline

1. Introduction

2. Solar Tachocline

3. L-H transition in tokamak

4. Conclusion

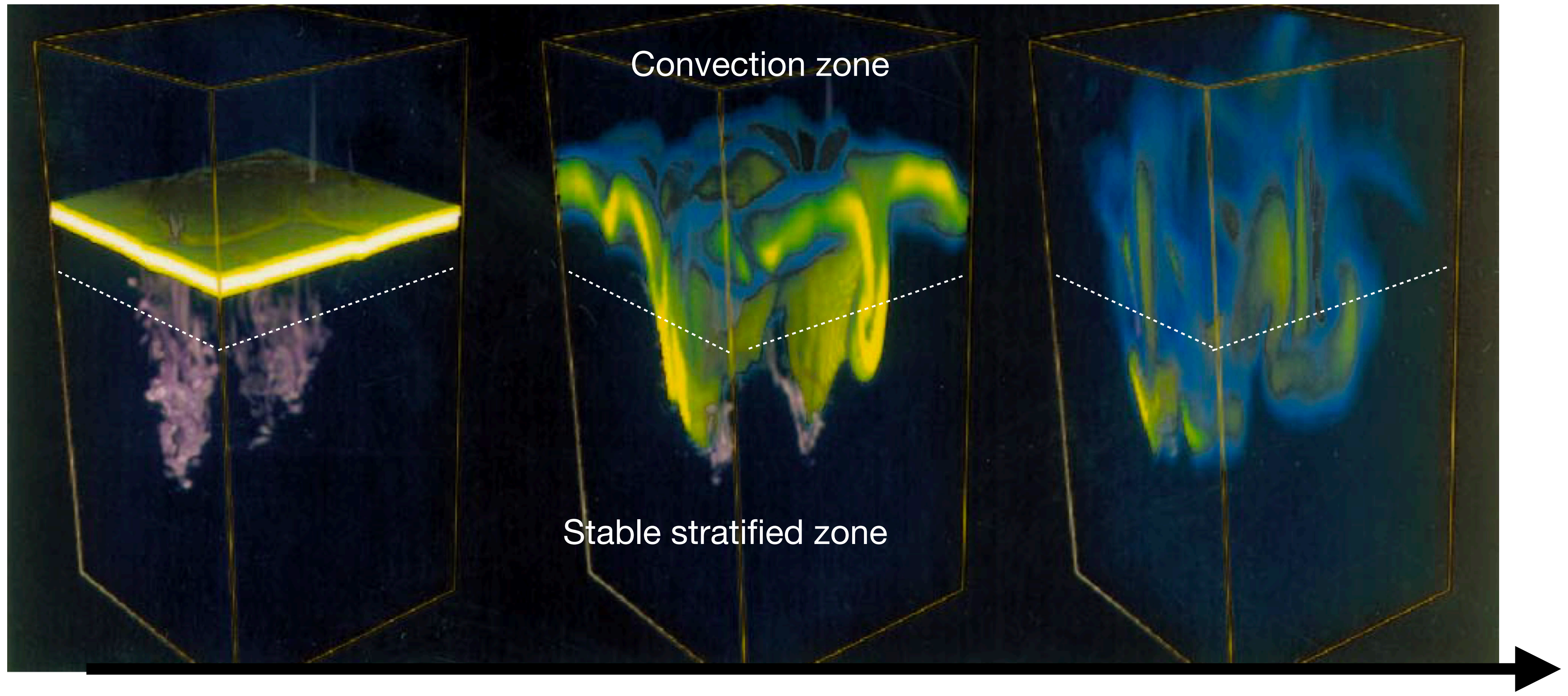
The physics of stochastic fields interaction with zonal flow in the solar techocline and at the edge of tokamak share fundamental elements.

- **Why study disordered magnetic fields?**

Disordered magnetic fields are frequently encountered.

The solar Tachocline

Weak mean magnetization

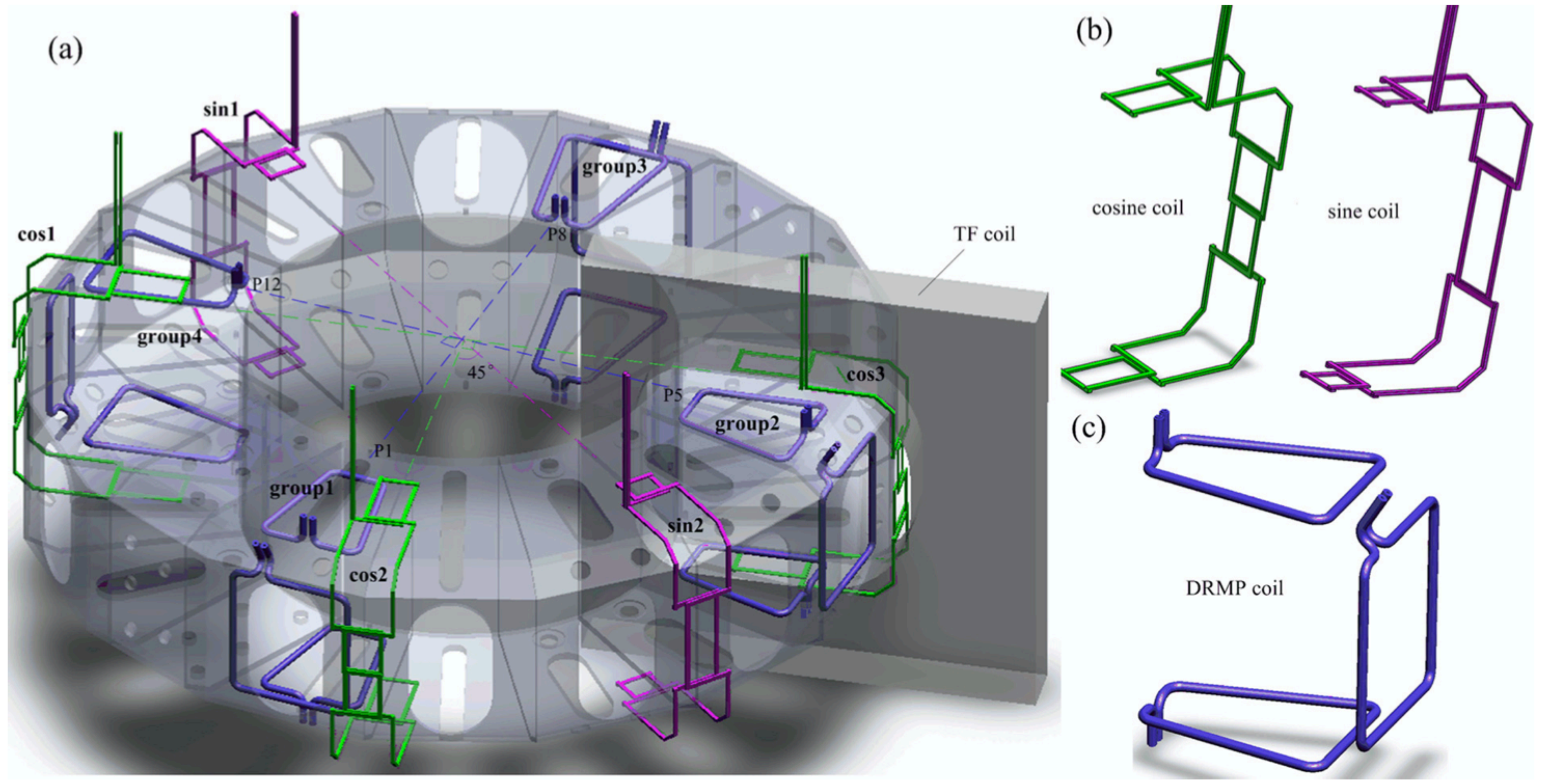


Simulation: the stochastic magnetic field has been "pumped" from the convection zone into the stably stratified region.

Quasi-2D

The tokamak

Strong mean magnetization



(J-TEXT)

The resonant magnetic perturbation (RMP) raises L-H transition power threshold.

3D with $\underline{k} \cdot \underline{B} = 0$ resonance

PV mixing in a disordered field **is a generic problem!**

Introduction

What is Potential Vorticity (PV)?

1. Potential Vorticity is a generalized vorticity.

$PV \equiv \zeta \equiv \nabla \times \mathbf{v}$ **(pure 2D fluid)**

$PV \equiv \zeta + 2\Omega \sin \phi_0 + \beta y$ **(on the β-plane)**

$PV \equiv (1 - \rho_s^2 \nabla^2)\frac{|e|\phi}{T} + \frac{X}{L_n}$ **(Hasegawa-Mima eq. for tokamak)**

2. It is conserved along the fluid — acts as conserved phase space density. Magnetic fields will break PV conservation.

How the zonal flow evolves?

1. Flux of the potential vorticity $\equiv \langle \tilde{u}\tilde{\zeta} \rangle$
2. Taylor Identity and the evolution of zonal flow.

Taylor Identity: $\underbrace{\langle \tilde{u}_y \tilde{\zeta} \rangle}_{PV\ flux} = \underbrace{-\frac{\partial}{\partial y}\langle \tilde{u}_y \tilde{u}_x \rangle}_{Reynolds\ force}$

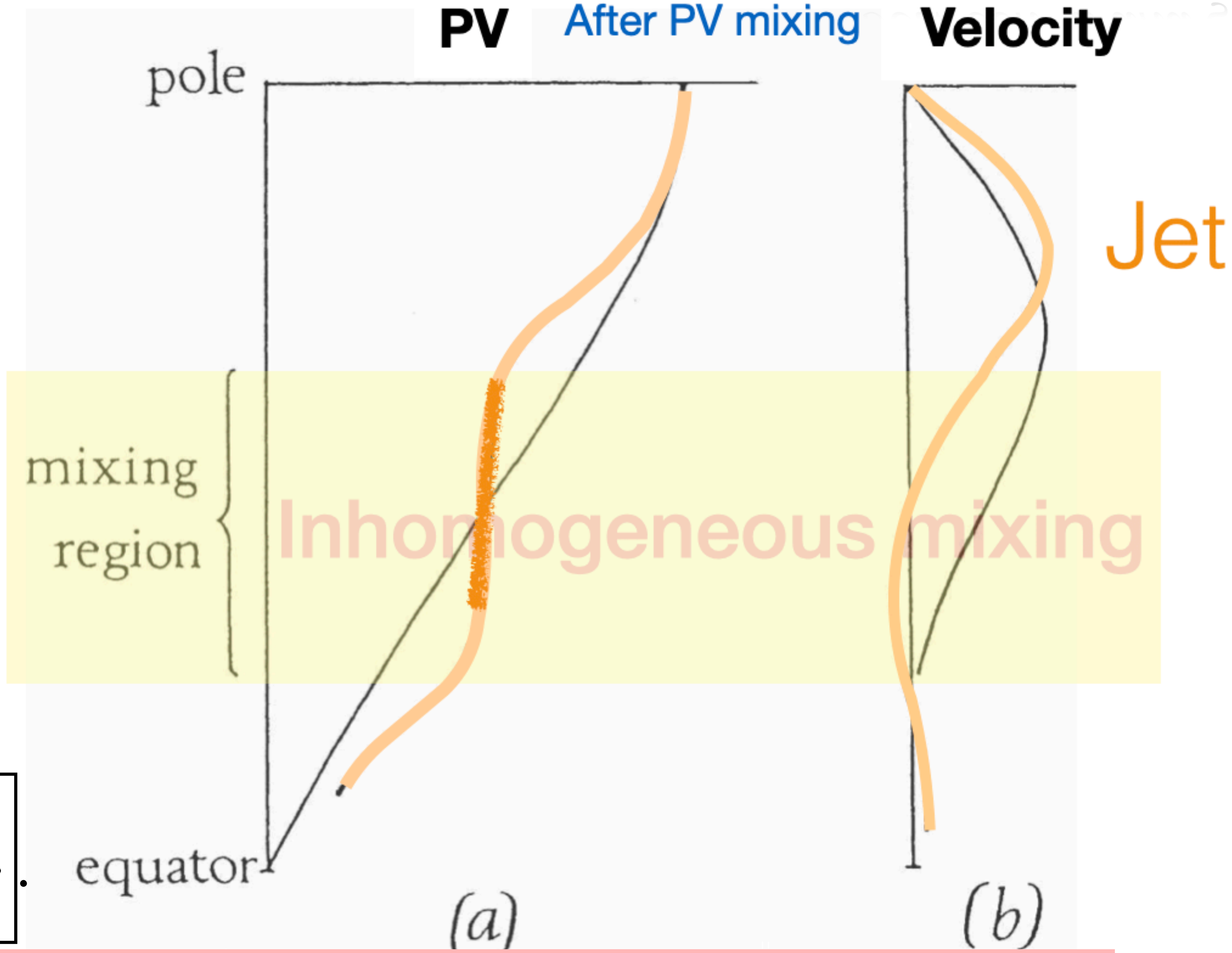
Evol. of zonal flow: $\frac{\partial}{\partial t}\langle u_x \rangle = \langle \tilde{u}_y \tilde{\zeta} \rangle = -\frac{\partial}{\partial y}\langle \tilde{u}_y \tilde{u}_x \rangle.$

What is inhomogeneous PV mixing?

Local PV mixing causes changes in flow structure.

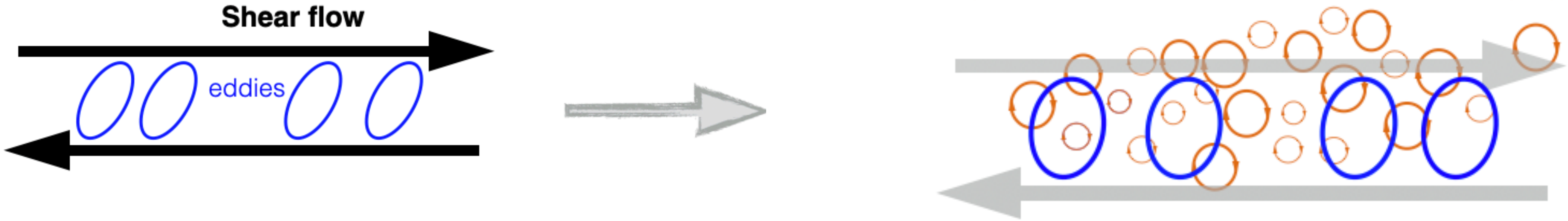
$$\textbf{PV flux} \equiv \underbrace{\langle \tilde{u}_y \tilde{\zeta} \rangle}_{} \neq 0$$

phase correlation between u and ζ

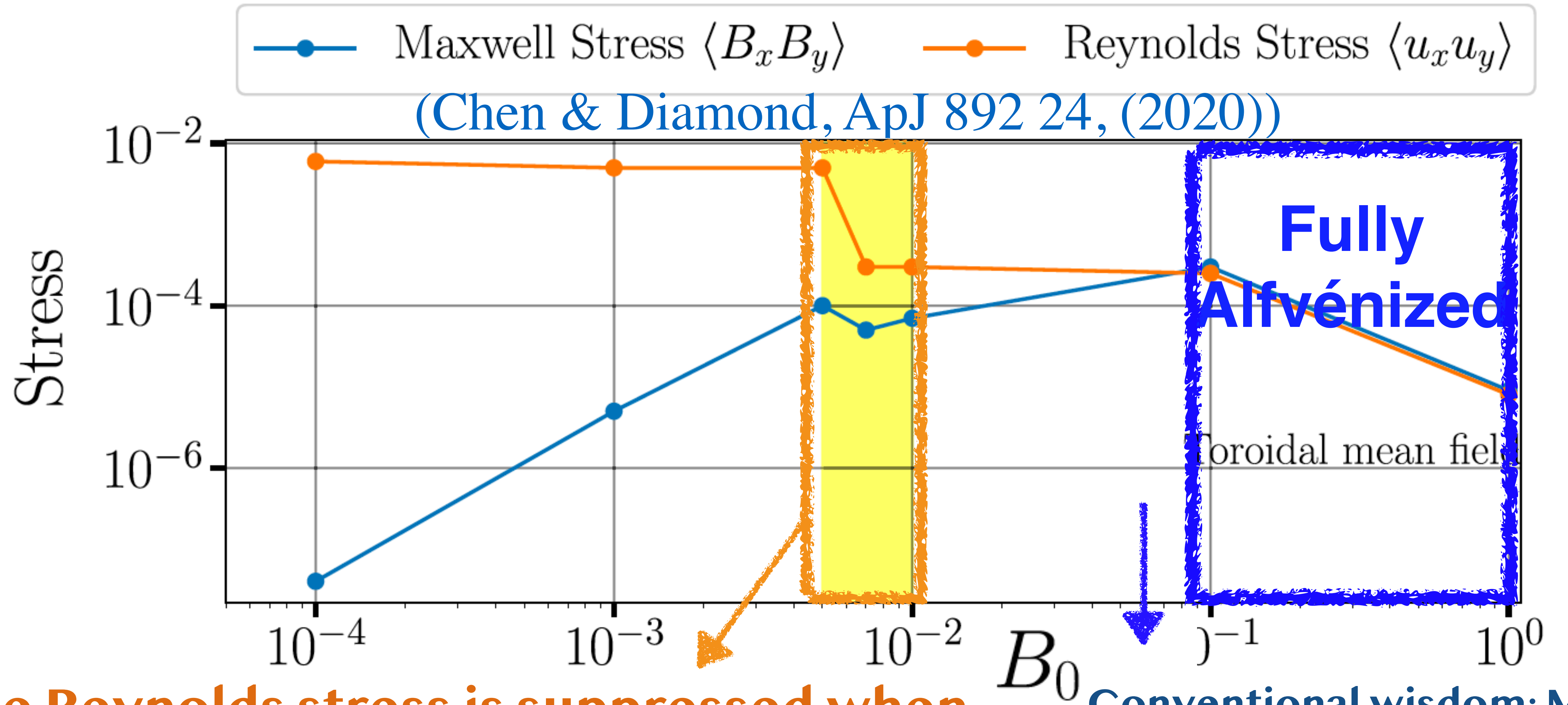


Stochastic Decoherence

- **Reynolds stress suppressed by stochastic dephasing:**



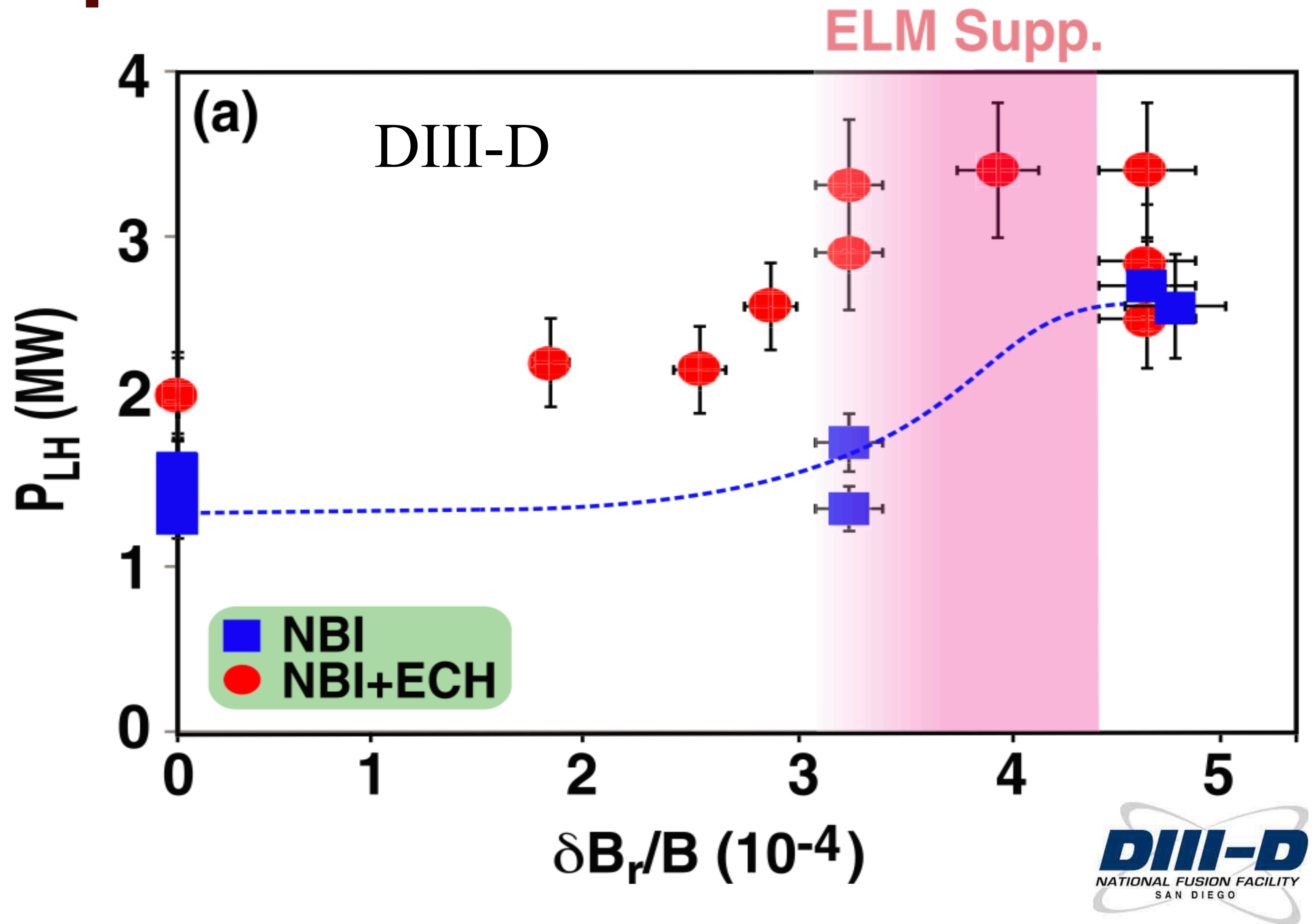
- **Simulation result in solar tachocline:**



The Reynolds stress is suppressed when mean field is weak, before the mean field is strong enough to fully Alfvénize the system.

Conventional wisdom: Maxwell/ Reynolds stress balance when the system is Alfvénized.

- **Experimental result from DIII-D:**



(L. Schmitz et al, NF **59** 126010 (2019))

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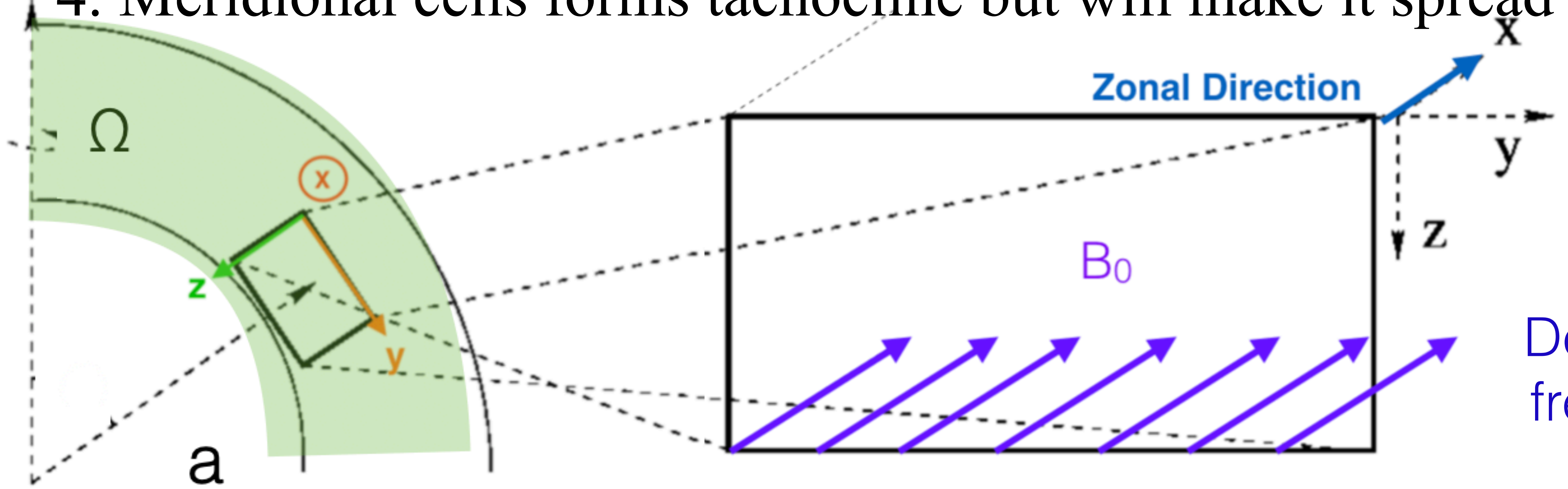
β - plane Model

Properties:

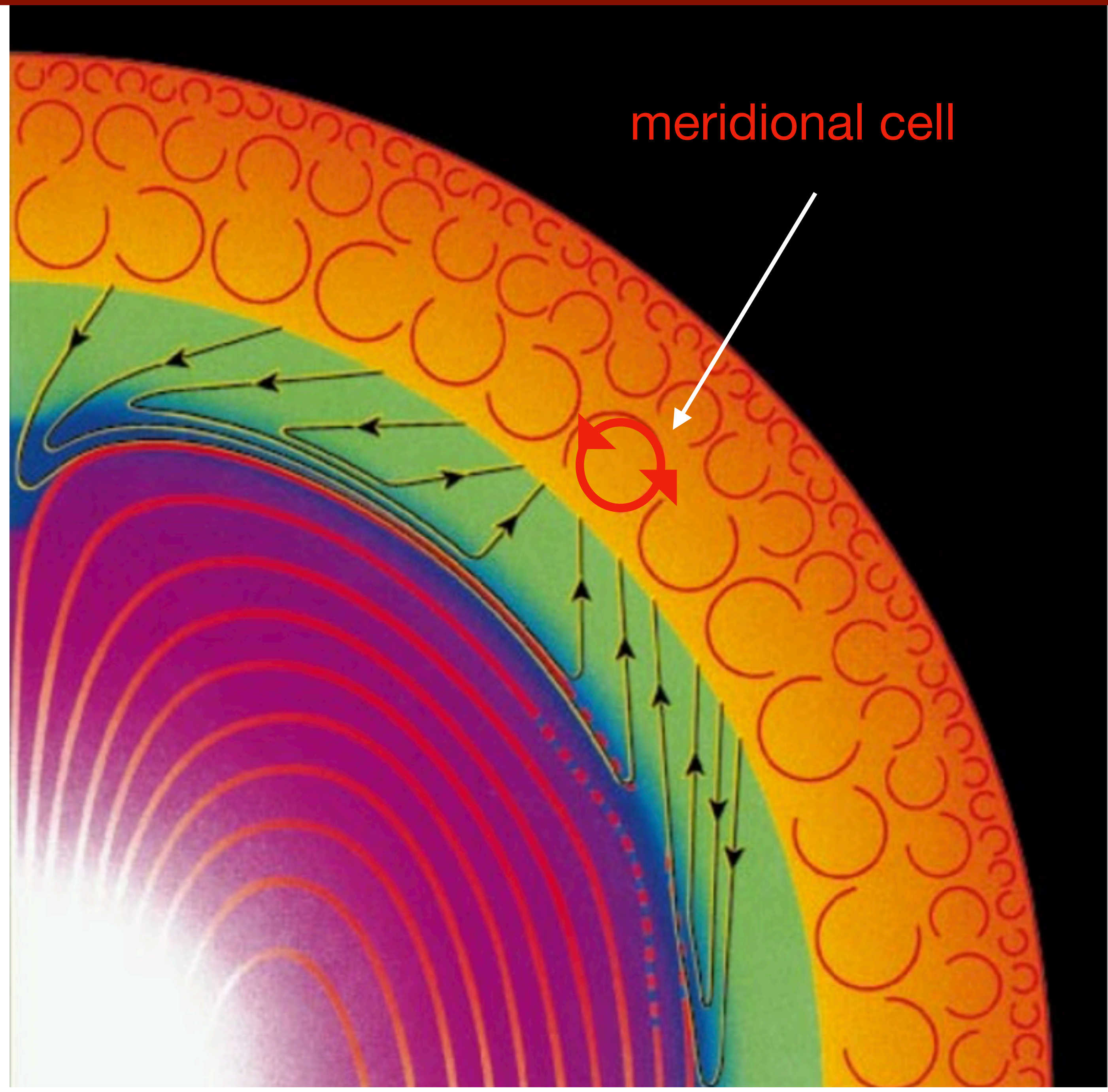
1. Strongly Stratified (β-plane model)
2. Zonal Flow and Rossby wave — as in the Jovian Atmosphere.
3. Large magnetic perturbation — large magnetic Kubo number.
4. Meridional cells forms tachocline but will make it spread



Rossby Parameter (β):

$$\beta = \frac{df}{dy}\Big|_{\phi_0} = 2\Omega cos(\phi_0)/a$$

Derivative of angular frequency f (Coriolis parameter) — latitude — rotation — radius



The tachocline formation:

➤ Spiegel & Zahn (1992) —
Spreading of the tachocline is opposed by turbulent viscous diffusion of momentum in latitude.

➤ Gough & McIntyre (1998) —
Spreading of the tachocline is opposed by a hypothetical fossil field in the radiational zone.

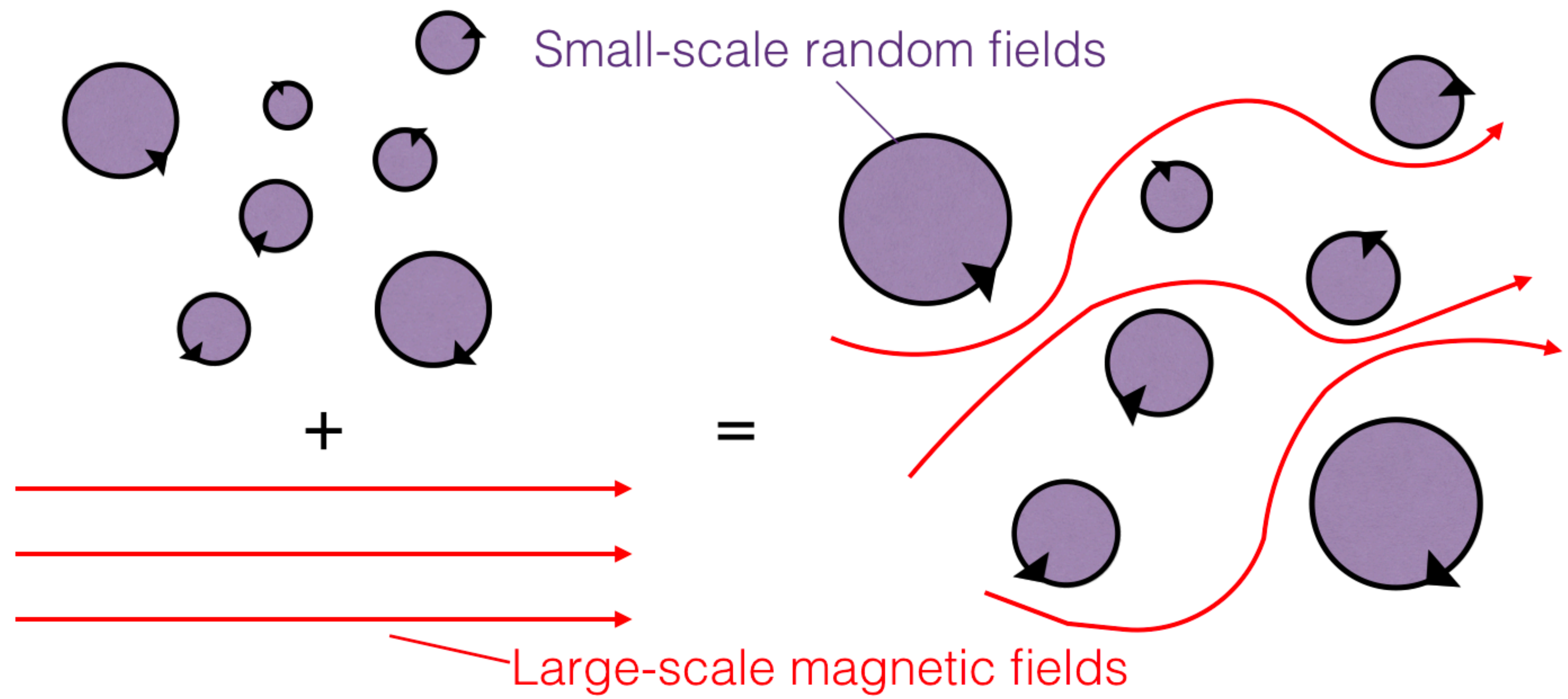
"At the heart of this argument, therefore, is the role of the fast turbulent processes in redistributing angular momentum on a long timescale." **— (Tobias et al. 2007)**

These two models ignore the strong predicted stochasticity of the tachocline magnetic field.

How we describe stochastic fields?

Quasi-2D

Magnetic field = mean field + stochastic field $B = B_0 + \widetilde{B}$



The large-scale magnetic field is distorted by the small-scale fields. The system thus is the 'soup' of cells threaded by sinews of open field line (Zel'dovich, 1983).

A weak mean field— large **magnetic Kubo number, if** $|\widetilde{B}^2|/B_0^2 \gg 1$.

- **Fluid Kubo number:** $Ku_f \equiv \frac{\delta_l}{\Delta_\perp} \sim \frac{\tilde{v}\tau_{ac}}{\Delta_\perp} \sim \frac{\tau_{ac}}{\tau_{eddy}} < 1,$

  Auto correlation time (τ_{ac})

  Eddy turnover time (τ_{eddy})

- **Magnetic Kubo number:** $Ku_{mag} \equiv \frac{\delta_l}{\Delta_{eddy}} = \frac{l_{ac}|\widetilde{\mathbf{B}}|}{\Delta_{eddy}B_0}$

$$Ku = \begin{cases} < 1, & \text{Quasi-linear theory} \\ > 1, & \text{Quasi-linear theory fails} \end{cases}$$

"Simple" quasi-linear theory can fail.

How we describe stochastic fields?

Truth in Advertising

The system is **strongly nonlinear** and **simple quasi-linear method fails**.

A "frontal assault" on calculating PV transport in an ensemble of tangled magnetic fields is a daunting task.

Rechester & Rosenbluth (1978) suggested replacing the "full" problem with one where waves, instabilities, and transport are studied in the presence of **an ensemble of prescribed, static, stochastic fields.**

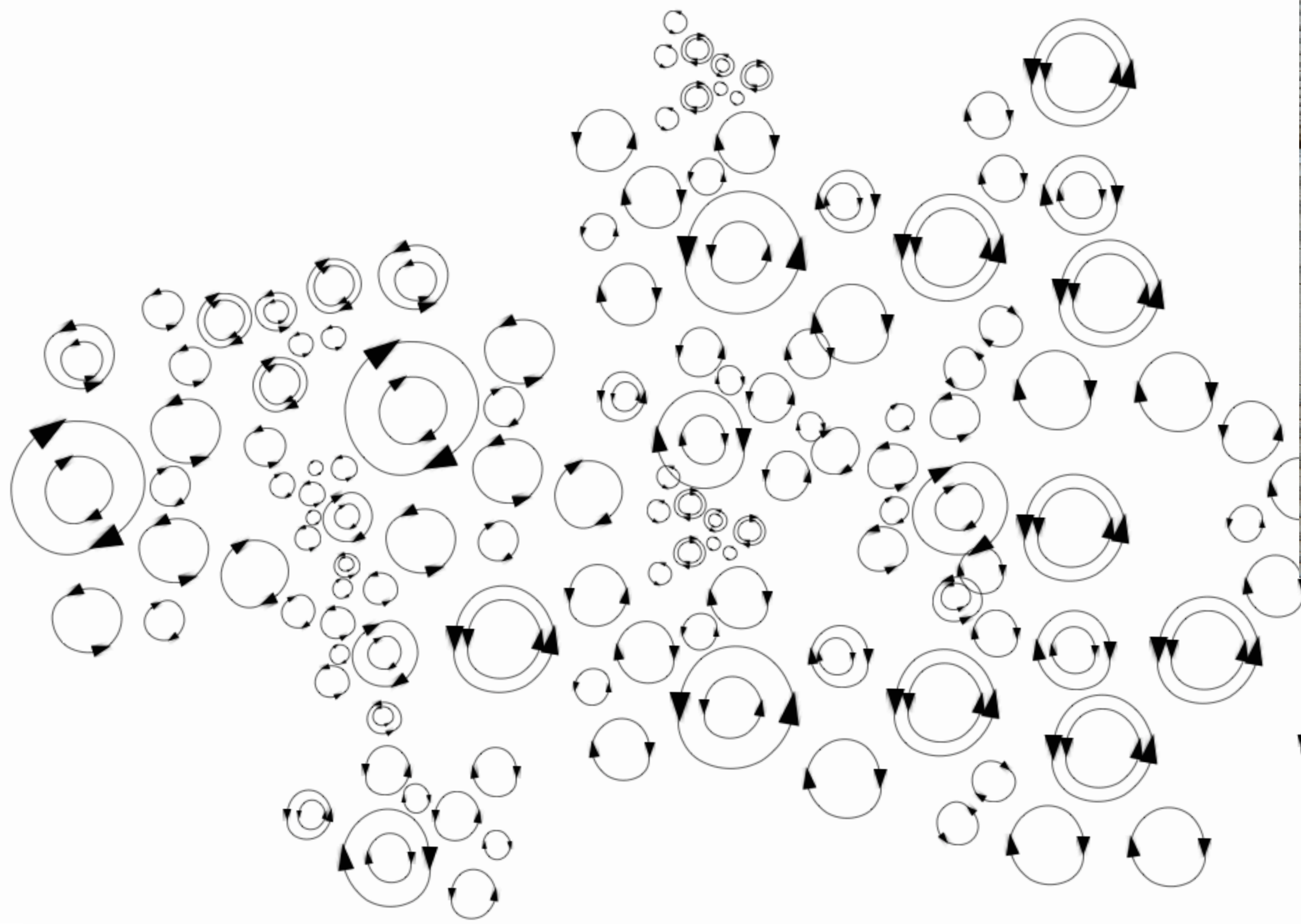
Assumptions:

1. Amplitudes of random fields distributed statistically.
2. Auto-correlation length of fields is small (delta correlation $l_{ac} \to 0$, such that

$$Ku_{mag} \equiv \frac{\tilde{u}\tau_{ac}}{\Delta_{eddy}} = \frac{l_{ac}\,|\,\widetilde{\mathbf{B}}\,|}{\Delta_{\perp}B_0} < 1, \text{ even}$$

$\widetilde{B}/B_0 > 1$.)

➤ Closure theory.



Order of Scale in β-plane MHD

Parameters:

Stream Function $\psi = \psi(x, y, z)$

Velocity field $\boldsymbol{u} = (\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0)$

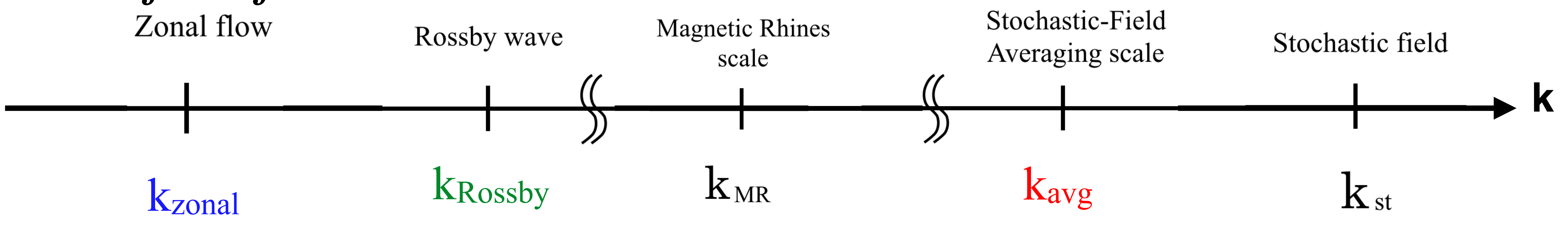
Fluid Vorticity $\boldsymbol{\zeta} = (0,0,\zeta)$

Potential Field $\boldsymbol{A} = (0,0,A)$

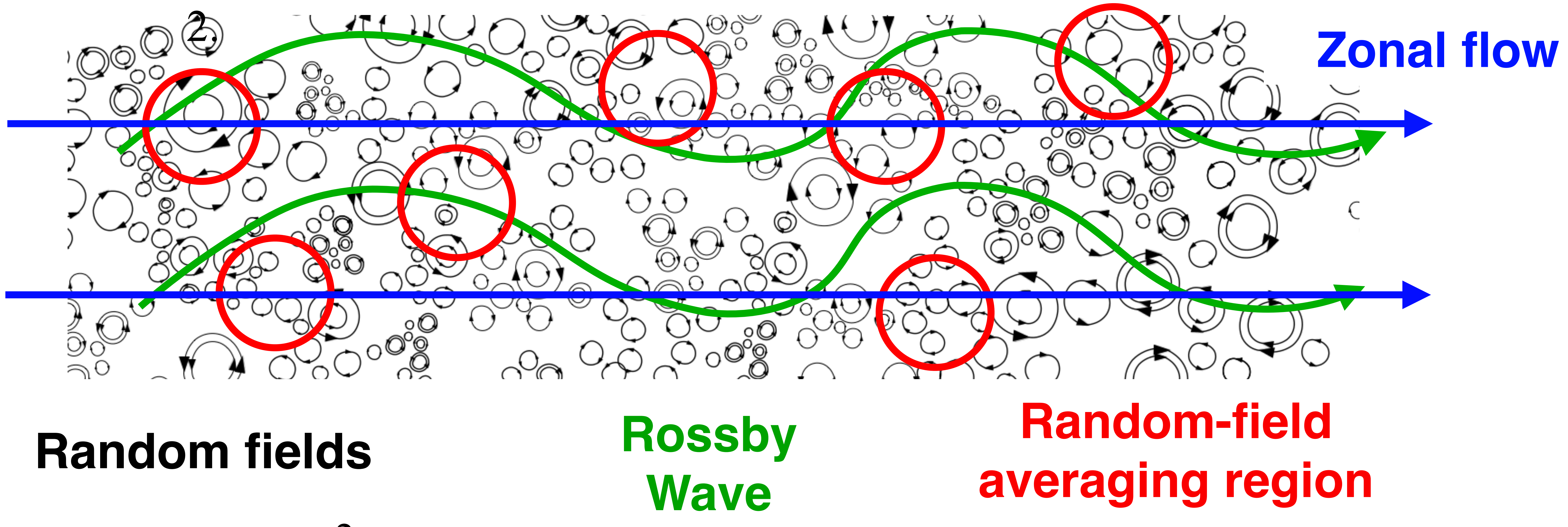
Magnetic Field $\boldsymbol{B} = (\frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x}, 0)$

Two-average Method:

1. $\overline{F} = \int dR^2 \int dB_{st} \cdot P_{(B_{st,x},B_{st,y})} F$ $\quad \langle\ \rangle = \frac{1}{L}\int dx \frac{1}{T}\int dt$ ensemble average over the zonal scales



2.



Two main equations:

Quasi-linear closure:

Function of fields $\mathbf{F} = \mathbf{F_0} + \widetilde{\mathbf{F}} + \mathbf{F_{st}}$

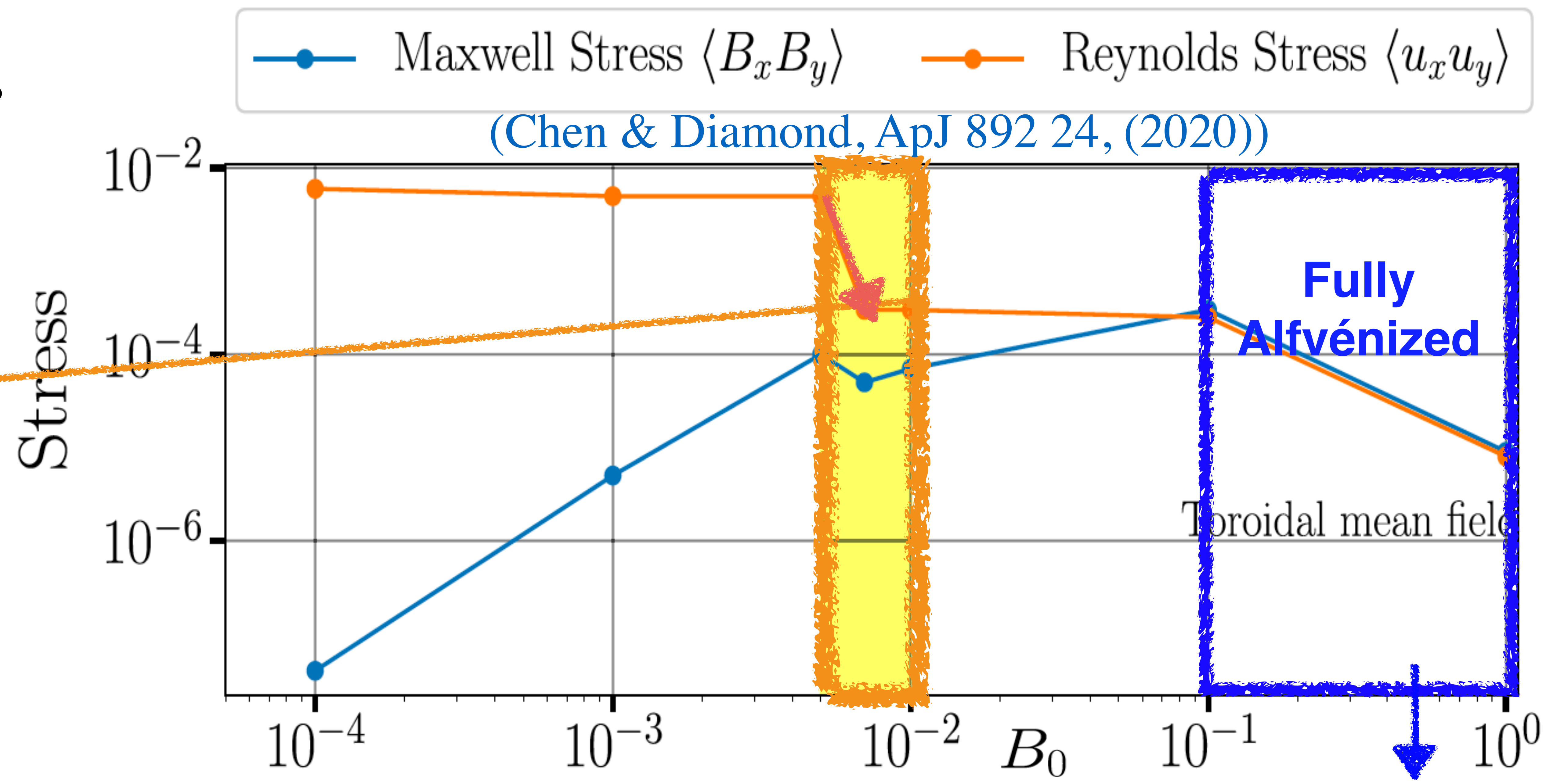
Vorticity Eq: $(\frac{\partial}{\partial t} + \mathbf{u}_\perp \cdot \nabla_\perp)\zeta - \beta\frac{\partial\psi}{\partial x} = -\frac{(\mathbf{B}\cdot\nabla)(\nabla^2 A_z)}{\mu_0\rho} + \nu(\nabla\times\nabla^2\mathbf{u})$

Induction Eq: $(\frac{\partial}{\partial t} + \mathbf{u}_\perp \cdot \nabla_\perp)A = B_l\frac{\partial\psi}{\partial x} + \eta\nabla^2 A,$

Simulation Results

Conventional wisdom– what "fully Alfvénization" means?

1. All the wave energy transferred into Alfvén wave (dominated mode of the wave is Alfvén frequency).
2. The wave and magnetic energy reach equi-partition.
3. Then the Maxwell stress cancels the Reynolds stress



Conventional wisdom: Maxwell/Reynolds stress balance when the system is Alfvénized.

What really happens…

The Reynolds stress is suppressed when mean field is weak, before the mean field is strong enough to fully Alfvénize the system.

Suppression of zonal flow:

- ➤Random magnetic fields modify the **PV flux.**
- ➤The dissipative nature of **the wave-field coupling** induces **a magnetic drag** on the mesoscale flows.

$$\frac{\partial}{\partial t}\langle u_x \rangle = \langle \overline{\Gamma} \rangle - \frac{1}{\mu_0 \rho \eta}\langle \overline{B_{st,y}^2} \rangle \langle u_x \rangle + \nu \nabla^2 \langle u_x \rangle$$

PV flux

Magnetic drag force $(J_{st} \times B_{st})$

Results — Multi-scale Dephasing

Multi-scale Dephasing:

Mean PV Flux (Γ) and PV diffusivity (D_{PV}).

PV Diffusivity $\mathbf{D_{pv}}$

Mean field

$B_0^2 < \overline{B_{st}^2}$ small-scale random fields

$$\overline{\Gamma} = -\sum_k |\tilde{u}_{y,k}|^2 \frac{\nu k^2 + (\frac{B_0^2 k_x^2}{\mu_0 \rho})\frac{\eta k^2}{\omega^2+\eta^2 k^4} + \frac{\overline{B_{st,y}^2} k^2}{\mu_0 \rho \eta k^2}}{\left(\omega - (\frac{B_0^2 k_x^2}{\mu_0 \rho})\frac{\omega}{\omega^2+\eta^2 k^4}\right)^2 + \left(\nu k^2 + (\frac{B_0^2 k_x^2}{\mu_0 \rho})\frac{\eta k^2}{\omega^2+\eta^2 k^4} + \frac{\overline{B_{st,y}^2} k^2}{\mu_0 \rho \eta k^2}\right)^2}\left(\frac{\partial}{\partial y}\overline{\zeta} + \beta\right)$$

The large- and **small-scale magnetic fields** have a **synergistic effect on the cross-phase** in the Reynolds stress.

Dispersion relation of the Rossby-Alfvén wave with stochastic fields:

(mean square) **(square mean)**

$$\left(\omega - \omega_R + \frac{i\overline{B_{st,y}^2} k^2}{\mu_0 \rho \eta k^2} + i\nu k^2\right)\left(\omega + i\eta k^2\right) = \frac{B_{0,x}^2 k_x^2}{\mu_0 \rho}$$

$$\frac{\text{sping constant}}{\text{dissipation}} = \frac{\overline{B_{st}^2} k^2/\mu_0 \rho}{\eta k^2}$$

AW of the large-scale magnetic field

Dissipative response to Random magnetic fields

Rossby frequency $\omega_R \equiv -\beta k_x / k^2$

➤ **Drag+dissipation effect**

→ this implies that the tangled fields and fluids define a **resisto-elastic medium.**

Results — Resisto-Elastic Medium

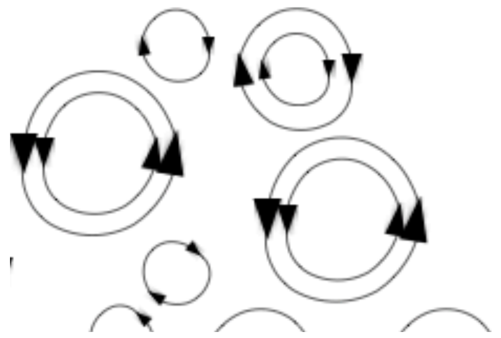
$\overline{B_{st}^2}$ - Resisto-elastic Medium:

Rossby frequency $\omega_R = 0$

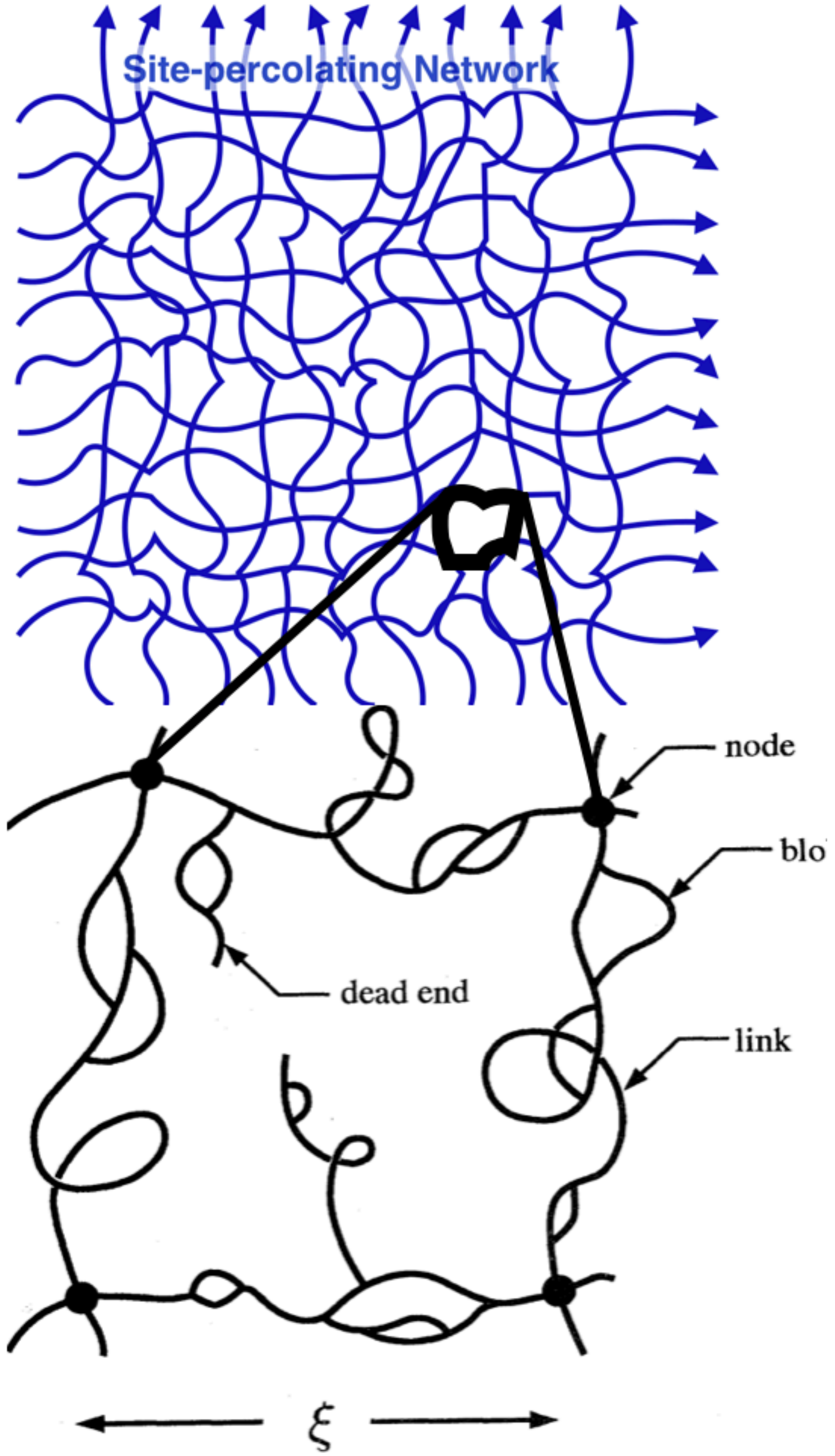
$$\omega^2 + i(\alpha + \eta k^2)\omega - \left(\frac{\overline{B_{st,y}^2}k^2}{\mu_0\rho} + \frac{B_0^2 k_x^2}{\mu_0\rho}\right) = 0,$$

Small-scale field spring constant

Large-scale field spring constant



Alfvénic loops + elastic wave = resisto-elastic medium



Schematic of the nodes-links-blobs model (Nakayama & Yakubo 1994).

- Fluids couple to network elastic modes. Large elasticity increases memory.
- This network can be fractal (multi-scale) and intermittent (→ packing factor: $\overline{B_{st}^2} \to p\overline{B_{st}^2}$) → "fractons" (Alexander & Orbach 1982).
- **Similar physics— polymeric liquids.** We can calculate the effective spring constant, effective Young's Modulus of elasticity.

Conclusion

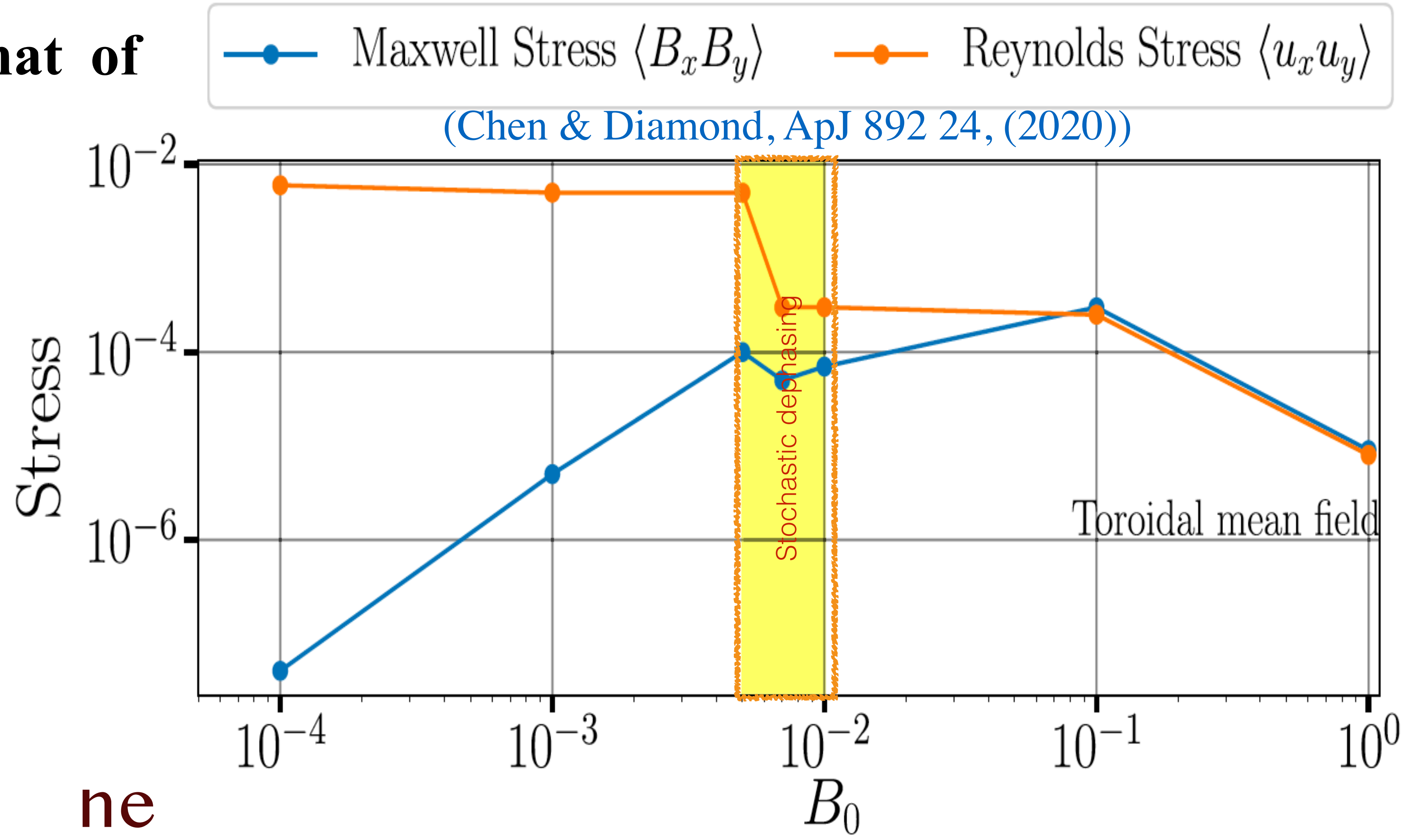
What studies have shown:

Reynolds stress will undergo decoherence at levels of field intensities **well below that of Alfvénization** (where Maxwell stress balances the Reynolds stress).

The flow generated by PV mixing/Reynolds force are reduced by:

$$\frac{\partial}{\partial t}\langle u_x\rangle = \langle\overline{\Gamma}\rangle - \frac{1}{\eta\mu_0\rho}\langle\overline{B_{st,y}^2}\rangle\langle u_x\rangle + \nu\nabla^2\langle u_x\rangle$$

1. Coupling to resisto-elastic waves, which is $\overline{B_{st}^2}$ dependent.
2. Increase of the magnetic drag.



- Both Spiegel & Zahn (1992) and Gough McIntyre (1998) Models fc ne solar Tachocline **are not correct**. The truth here is 'neither pure nor simple' (apologies to Oscar Wilde).

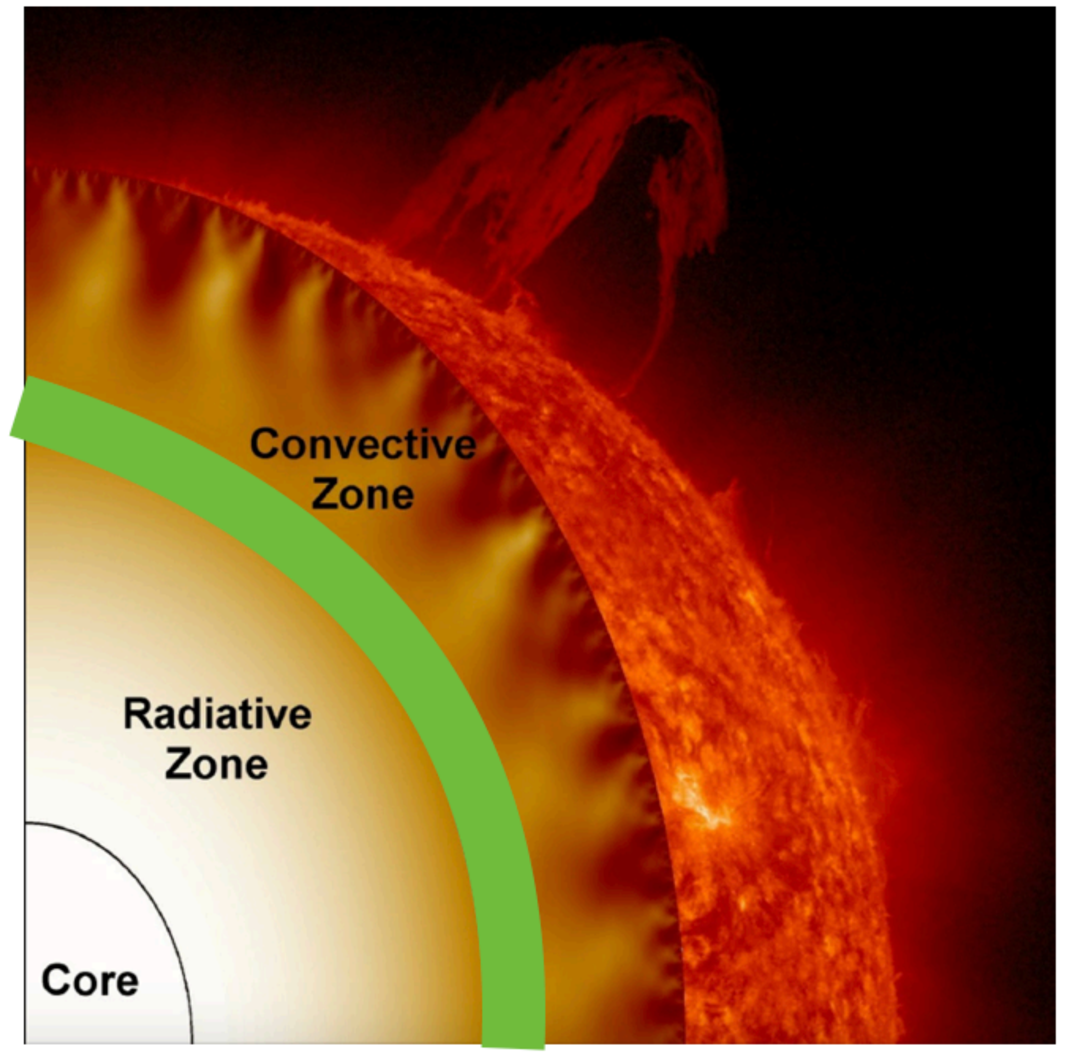
These two models both ignore strong stochastic fields of the tachocline.

Suggestions for future simulation works:

1. A third axis: Rossby parameter β.
2. Include a static tangled field.

➤ Maya Katz, Robin Heinonen, and Patrick Diamond (TO-16. Nov. 12, 10:30 am).
Cross-Helicity Generation and Structure Formation in β-plane MHD Turbulence.

Solar Tachocline



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3D

The model (Cartesian Coordinate):

1. Strong mean field (3D).
2. $\underline{k} \cdot \underline{B} = 0$ (or $k_{\|} = 0$) **resonant at rational surface has third direction —** $\omega \to \omega \pm v_A k_z$.
3. Kubo number: $Ku_{mag} = \frac{l_{ac}|\widetilde{\mathbf{B}}|}{\Delta_{\perp} B_0} < 1$).
4. Four-field equations

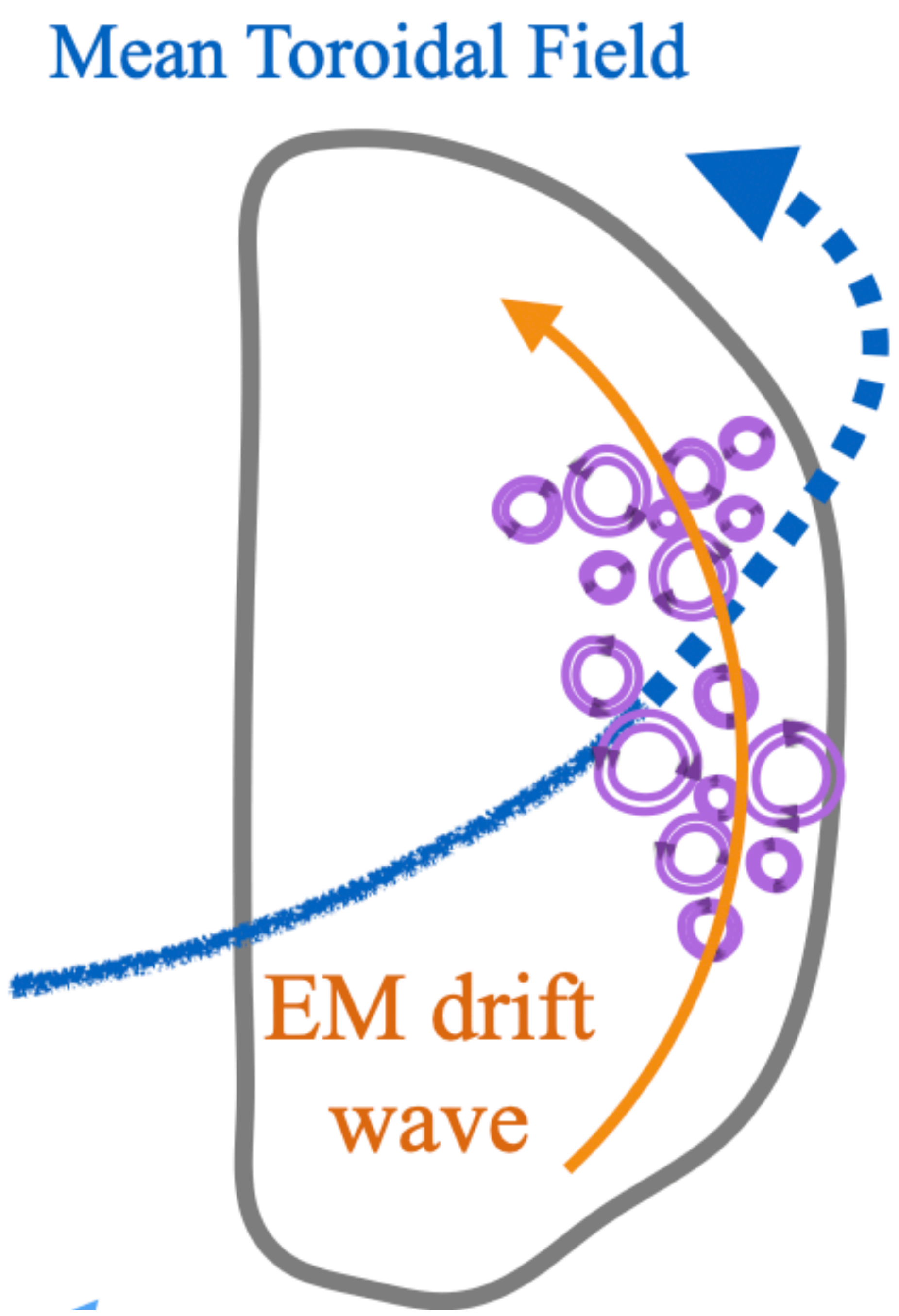
→

(a) Vorticity equation — vorticity $-\nabla^2 \psi \equiv \zeta$

(b) Induction equation — **A, J**

(c) Pressure equation — **P**

(d) Parallel flow equation — $\mathbf{v}_{\|}$



Mean-field Approximation:

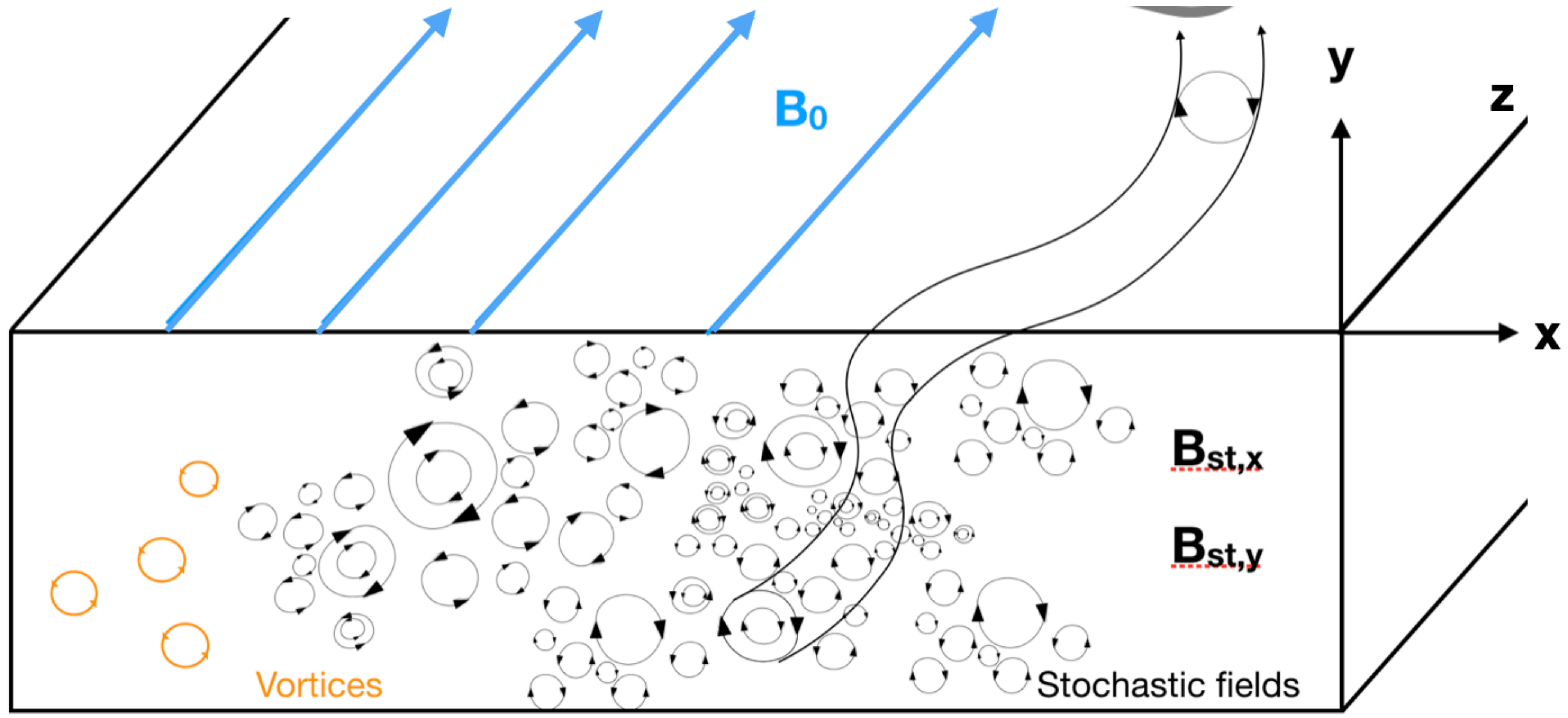
$$\zeta = \langle \zeta \rangle + \tilde{\zeta}$$
$$\psi = \langle \psi \rangle + \tilde{\psi}$$
$$A = \langle A \rangle + \tilde{A}$$

$$, where \;\; \langle\ \rangle = \frac{1}{L}\int dx \frac{1}{T}\int dt$$

Perturbations produced by turbulences

ensemble average over the zonal scales

$$b^2 \equiv \overline{B_{st}^2}/B_0^2$$



Scales

When does stochastic Fields dephasing become effective?

Basic scales:

Shear flow rate	Alfvénic Dispersion	Natural linewidth	Stochastic broadening
$k_\theta \Delta x \frac{\partial}{\partial x} u_y$	$v_A \mid \Delta k_\parallel \mid$	$\Delta\omega$	$Dk_\perp^2$

ω

(non-linear micro-instability process)

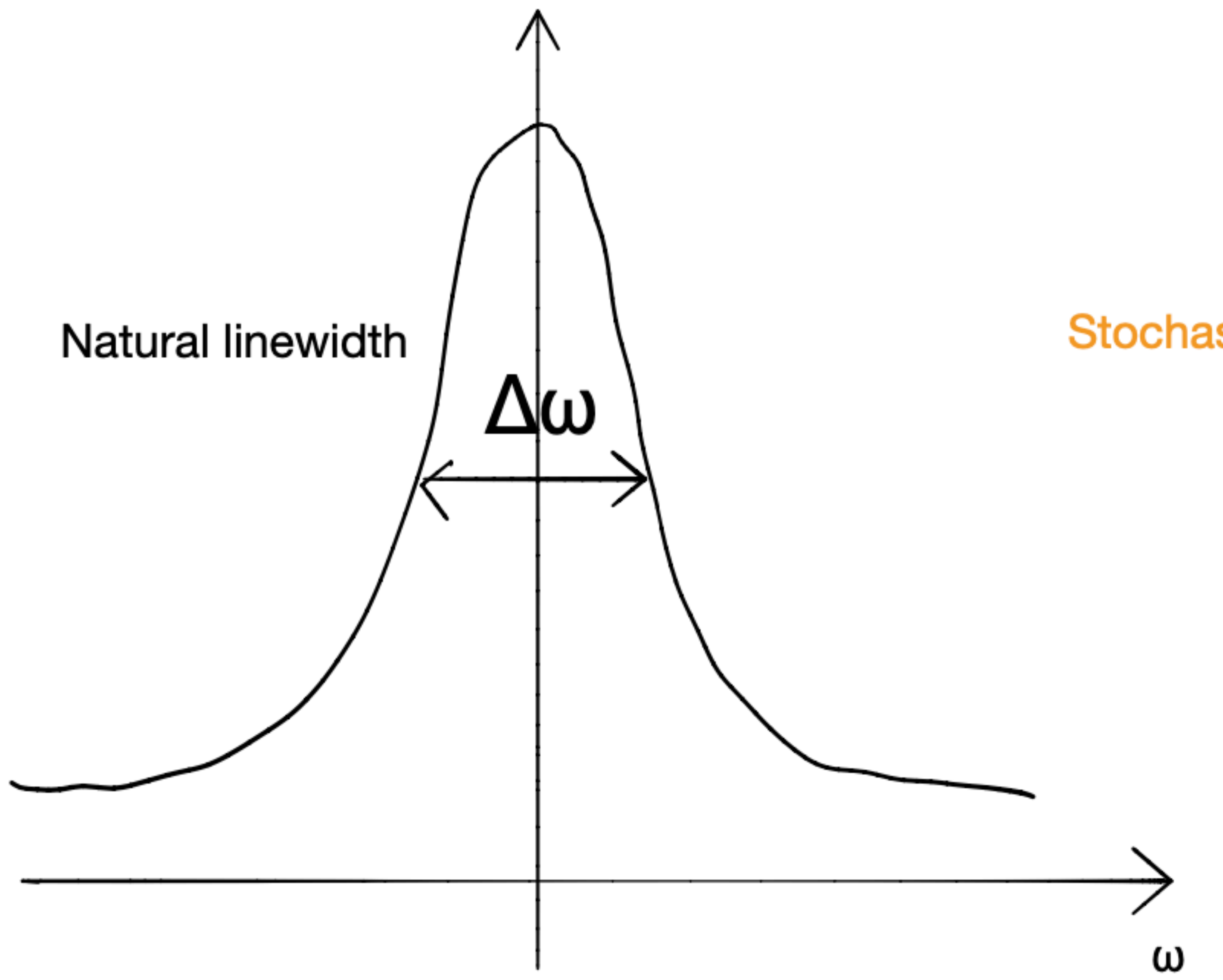
Stochastic field decoherence beats self-decoherence.

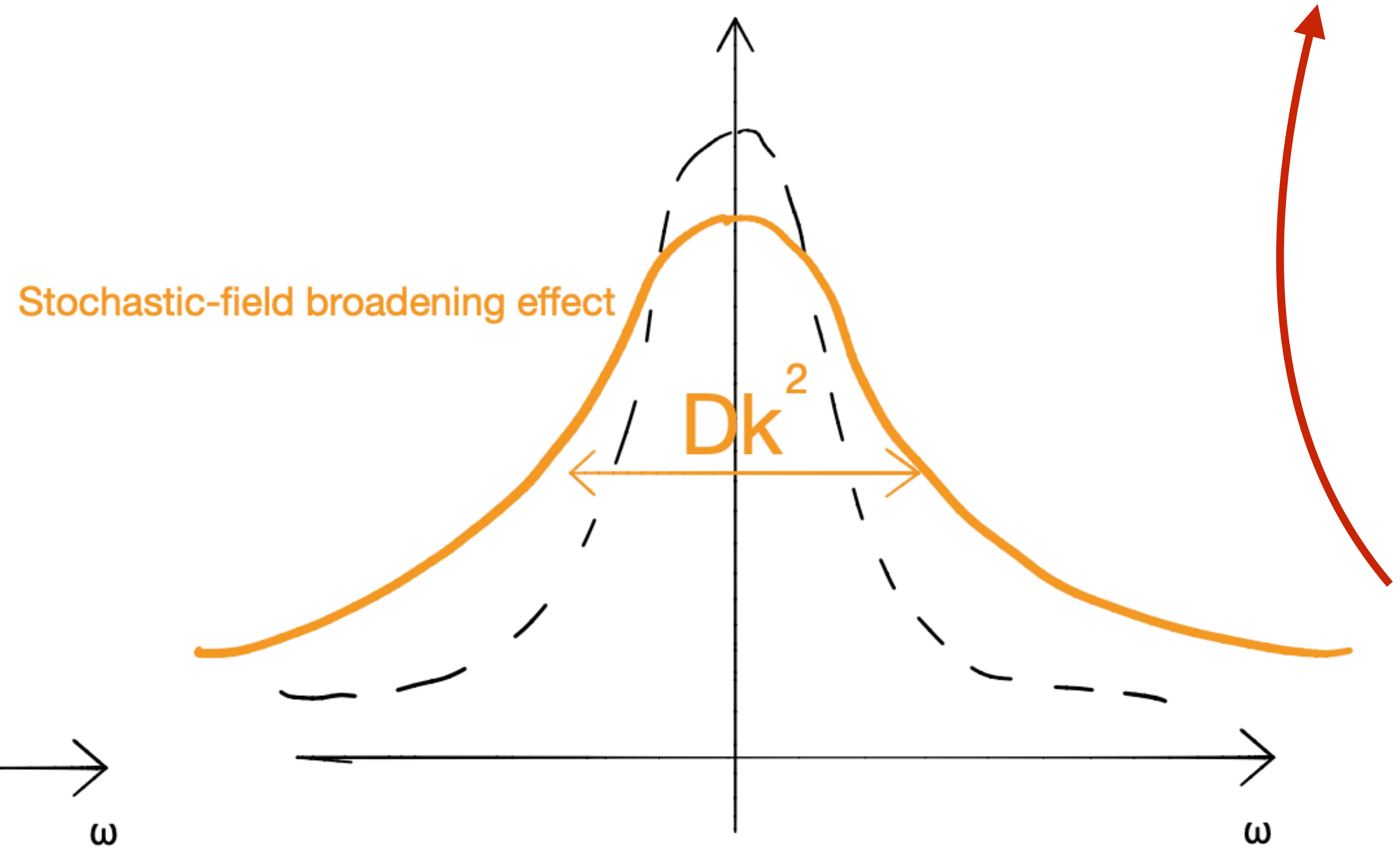
$$D \equiv v_A D_M = v_A \sum_k \pi\delta(k_\parallel) b_k^2$$

Magnetic diffusivity

Auto-correlation length l_{ac}

Alfvén wave propagate via stochastic fields
→ characteristic velocity from $\underline{\nabla} \cdot \underline{J} = 0$





Scales

- $Dk_\perp^2 > \Delta\omega$ **gives a dimensionless parameter (α):**

1. $$\begin{cases} l_{ac} \simeq Rq \\ \epsilon \equiv L_n/R \sim 10^{-2} \\ \beta \simeq 10^{-2\sim-3} \\ \rho_* \equiv \frac{\rho_s}{L_n} \simeq 10^{-2\sim-3} \end{cases}$$

$$b^2 \equiv \left(\frac{\delta B_r}{B_0}\right)^2 > \sqrt{\beta}\rho_*^2 \frac{\epsilon}{q} \sim 10^{-7}$$

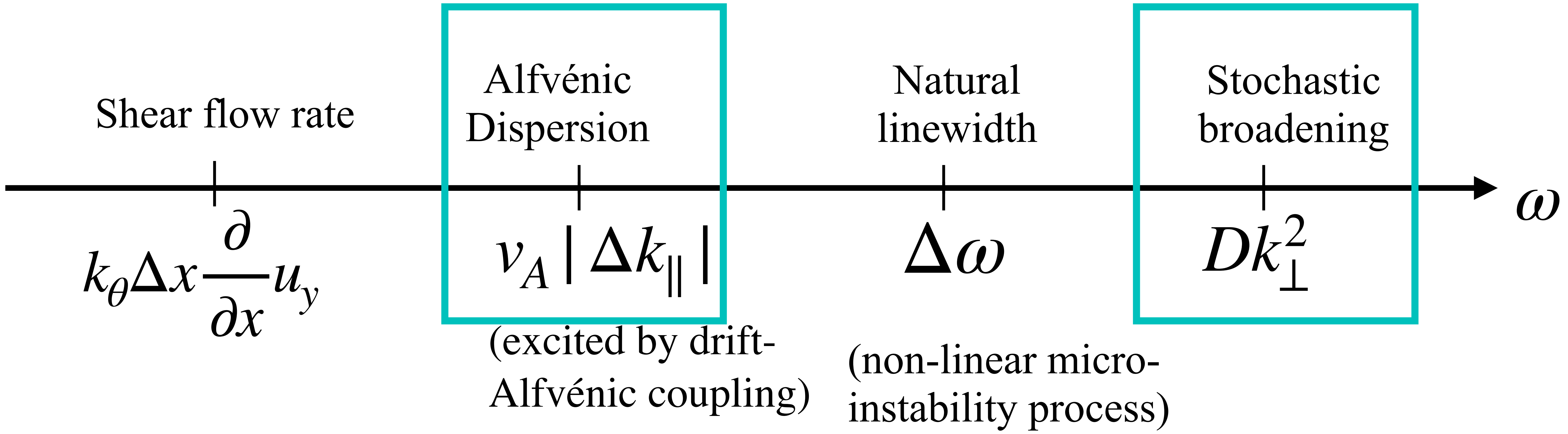
2. $$\alpha \equiv \frac{b^2}{\rho_*^2\sqrt{\beta}}\frac{q}{\epsilon} > 1$$

Extended Kim-Diamond Model

Criterion for stochastic fields effect important to L-H transition.

- **How ‘stochastic’ is this? Magnetic Kubo number?**

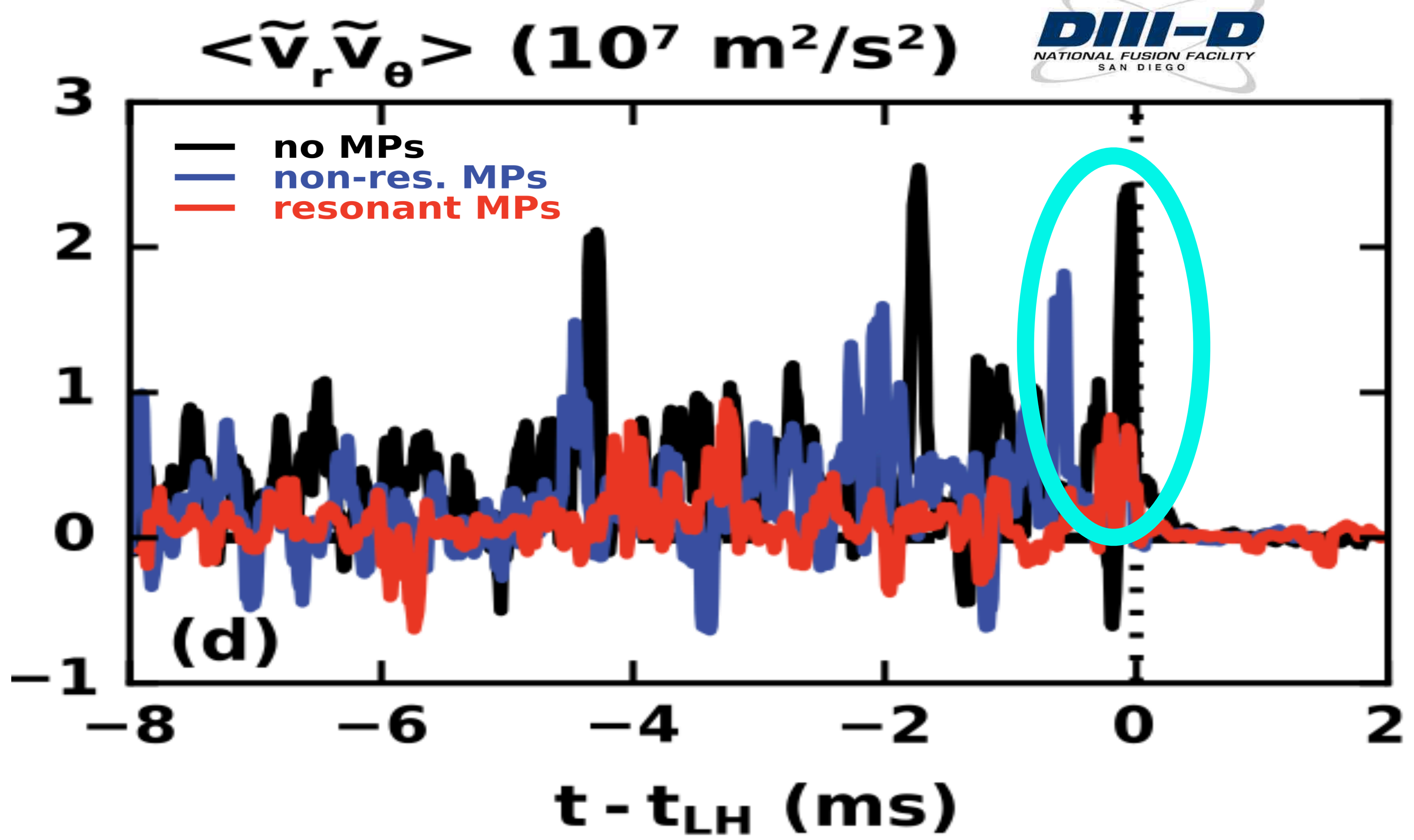
Basic scales:



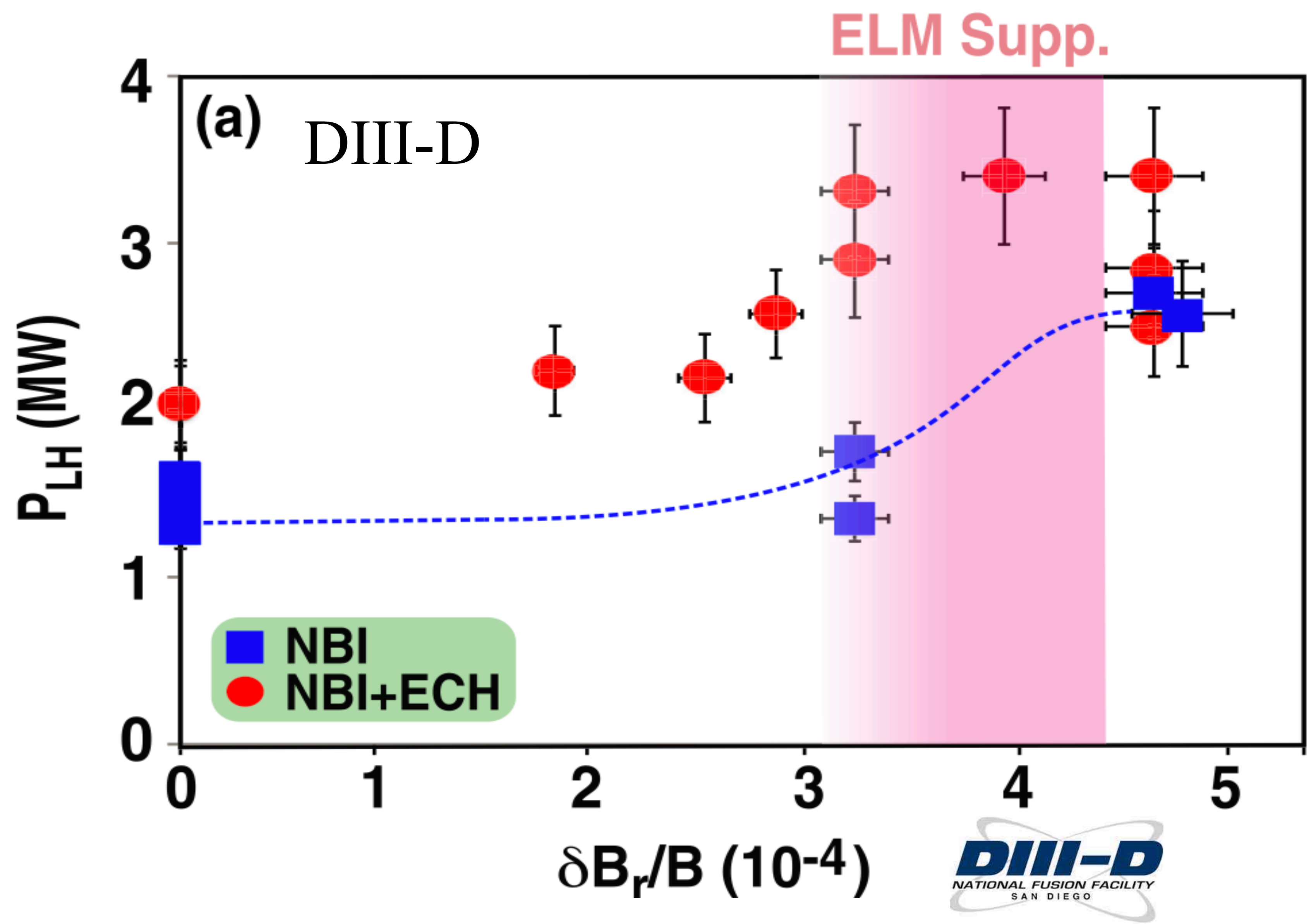
$$Ku_{mag} \equiv \frac{\textbf{stochastic field scattering length}}{\textbf{perpendicular magnetic fluctuation size}} \simeq 1$$

Experimental Results

- **Experimental results in L-H transition (DIII-D):**



(D. Kriete et al, PoP **27** 062507 (2020))



(L. Schmitz et al, NF **59** 126010 (2019))

- **Suppression of poloidal Reynolds stress:** $\bar{\omega} \equiv \omega - \langle u_y \rangle k_y$

$$\langle \tilde{u}_x \tilde{u}_y \rangle = -D_{PV} \frac{\partial}{\partial x} \langle u_y \rangle + F_{res} \kappa \langle p \rangle$$

Residual Stress   Curvature

Suppressed by stochastic fields

$$D_{PV} = \sum_{k\omega} |\tilde{u}_{x,k\omega}|^2 \frac{v_A b^2 l_{ac} k^2}{\bar{\omega}^2 + \left(v_A b^2 l_{ac} k^2 \right)^2}$$

Reynolds stress will be suppressed as stochastic fields via PV diffusivity and residual stress.

This stochastic dephasing is insensitive to turbulent mode (e.g. ITG, TEM,…etc.).

Decoherence of eddy tilting feedback — the physics

- **Snell's law:**

$$\frac{d}{dt}k_x = -\frac{\partial \omega_k}{\partial x} = -k_y\frac{\partial u_y}{\partial x}$$ shear flow

Leads to non-zero $\langle k_x k_y \rangle$
$\rightarrow \langle \tilde{u}_x \tilde{u}_y \rangle \propto \langle k_x k_y \rangle$

- **Self-feedback of Reynolds stress:**

$$\langle \tilde{u}_x \tilde{u}_y \rangle \simeq -\sum_k \frac{|\tilde{\phi}_k|^2}{B_0^2}\left(k_y^2 \frac{\partial u_y}{\partial x}\tau_c\right)$$

The $E \times B$ shear generates the $\langle k_x k_y \rangle$ correlation and hence support the non-zero Reynolds stress.

The Reynold stress modifies the shear via momentum transport.

➤ **The shear flow reenforce the self-tilting.**

- **Now, the dispersion relation with drift-Alfvén coupling is:**

$$\omega^2 - \omega_D \omega - k_\parallel^2 v_A^2 = 0$$

$$k_\parallel = k_\parallel^{(0)} + \underline{b}\cdot \underline{k}_\perp$$

Drift-wave frequency

$$\omega = \omega_D + \delta\omega$$

Frequency shift induced by b^2

$$(\omega_D + \delta\omega)^2 - \omega_D(\omega_D + \delta\omega) - (k_\parallel + \underline{b}\cdot \underline{k}_\perp)^2 v_A^2 = 0$$

Decoherence of eddy tilting feedback — the physics

- **Stochastic fields dephase the self-feedback loop of Reynolds stress:**

Expectation of frequency in stochastic fields: $\langle \omega \rangle = \langle \omega_0 \rangle + \langle \delta\omega \rangle$.

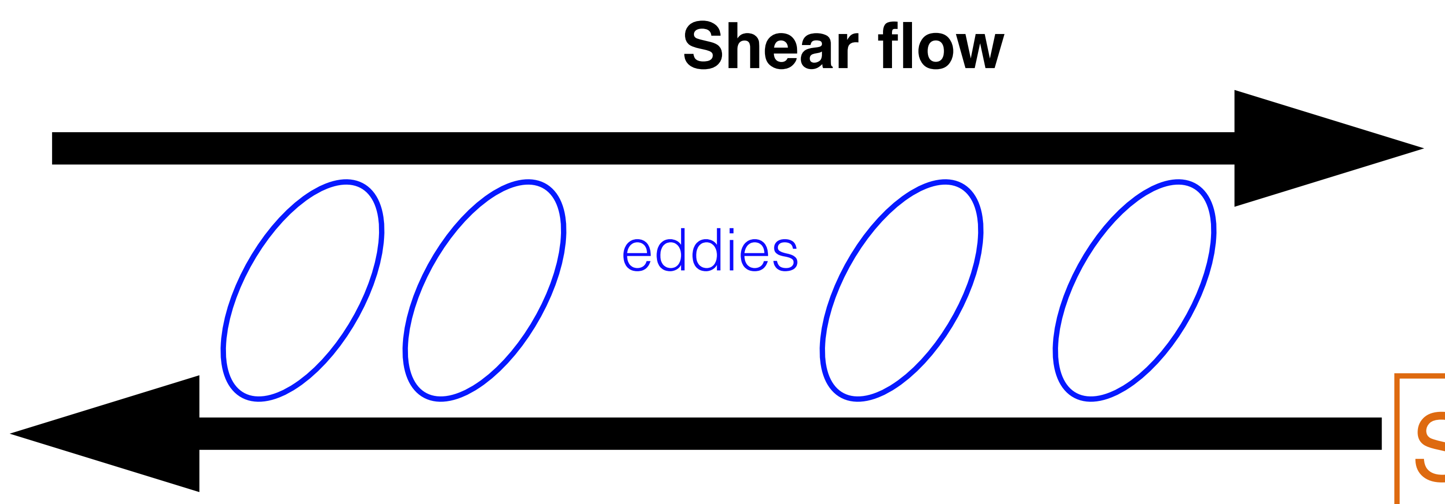
$$\langle \omega \rangle \simeq \omega_D + \frac{1}{2}\frac{v_A^2}{\omega_0} b^2 k_\perp^2$$

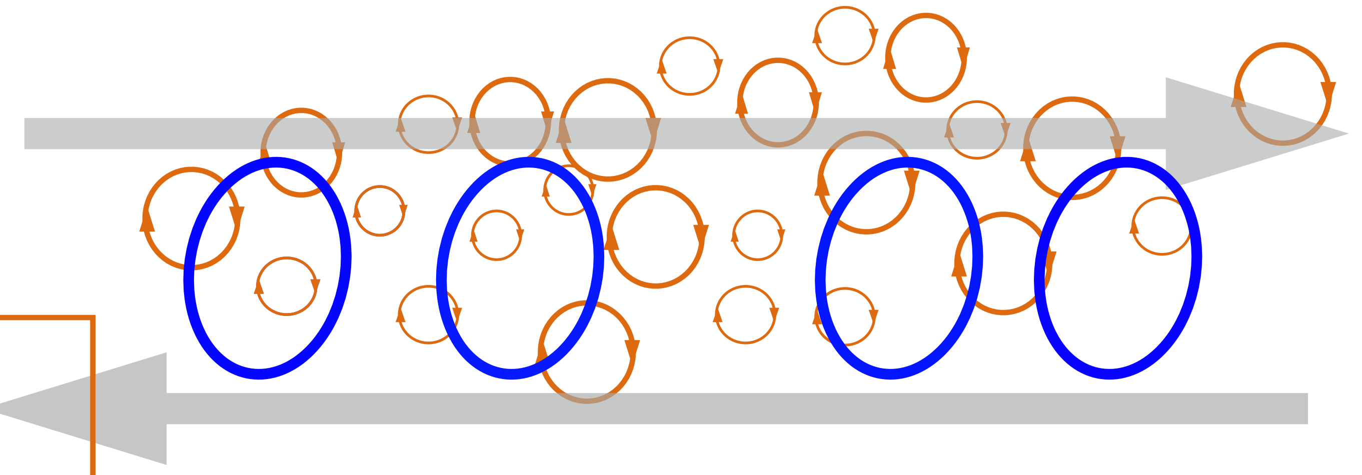
← **Ensemble average frequency shift**

$$\langle \tilde{u}_x \tilde{u}_y \rangle \simeq -\sum_k \frac{|\tilde{\phi}_k|^2}{B_0^2} \left(k_y^2 \frac{\partial u_y}{\partial x}\tau_c - \frac{1}{2}\frac{v_A^2 k_\perp^2}{\omega_0}\frac{\partial b^2}{\partial x}\tau_c\right)$$

← **Stochastic dephasing**

When these two are comparable, the feedback loop will be broken.





Stochastic fields act as elastic loops and resist the tilting of eddies.

Stochastic fields interfere with the shear-tilting feedback loop.

Results — Increment of P_{LH}

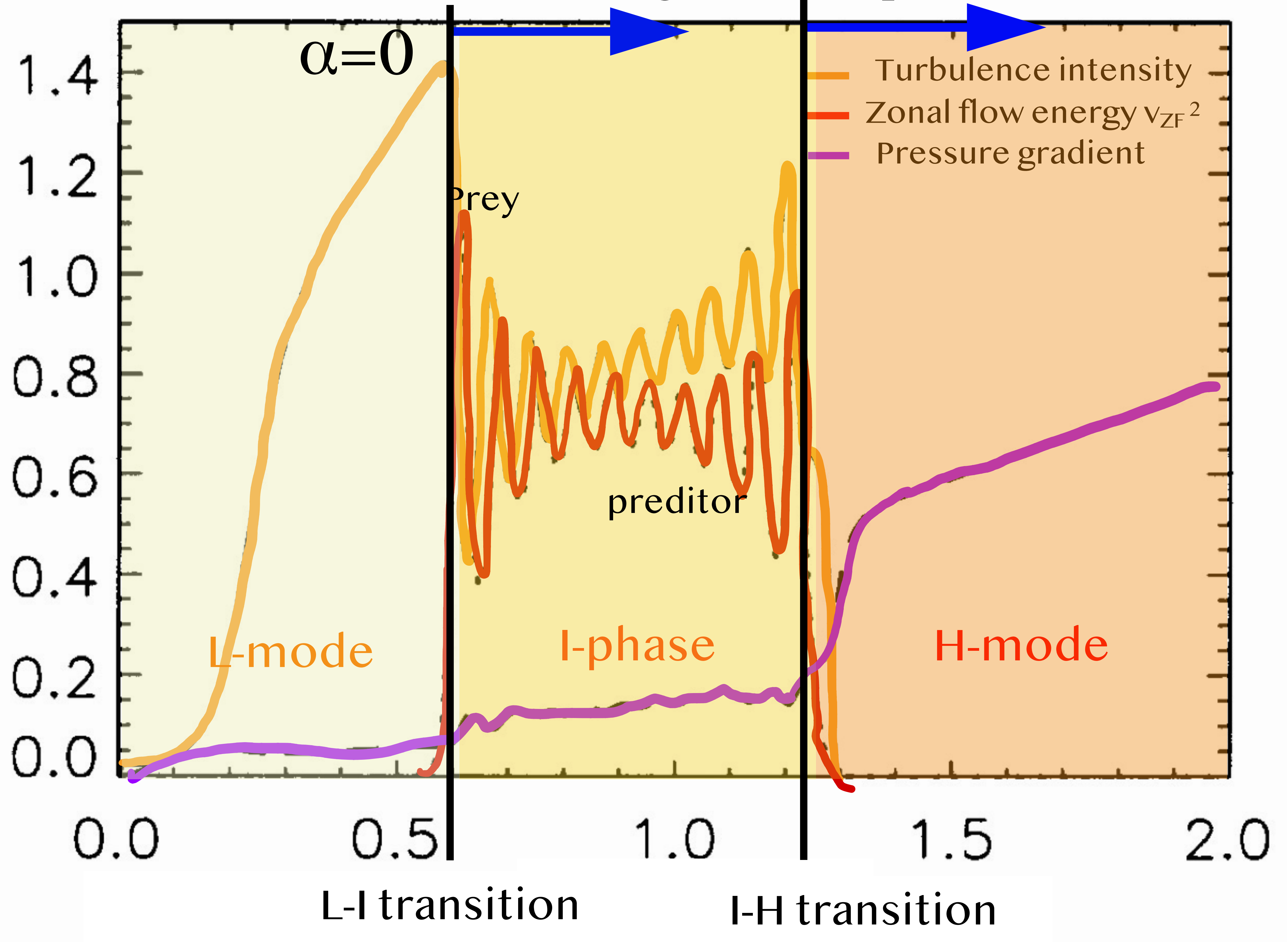
Macroscopic Impact

- **Extended Kim-Diamond Model (Simple reduced model):**

  Stochastic fields broadening effect requires: $\Delta\omega \le k_\perp^2 D$. This gives dimensionless parameter (α): $\alpha \equiv \frac{b^2}{\sqrt{\beta}\rho_*^2}\frac{q}{\epsilon} > 1$

- **1D Theory of power threshold:** M. A. Malkov et al. (PoP **22**, 032506 (2015)).

  Kim-Diamond model is useful for testing trends in power threshold increment induced by stochastic fields.

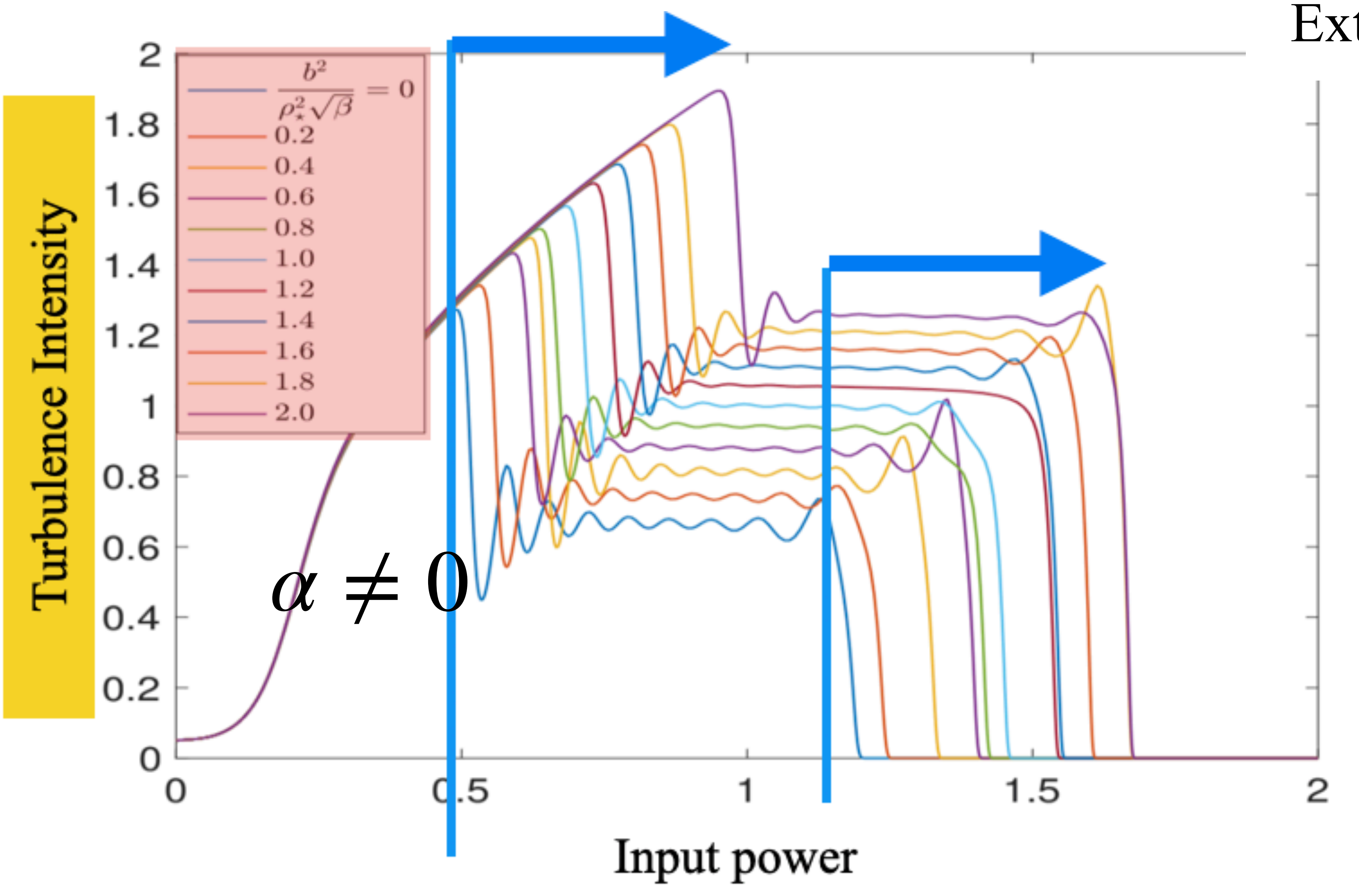


➤ **We expect stochastic fields to raise L-I and I-H transition thresholds.**

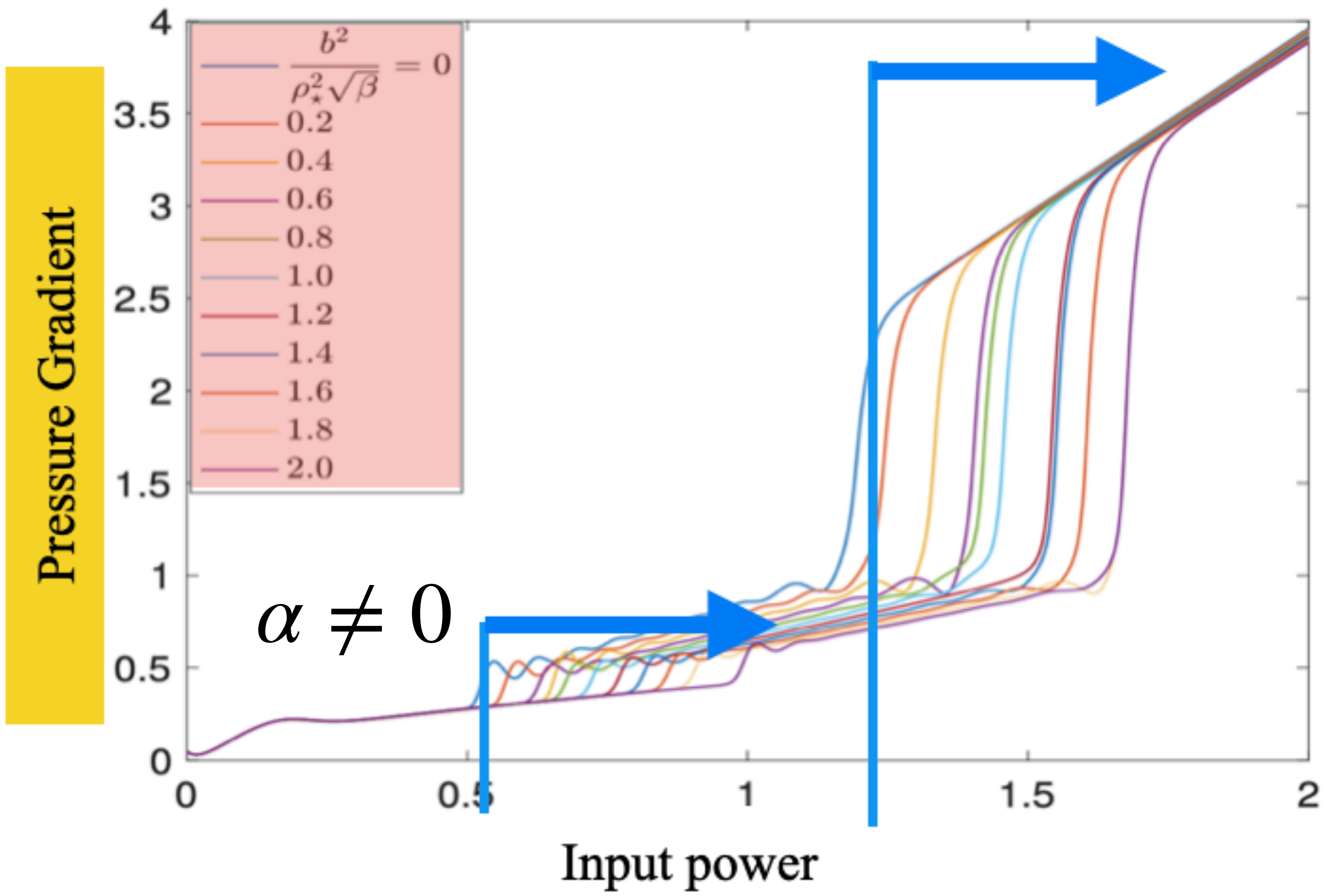
$$\alpha \equiv \frac{b^2}{\sqrt{\beta}\rho_*^2}\frac{q}{\epsilon}$$

quantifies the strength of stochastic dephasing.

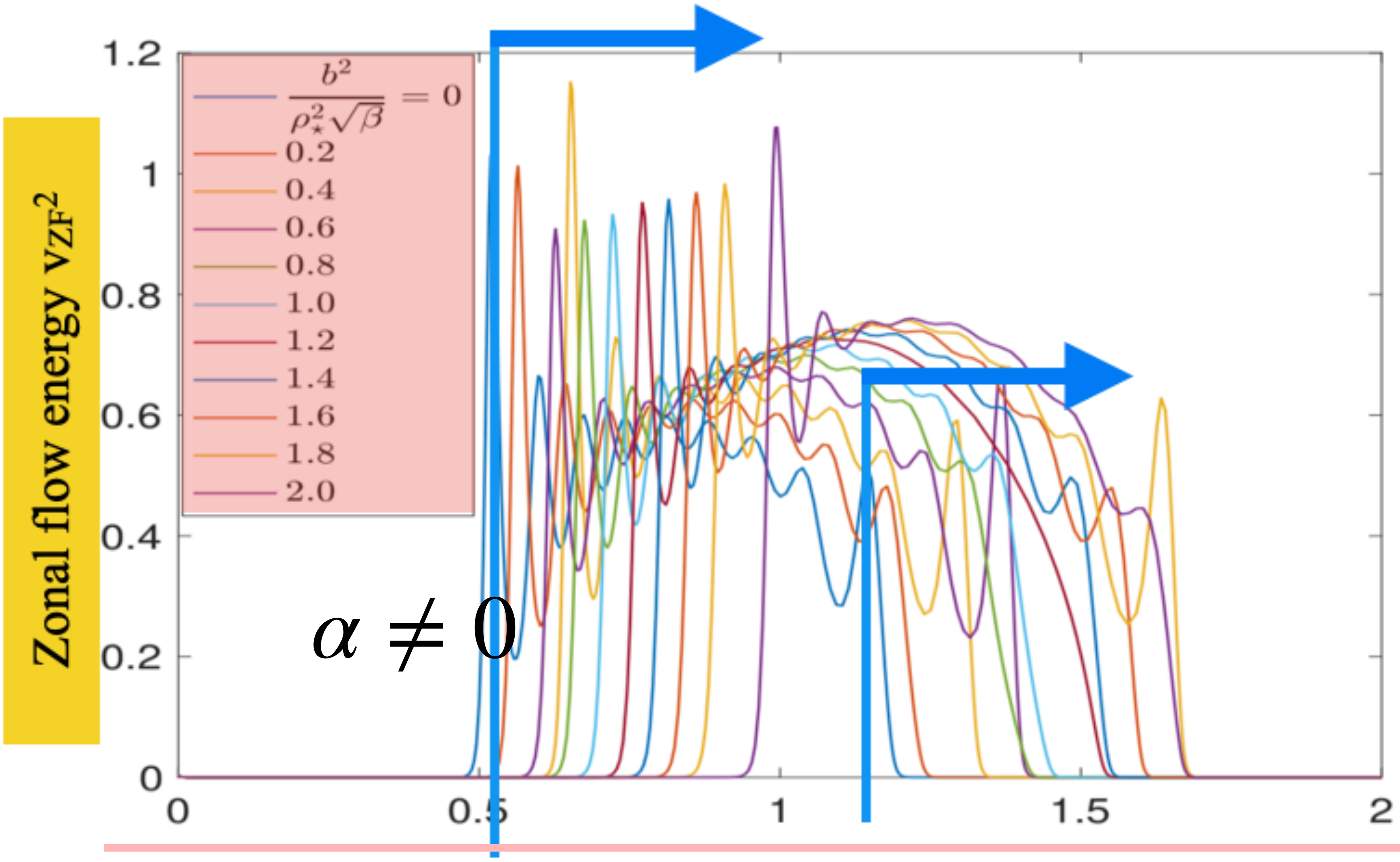
Results — Increment of P_{LH}



Extended Kim-Diamond Model with b^2



$$\alpha \equiv \frac{b^2}{\sqrt{\beta}\rho_*^2}\frac{q}{\epsilon}$$

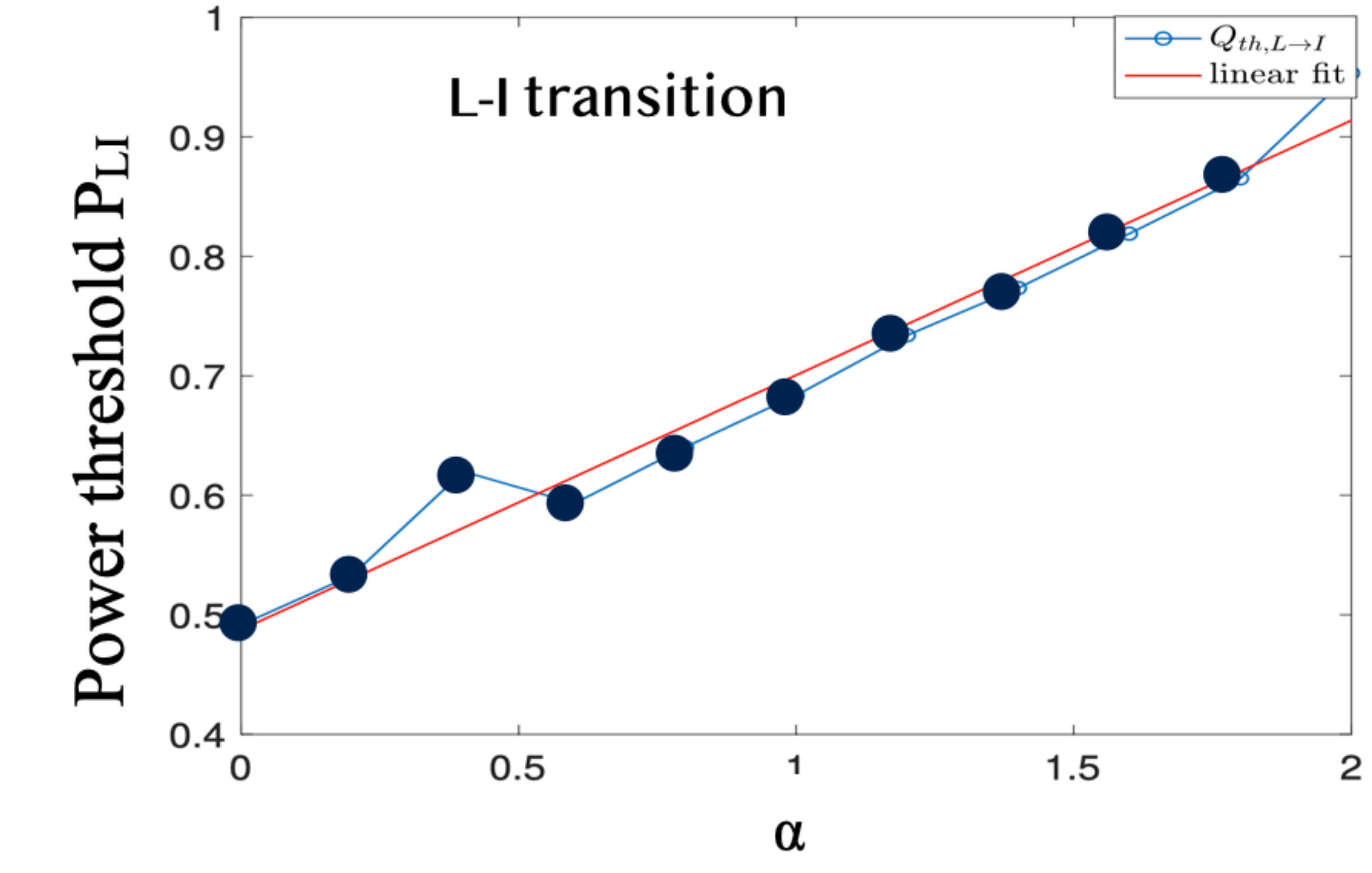


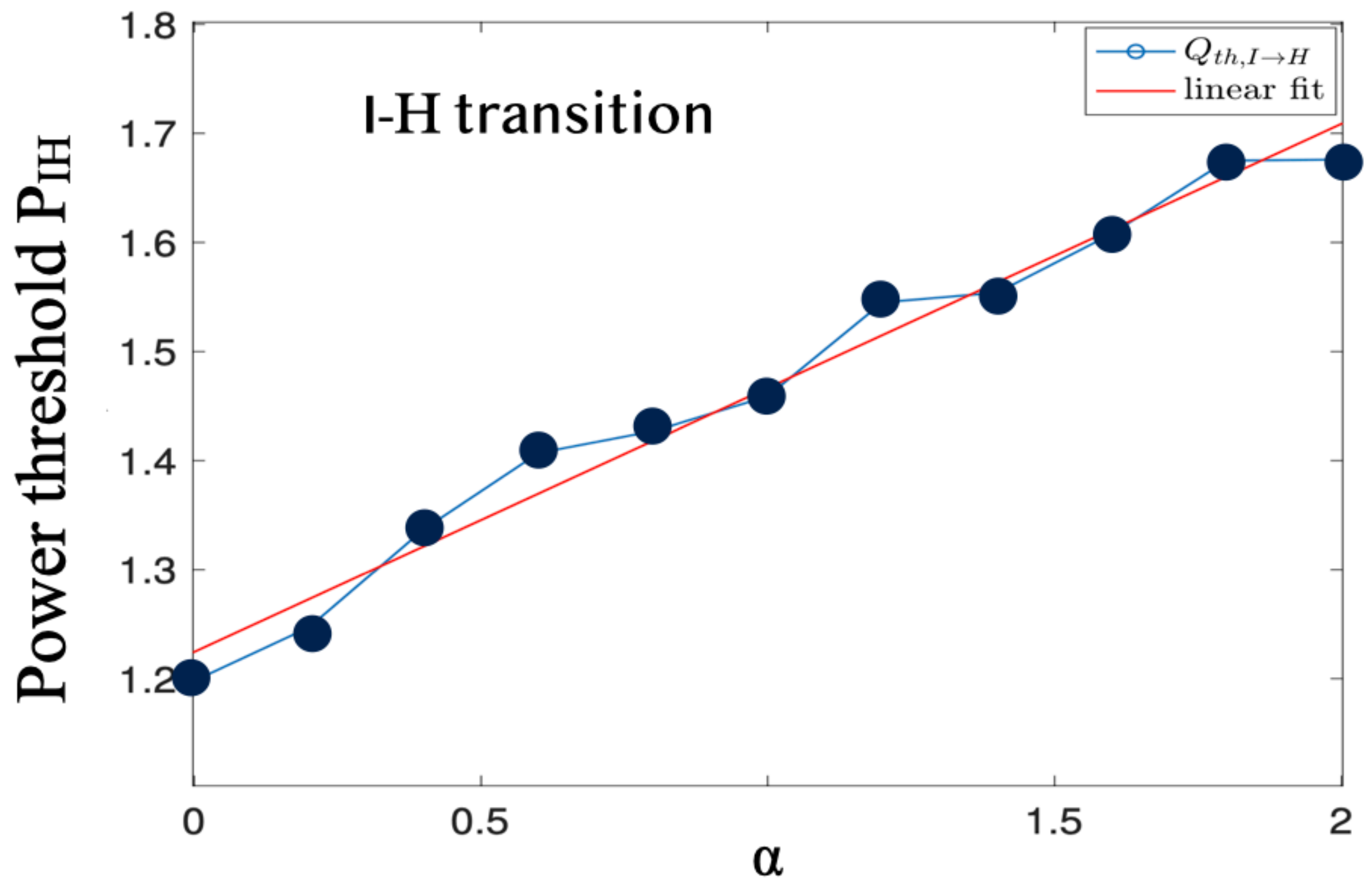
The threshold increase due to stochastic dephasing effect is seen in turbulence intensity, zonal flow, and pressure gradient.

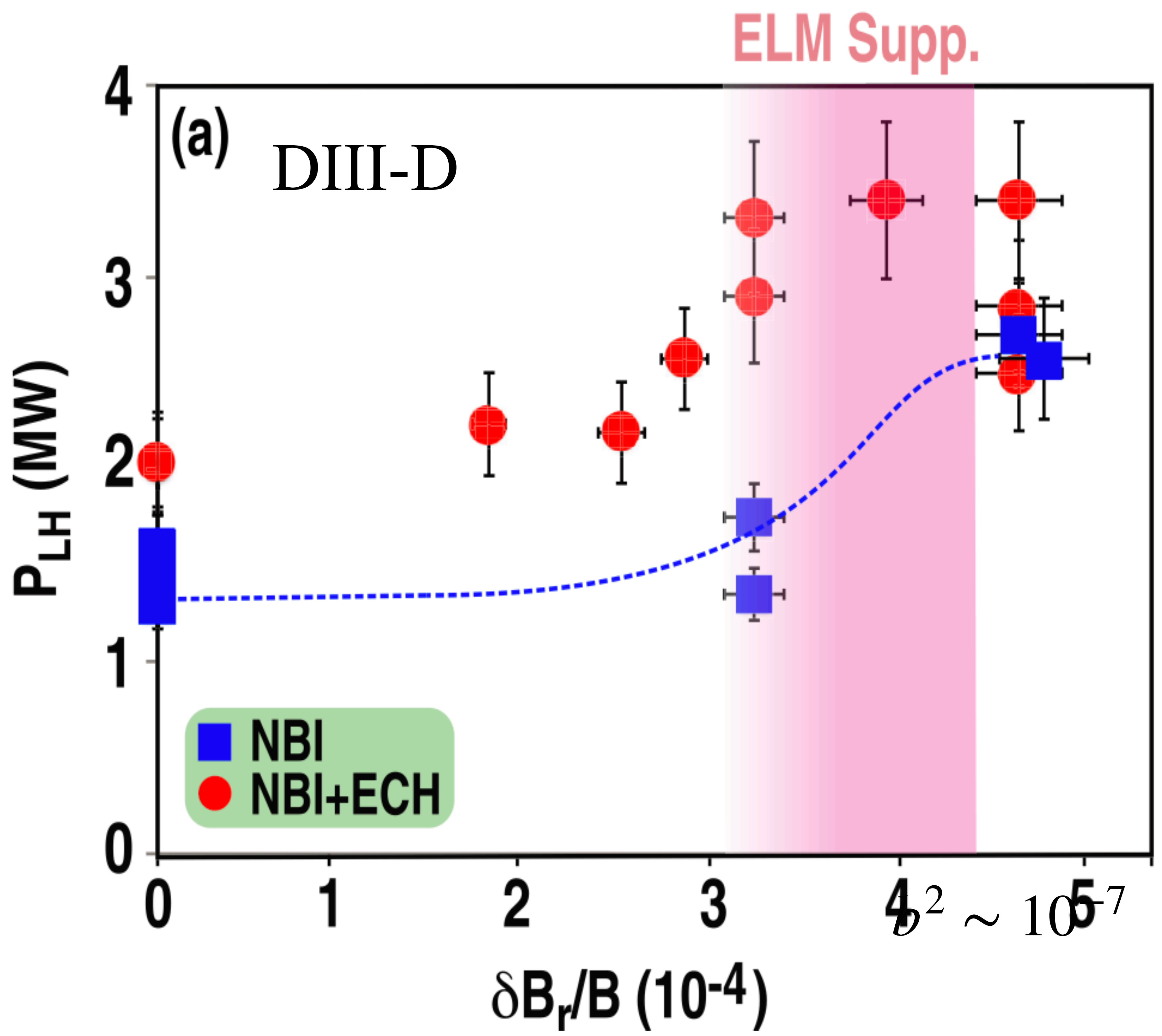
Results — Increment of P_{LH}

Increment of Power threshold:

The power threshold increases linearly with the increment of stochastic fields intensity $\alpha \equiv \frac{b^2}{\sqrt{\beta}\rho_*^2}\frac{q}{\epsilon}$.







(L. Schmitz et al, NF **59** 126010 (2019))

$\overline{b^2}$ **shift L-H, I-H thresholds to higher power, in proportional to α.**

Outline

1. Introduction

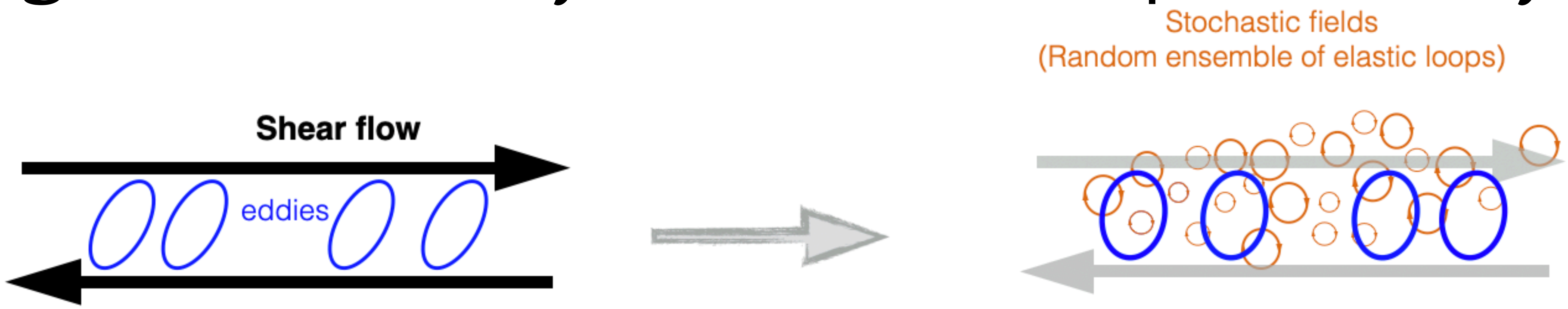
2. Solar Tachocline

3. L-H transition in tokamak

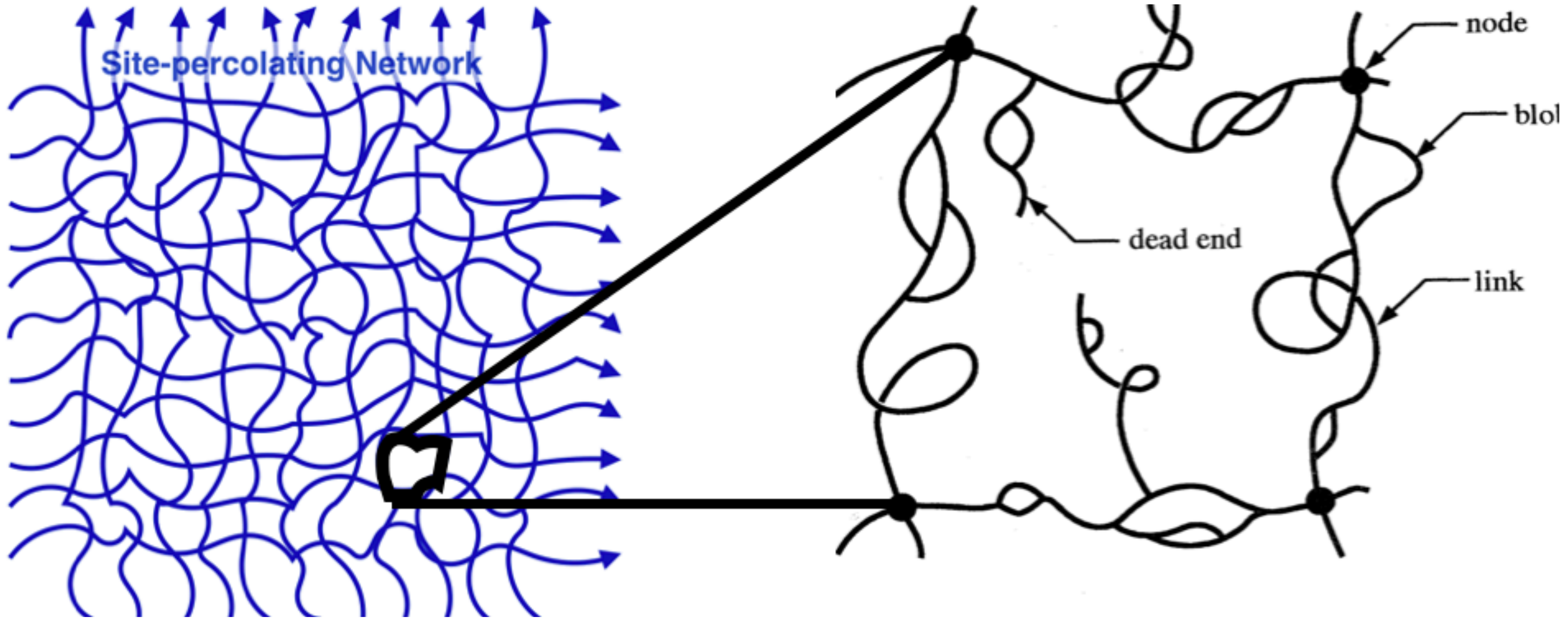
4. **Conclusion**

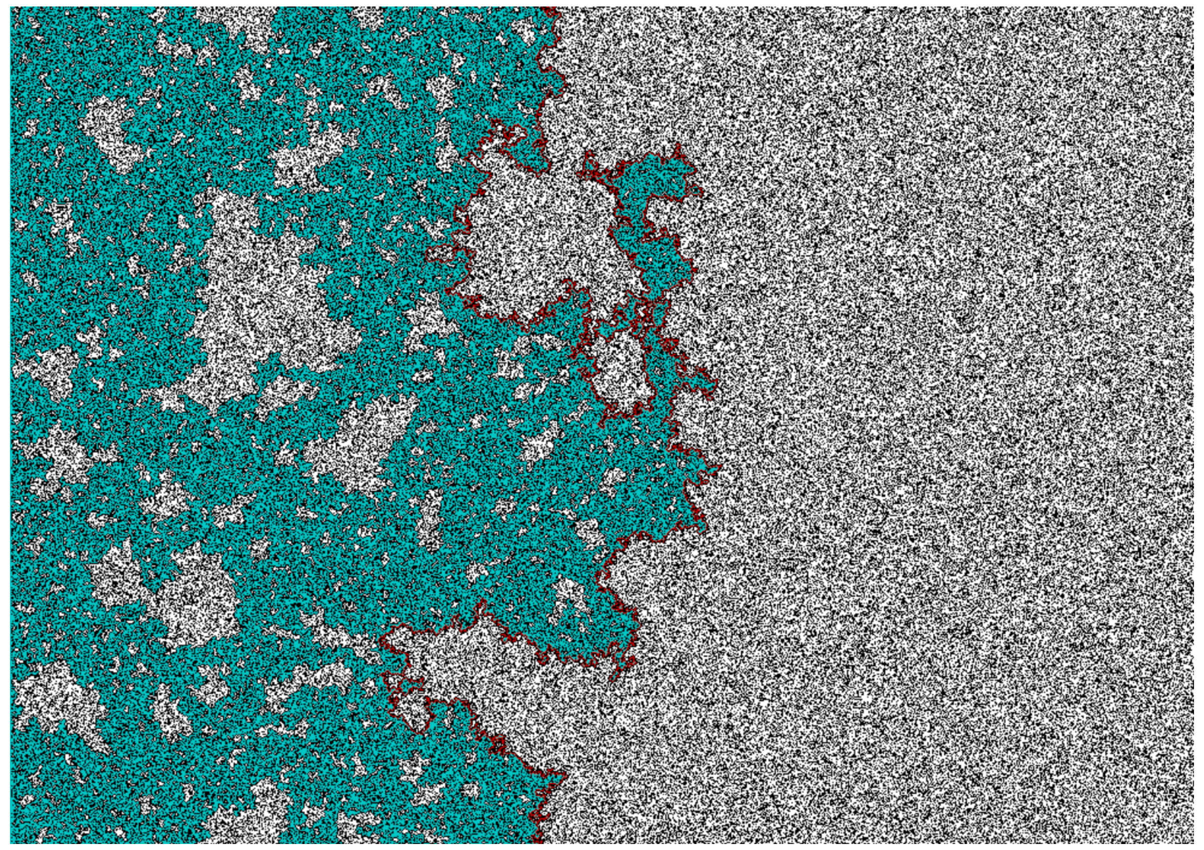
Conclusion — General Ideas

- **Dephasing effect** caused by stochastic fields quenches Reynolds stress.



- Stochastic fields can form a **fractal, elastic network**. Strong coupling of flow turbulence to the fractal network **prevents** PV mixing and hence zonal flow formation.





PV dynamics with a tangled field relevant, is a broadly applicable paradigm!

Thank you!