Thesis defense:
“Topics in mesoscopic turbulent transport”

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Introduction

Three projects on plasma turbulence. Unifying feature: interaction of turbulent microscales \[\Rightarrow\] meso-/macro-scale transport

1. Use machine learning to find reduced model for particle/momentum transport in drift-wave turbulence
2. New model for turbulence spreading and avalanching
3. Study relationship between cross-helicity and momentum transport in $\beta$-plane MHD
Background: drift wave turbulence
Tokamak physics basics

- Toroidal fusion device that uses strong helical magnetic field to confine plasma
- Key challenge: \( \langle n \rangle \langle T \rangle \tau_E > 10^{21} \text{ keV s/m}^3 \) (Lawson criterion) → maximize confinement time \( \tau_E \) → minimize losses due to transport
- But: \( n, T \) gradients → instabilities → turbulence → anomalous transport. How to understand?
Drift waves

- Drift wave turbulence is useful paradigm for turbulence due to gradient instabilities (universal)
- Drift wave: collective oscillations associated with ion/electron diamagnetic drifts, which form in response to temperature/density gradients \( v_d = 1/(qnB^2) \nabla p \times B \)
- Structure: cell convecting around \( \tilde{n} \) at \( v_E = -c/B^2 \nabla \phi \times B \), traveling at \( v_d \)

force balance \( q(E + \mathbf{v} \times B) = \nabla p / n \)

FIG. 1. Drift-wave mechanism showing \( E \times B \) convection in a nonuniform, magnetized plasma.
Drift wave turbulence

- $\tilde{n}$ coupled tightly to $\tilde{\phi}$ by fast parallel “Boltzmann” electron response (from force balance $n_e e \partial_z \tilde{\phi} = T_e \partial_z n_e$)

$$n_e \approx n_0 \exp(e \tilde{\phi}/T_e) \rightarrow \tilde{n}/n_0 \approx e\tilde{\phi}/T_e$$

- Collisions and resonances $\rightarrow$ phase shift $\tilde{n}_k/n_0 \approx e\tilde{\phi}_k/T_e(1 - i\delta_k) \rightarrow$ instability!

- Turbulence results when many drift modes unstable, nonlin. interaction becomes important

FIG. 1. Drift-wave mechanism showing $E \times B$ convection in a nonuniform, magnetized plasma.
Zonal flows

- Special modes with $m = n = 0, \omega \approx 0$. Turbulence-driven, sheared poloidal flows
- In certain regime, spontaneously build up via secondary instability (multiscale interaction)
- No radial flow → do not cause harmful transport. “benign” free energy repository
- ZF shear stretches turbulent eddies → regulate turbulence
- Extremely important for confinement problem: zonal flows induce L-H transition

Figure ZFs also important in geophysical flows
Hasegawa-Wakatani model

- Simplest realistic framework for understanding collisional drift wave/zonal flow system.
- Coupled dynamics for potential $\phi$, electron density $n$ (dimensionless units):

$$\begin{align*}
\frac{dn}{dt} &= \alpha(\phi - n) + D\nabla^2 n \\
\frac{d\nabla^2 \phi}{dt} &= \alpha(\phi - n) + \mu \nabla^4 \phi \\
\frac{d}{dt} &\equiv \frac{\partial}{\partial t} + (\hat{z} \times \nabla \phi) \cdot \nabla
\end{align*}$$

- $\alpha \equiv k^2 T_e / (n_0 \eta \Omega_i e^2)$ “adiabaticity parameter,” measures parallel electron response
- $\phi$ is stream function for flow $\mathbf{v}$
Motivation: mean-field Hasegawa-Wakatani

- Want theory for radial transport
- Averaging over symmetry directions ($\langle \cdots \rangle$) yields

\[
\begin{align*}
\partial_t \langle n \rangle + \partial_x \Gamma &= \text{dissipation} \\
\partial_t \langle \nabla^2 \phi \rangle - \partial_x^2 \Pi &= \text{dissipation} \\
\partial_t \varepsilon + 2\varepsilon (\Gamma - \partial_x \Pi) (\partial_x \langle n \rangle + \partial_x^3 \langle \phi \rangle) &= -\gamma \varepsilon - \gamma_{NL} \varepsilon^2
\end{align*}
\]

where $\Gamma = \langle \tilde{n} \tilde{v}_x \rangle$ (particle flux) and $\Pi = \langle \tilde{v}_x \tilde{v}_y \rangle$ (poloidal momentum flux or "Reynolds stress").

- $\varepsilon = \langle (\tilde{n} - \nabla^2 \tilde{\phi})^2 \rangle$ is turbulent potential enstrophy. Proxy for turbulence intensity

- Seek mean-field closure: $\Gamma, \Pi$ as function of $\langle n \rangle, \langle \phi \rangle, \varepsilon$, radial derivatives. Idea: use supervised learning. Can we do better than simple QLT?
Feature selection: what do we want our model to look like?

- Assume a **local** model: local mean fields (in space and time) suffice to specify the local fluxes
- HW invariant under uniform shifts $n \rightarrow n + n_0$ and $\phi \rightarrow \phi + \phi_0 \implies$ eliminate dependence on $\langle n \rangle, \langle \phi \rangle$
- Invariant under poloidal boosts

$$
\begin{align*}
\phi & \rightarrow \phi + v_0 x \\
y & \rightarrow y - v_0 t
\end{align*}
$$

$\implies$ eliminates dependence on ZF speed $V_y = -\partial_x \langle \phi \rangle$.

- Confine ourselves to adiabatic regime so $\tilde{n} \sim \tilde{\phi} \implies \varepsilon$ reasonably suffices to specify intensity
- Anticipate that hyperviscosity necessary to regularize ZF, so need derivatives up to $V''_y$
Methods

- Thus choose minimal set of inputs $N', U, U', U'', \varepsilon (N = \langle n \rangle, U = V'_y)$
- 32 simulations of 2D HW, with $\alpha = 2$, various initial conditions for mean density, flow
- Postprocess to compute inputs, $\Gamma$, $\Pi$. Key: locality means each point in space, time treated on equal footing $\rightarrow$ lots of data per simulation
- Train neural network to output fluxes as functions of inputs
- Exploit/enforce 3 reflection symmetries via data augmentation

Figure Schematic of deep learning method
Neural networks 101

- **Bottom line:** simply a proven form of nonparametric, multivariate regression
- Use simplest form (multilayer perceptron)
- Inputs \( x \) repeatedly transformed
  \[ x_j^{(n+1)} = \sigma(W_{ij}^{(n)}x_i^{(n)} + b_j) \]
  where \( \sigma \) is a nonlinear function ("activation")
- Weights \( W^{(n)} \), biases \( b \) are "trained" using sophisticated algorithm to minimize loss function which measures deviation from labeled samples

**Figure** Diagram of MLP, shamelessly stolen from the internet
Particle flux results

DNN learns a model roughly of the form (for small gradients)

$$\Gamma \simeq -D_n \varepsilon N' + D_U \varepsilon U'$$

Diffusive term $\propto N'$ is well-known, tends relax driving gradient. Second (non-diffusive) term not well-known, driven by vorticity gradient!

**Figure** Particle flux at constant $\varepsilon$ as function of density and vorticity gradients
Derivation of nondiffusive term

$\alpha \to \infty$ calculation reproduces nondiffusive term. Need include frequency shift due to ZF! (quasilinear treatment, i.e. flux assumed due to coherent unstable drift waves)

$$\omega_k = \frac{k_y}{1 + k^2} (N' + U') + O(\alpha^{-2})$$

$$\gamma_k = \frac{k_y^2}{\alpha (1 + k^2)^3} (N' + U') (k^2 N' - U') + O(\alpha^{-2})$$

$$\Gamma = \text{Re} \sum_k -ik_y \tilde{n}_k \tilde{\phi}_k^*$$

$$= \sum_k -k_y^2 \partial_x n (\gamma_k + \alpha) + \alpha k_y \omega_{r,k} \left| \tilde{\phi}_k \right|^2$$

$$= \sum_k \frac{k_y^2}{\omega_{r,k}^2 + (\gamma_k + \alpha)^2}$$

$$= \frac{1}{\alpha} \sum_k -\frac{k_y^2}{1 + k^2} (k^2 N' - U') \left| \tilde{\phi}_k \right|^2 + O(\alpha^{-2})$$
Comparison to theory (diffusive term)

Compare DNN result to theory result using spectrum centered at most unstable $k$ for $U' = 0$

$$\varepsilon_k = \frac{\langle \varepsilon \rangle}{2\pi^2 \Delta k_x \Delta k_y} \frac{1}{1 + k_x^2 / \Delta k_x^2} \left( \frac{1}{1 + (k_y - \sqrt{2})^2 / \Delta k_y^2} + \frac{1}{1 + (k_y + \sqrt{2})^2 / \Delta k_y^2} \right)$$

**Figure** Curves (at fixed $U = U' = U'' = 0$, and various $\varepsilon$) of $\Gamma$ vs density gradient from DNN

**Figure** Corresponding curves from QLT+ansatz with $\Delta k_x = \Delta k_y = 0.8$
Comparison to theory (nondiffusive term)

**Figure** Curves (at fixed $N' = U = U'' = 0$, and various $\varepsilon$) of $\Gamma$ vs $U'$ from DNN

Good agreement when $\partial_x n, \partial_x U$ are small!

**Figure** Corresponding curves from QLT+ansatz with $\Delta k_x = \Delta k_y = 0.8$
Implications of nondiffusive term

- Neglected in literature, but coupling same order of magnitude ($\sim 0.5$) that of usual $N'$ term. Stronger than coupling to shear!

- Consequence: ZF can induce “staircase” pattern on profile. If $V_y = V_0 \sin(qx)$, $U'$ term will contribute

$$\partial_t \langle n \rangle \sim - \frac{k_y^2 q^3 V_0 \langle \varepsilon \rangle}{\alpha(1 + k^2)^3} \cos(qx)$$

- Previous explanation for staircase is some form of bistability. This mechanism is distinct.

**Figure** Cartoon indicating how ZF may induce profile staircase via nondiffusive flux/pinch
Reynolds stress results

- Learns model of (Cahn-Hilliard) form (leading order)

\[ \Pi = \varepsilon (-\chi_1 U + \chi_3 U^3 - \chi_4 \partial_x^2 U) \]

with \( \chi_1, \chi_3, \chi_4 > 0 \)

- \( \partial_t U = \partial_x^2 \Pi \sim \chi_1 \varepsilon k^2 U \). Zonal flow generation by negative viscosity \( \varepsilon \chi_1 \)

- Large \( U \) stabilized by nonlinearity \( \propto U^3 \), small scales by hyperviscosity \( \chi_4 \) (not shown)

- Agrees with previous theoretical models for zonal flow generation

- Recovery of hyperviscous is sensitive test of method

Figure Reynolds stress as function of \( U \), at fixed \( U', U'' \)
Reynolds stress: gradient corrections

- Learned dependence well-described by overall suppression factor $f \approx \frac{1}{1 + 0.04(N' + 4U')^2}$, i.e. gradients generally reduce Reynolds stress.
- Found to be crucial for stability of learned model. Kinks tend to form in flow in its absence.

![Figure](image.png) Reynolds stress dependence on gradients at fixed $\varepsilon, U, U''$
Numerical solution of reduced 1-D model

Choose analytical expressions to match deep learning results, solve using implicit scheme
Conclusions

- ML recovers CH theory for ZF generation, while finding nontrivial gradient corrections
- Highlights rarely-discussed coupling of profile to flow, which induces profile layering
- Were confined to single adiabatic \( \alpha, N' \lesssim 3 \). Otherwise, vortex interactions \( \rightarrow 1D \) model doesn’t make sense
- Test of concept for more complex applications. Geometry? 3D? \( T, B \) coupling?
- May need to relax some assumptions: multiple intensities? Spatial and/or temporal nonlocality?
- Tradeoff b/t complexity and interpretability
- Spreading???
Introduction

- Turbulence spreading = radial self-propagation of turbulence. Important in DWT
- Nonlinear coupling of microscales to mesoscopic envelope scale. Closure of $E \times B$ with envelope:

$$\partial_t \varepsilon_k \sim - \sum_{k'} (k \cdot k' \times \hat{z})^2 |\tilde{\phi}_{k'}|^2 R(k, k') l_k \rightarrow \frac{\partial}{\partial x} D_x(l_k) \frac{\partial}{\partial x} l_k - kk : D l_k$$

$$D_x = \sum_{k'} k'_y^2 |\phi_{k'}|^2 R(k, k')$$

- Decouples flux-gradient relation: local turbulence intensity now depends on global properties of the profiles
- Fluctuations in linearly stable regions!
Figure Spatiotemporal evolution of flux-surface-averaged turbulence intensity in toroidal GK simulation. Linearly unstable region is $0.42 < r < 0.76$. From [Wang et al., 2006]
Avalanches

- Fast, intermittent transport events. Can account for a large percentage of total flux!
- Initially localized fluctuation cascades through neighboring cells via gradient coupling. Cell microscales couple with mesoscopic avalanche scale
- Associated with profile relaxation, SOC
- Closely related to spreading: both result in fast, mesoscopic turb front propagation. Unified model?

Figure Heat flux spectrum from GK simulation showing $1/f$ scaling
Conventional wisdom for spreading is Fisher-type equation for turbulence intensity:

\[
\frac{\partial_t I}{I} = \gamma_0 I - \gamma_{nl} I^2 + \partial_x \left( D_0 I \partial_x I \right)
\]

- \( \gamma_0 I \) local linear growth/decay
- \( -\gamma_{nl} I^2 \) local nonlinear coupling to dissipation
- \( \partial_x \left( D_0 I \partial_x I \right) \) nonlinear diffusion of turbulence energy

For \( \gamma_0 > 0 \), dynamics characterized by traveling fronts connecting unstable “laminar root” \( I = 0 \) and saturated “turbulent root” \( I = \frac{\gamma_0}{\gamma_{nl}} \) with speed \( c = \sqrt{\frac{D_0 \gamma_0^2}{2 \gamma_{nl}}} \)
Depiction of Fisher evolution

**Figure** Evolution of traveling turbulence front in Fisher model. From [Gürcan and Diamond, 2006]
Problems with Fisher

- Weak spreading into stable zone (few $\Delta_c$). Dubiously consistent with experiment?

- If unstable, *why didn’t noise already excite the whole system to turbulence?*

- Unless $\Delta x^2 \gamma_{nl} \ll D_0$, physical fronts require *bistability à la* [Pomeau, 1986]

- Growing body of evidence for bistable MF turbulence e.g. [Biskamp and Walter, 1985, Drake et al., 1995, Barnes et al., 2011, van Wyk et al., 2016]

**Figure** Experiment by Nazikian et al. 2005 clearly showing fluctuations in stable zone
Bistable model

- Propose phenomenological model of form

\[ \partial_t I = \gamma_1 I + \gamma_2 I^2 - \gamma_3 I^3 + \partial_x(D(I)\partial_x I) \]

- take \( D(I) = D_0 I \)

- New physics: nonlinear turbulence drive \( \propto I^2 \). Can sustain sufficiently large fluctuations even when linearly damped

- Bistable in weak damping regime

- Estimate \( \gamma_1 \sim \epsilon \omega_* \), \( \gamma_{2,3} \sim \omega_* \), \( D_0 \sim \chi_{GB} \)
Model analysis I

\[ \partial_t l = \gamma_1 l + \gamma_2 l^2 - \gamma_3 l^3 + \partial_x (D(l) \partial_x l) \]

- Qualitatively similar to Fisher EXCEPT in weak damping case
  - \( \gamma_1 < 0 \) and \( \gamma_2^2 > 4|\gamma_1|\gamma_3 \)
- Can then transform to Zel’dovich/Nagumo equation
  \[ \partial_t l = f(l) + \partial_x (Dl \partial_x l) \]
  \[ f(l) \equiv \gamma l(l - \alpha)(1 - l) \]

where \( \alpha \equiv l_- / l_+ \), \( \gamma \equiv l_+^2 \gamma_3 \), \( D \equiv l_+ D_0 \), \( l_\pm \equiv (\gamma_2 \pm \sqrt{\gamma_2^2 - 4|\gamma_1|\gamma_3})/2\gamma_3 \)
Unlike Fisher, traveling fronts admitted in weak damping case!

- Propagation speed $c \sim \sqrt{D\gamma}$ (depends on $\alpha$), characteristic scale $\ell \sim \sqrt{D/\gamma}$
- “Maxwell construction” for speed

\[
c \int_{-\infty}^{\infty} D(I(z))I'(z)^2 \, dz = \int_{0}^{1} D(I)f(I) \, dl
\]

$z = x - ct$

- Thus turbulence spreads if $\alpha < \alpha^*$, recedes if $\alpha > \alpha^*$. Also corresponds to (meta)stability of fixed points (Lyapunov functional)
Penetration into stable zone

- Fisher model: evanescent penetration, depth $\ell \sim \rho_s$
- Our model: new front with reduced speed/amplitude forms in second region if weakly damped (i.e. $\gamma_d$ is small enough that $\alpha < \alpha^*$)
- Hence: can have ballistic propagation even in stable zone!
  Much stronger penetration, delocalization
Figure Spreading into stable zone in GK simulation with magnetic shear [Yi et al., 2014]. Ballistic propagation???
Avalanche threshold I

- In contrast to Fisher, sufficiently large localized puff of turbulence will grow into front and spread. Suggestive of an avalanche triggered by initial seed

- How to determine threshold?

Two puffs differing only in spatial size are initialized; one grows and spreads, other collapses
Avalanche threshold II

- Obviously puff amplitude must exceed $l_0 = \alpha$ or else $\gamma_{eff} = (l - \alpha)(1 - l) < 0$
- Consider “cap” of puff (part exceeding $l = \alpha$)
- Size threshold governed by competition between diffusion of turbulence out of cap and total nonlinear growth in cap
- Sets scale $\sqrt{D/\gamma}$. Can derive $L_{min} \sim (l_0 - \alpha)^{-1/2}$
Avalanche threshold: analytical vs. simulation

Figure Numerical result for threshold at $\alpha = 0.3$ for three types of initial condition (Gaussian ($I_1$), Lorentzian ($I_2$), parabola ($I_3$)), compared with analytical estimate.
Bistable model: conclusions

- Bistable model rectifies issues with Fisher, is supported by evidence for subcritical turbulence.
- Provides simple framework for understanding avalanching: local, intermittent exceedance of nonlinear instability by turbulent puffs. Threshold weak near marginal → triggered by noise?
- Key testable predictions: ballistic spreading into weakly linearly damped regions, power-law threshold for spreading of puffs.
Note on experiments

- [Van Compernolle et al., 2015] created avalanches in experiment by locally perturbing plasma with source, measuring spatiotemporal response.
- Suggest testing avalanche threshold in similar manner. How intense/large must source be?
- [Inagaki et al., 2013]: purported hysteresis between fluctuation intensity and driving gradient (no TB present).
- But if bistable, why does intensity relax after source turned off?
- Suggest more experiments à la Inagaki to investigate bistability.

Figure: Hysteresis between intensity and gradient, flux and gradient.
Beta-plane MHD project
Solar tachocline

- Thin, radially-sheared layer at base of convection zone. Strongly turbulent
- Believed to be strongly involved in the solar dynamo
- Home to $\Omega$-effect: shear drags poloidal field lines originating from core, converts to strong toroidal field
Strong stratification in tachocline $\Longrightarrow$ quasi-2D

2D magnetized incompressible turbulence in presence of planetary vorticity (Coriolis force) gradient:

$$2\Omega = (0, 0, f + \beta y)$$

\[
\begin{align*}
\partial_t \nabla^2 \psi + \beta \partial_x \psi &= \{\psi, \nabla^2 \psi\} - \{A, \nabla^2 A\} + \nu \nabla^4 \phi + \tilde{f} \\
\partial_t A &= \{\psi, A\} + \eta \nabla^2 A
\end{align*}
\]

\[\mathbf{v} = (\partial_y \psi, -\partial_x \psi, 0), \quad \mathbf{B} = (\partial_y A, -\partial_x A, 0)\]

\[\{a, b\} = \partial_x a \partial_y b - \partial_y a \partial_x b\]

Note similarity to HW: $\beta$ plays the role of $\partial_x \langle n \rangle$
Tobias et al. (2007) assessed impact of weak mean field $b_0 \hat{x}$ on zonal flow formation.

Above a critical $b_0$, turbulence is “Alfvénized.” Reynolds-Maxwell stress

$$\langle \partial_x \psi \partial_y \psi \rangle - \langle \partial_x A \partial_y A \rangle \sim \sum_k (|v_k|^2 - |B_k|^2)$$

small $\Rightarrow$ no ZF

$\eta$ large enough $\Rightarrow$ quenches magnetic turbulence $\Rightarrow$ critical $b_0$ can be quite large.

**Fig. 5.**—Scaling law for the transition between forward cascades (diamonds) and inverse cascades (plus signs). The line is given by $B_0^2/\eta = \text{constant}$. 
Previous analytical studies have neglected the effect of cross-helicity $\langle \mathbf{v} \cdot \mathbf{B} \rangle = -\langle A \nabla^2 \psi \rangle$. Often frozen at zero for simplicity, invoking usual conservation law. However, Coriolis term explicitly breaks conservation:

$$\partial_t \langle A \nabla^2 \psi \rangle = -\beta \langle v_y A \rangle + \text{dissipation}$$

In this work: seek to elucidate the role of cross-helicity in this system. What is role in momentum transport?
Stationary value

As a start, can obtain stationary CH value from a simple calculation à la Zeldovich. Neglecting forcing:

\[
\frac{1}{2} \partial_t \langle A^2 \rangle = b_0 \langle A \partial_x \psi \rangle - \eta \langle (\nabla A)^2 \rangle
\]

\[
\Rightarrow \langle A \partial_x \psi \rangle_\infty = \frac{\eta}{b_0} \langle \tilde{b}^2 \rangle
\]

\[
\partial_t \langle A \nabla^2 \psi \rangle = -\beta \langle A \partial_x \psi \rangle + (\eta + \nu) \langle \nabla^2 \psi \nabla^2 A \rangle
\]

\[
\Rightarrow \langle A \nabla^2 \psi \rangle_\infty \approx \frac{\beta \langle \tilde{b}^2 \rangle \ell_B \ell_v}{b_0 (1 + \text{Pm})}
\]

where \( \text{Pm} \equiv \frac{\nu}{\eta} \)

Note appearance of “magnetic Rhines” scale \( k_{MR} = \sqrt{\frac{\beta}{b_0}} \), defines crossover of Rossby and Alfvén frequencies
Simulation results

- Simulate $\beta$-plane system with fixed $b_0 = 2$, $\eta = \nu = 10^{-4}$, $\epsilon = 0.01$, $k_f = 32$ at various $\beta$
- Transition to Rossby turb. begins around $k_{MR} = k_f$ ($\beta = b_0 k_f^2$)
- Good agreement with Zeldovich with $\ell = \ell_f$ (breaks down for large $\beta$ as $\ell_b < \ell_f$)
- Transition presaged by increasing mean CH — suggests CH plays a role?

![Graph showing simulation results](image-url)
Weak turbulence theory

- Need spectra to determine transport. Seek closure of spectral equations that treats cross-helicity on equal footing with energy spectra.
- Simplest approach: weak turbulence theory [Sagdeev and Galeev, 1969]. Treat nonlinear terms as triplet interactions between resonant linear modes.
- Downside: dubious for small $k_x$ or weak field.
- Two eigenmodes in this system (Rossby-Alfvén):

$$\omega_{\pm} = \frac{\omega_\beta \pm \sqrt{4\omega_A^2 + \omega_\beta^2}}{2}$$

with $\omega_\beta = -\beta k_x / k^2$, $\omega_A = k_x b_0$
Spectra I

- Can write down spectral equations for correlators $C_{k}^{\alpha\alpha'}$, but very complicated. Hard to make progress.
- Perturbation theory for small $\beta$ doesn’t work. $\beta$ changes topology of resonant surfaces.
- However, Rossby-Alfvén cross-correlator naturally oscillates at $\omega_+ - \omega_- = \Omega = \sqrt{4\omega_A^2 + \omega_B^2}$ → time average is zero!
- We have

$$k^2 \text{Re}(C_{k}^{++} e^{-i\Omega t}) = -\frac{1}{\Omega^2} \left( \omega_A^2 (|\tilde{v}_k|^2 - |\tilde{b}_k|^2) + \omega_B \omega_A \text{Re}\langle \tilde{v}_k \cdot \tilde{b}_{-k} \rangle \right)$$

$$\implies |\tilde{v}_k|^2 - |\tilde{b}_k|^2 = \frac{\beta}{b_0 k^2} \text{Re}\langle \tilde{v}_k \cdot \tilde{b}_{-k} \rangle$$

Time-averaged, stationary cross-helicity spectrum entirely determines momentum transport!

$$\langle \phi_k^{\alpha} \phi_{k'}^{\alpha'} \rangle = C_k^{\alpha\alpha'} \delta(k + k') e^{-i(\omega_k^{\alpha} - \omega_k^{\alpha'})t}$$
**Spectra II**

- Buildup of cross-helicity during transition thus linked to breakdown of Alfvénization condition $|\tilde{v}_k|^2 = |\tilde{b}_k|^2$

- Equivalently:

$$\frac{\langle \partial_t \tilde{v} \rangle_k}{\langle \partial_t \tilde{b} \rangle_k} = \frac{k_{MR}^2}{k^2}.$$  

$\implies$ Fluctuations kinetic for $\ell > \ell_{MR}$, magnetic for $\ell < \ell_{MR}$  

[Diamond et al., 2007]

- Also have estimate (for $\beta \lesssim b_0 k_f^2$):

$$\langle \tilde{v}^2 \rangle - \langle \tilde{b}^2 \rangle \simeq \frac{\beta^2}{b_0^2 k_f^4} \frac{\langle \tilde{b}^2 \rangle}{1 + \text{Pm}}$$

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**Figure** Time-averaged, $k_y$-averaged spectra from simulation, confirming calculation. Note that spectra don’t agree at $k_x = 0$ because $\Omega \to 0$
Conclusion

- Cross helicity is non-conserved in $\beta$-plane MHD. In presence of mean magnetic field, attains a finite stationary value.
- In weak turbulence theory, stationary cross-helicity spectrum equivalent to Maxwell-Reynolds stress $\rightarrow$ determines momentum transport.
- Have confirmed both of these calculations in simulation.
- $H = \frac{\beta \langle \tilde{b}^2 \rangle \ell_b \ell_v}{b_0(1+Pm)}$ could be very large for weak $b_0$, large $Rm$. Should study this case numerically! Flux of magnetic potential?
- CH spectrum related to turbulent emf, but need 3D to study dynamo.
Final remarks: where does ML fit in with the other projects?

- Had hoped to use machine learning approach to study interactions between spreading and ZF (spreading breaks up ZF, ZF limits spreading?).
- But: diffusive mean field model 
  \[
  \langle \tilde{v}_x (\tilde{n} - \nabla^2 \tilde{\phi})^2 \rangle = f(\varepsilon, \partial_x \varepsilon, \ldots)
  \]
didn’t work. Spreading not important in adiabatic HW? Spreading not described by local model?
- Given similarities between beta-plane MHD and HW, might consider applying ML
- Issues: no adiabaticity, need to specify forcing, 1D model only makes sense when \( k_{MR} \) is large
- Final outlook: would like to apply ML methodology to other systems. 2D HW with generic \( \alpha \), 3D HW lowest-hanging fruits. Other systems with special spatial DOF?
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Extra slides
Sketch of Hasegawa-Wakatani derivation

- Assume cold ions $T_i = 0$, use ion/electron continuity + $E \times B$ and ion polarization drifts + Ohm’s law for parallel electron current + quasineutrality

\[
\partial_t n_\alpha + \mathbf{v}_\alpha \cdot \nabla n_\alpha = 0
\]

force balance: $\mathbf{v}_i = -\frac{c}{B} \nabla \phi \times \hat{z} - \frac{c}{\omega_{ci} B} \frac{d\nabla \phi}{dt} + \frac{\mu c}{\omega_{ci} B} \nabla^2 (\nabla \phi)$

$\mathbf{v}_e = \mathbf{v}_E + \mathbf{v}_{e,\|}$ (ignore pol. drift due to mass ratio)

$\eta J_{e,\|} = -\nabla_{\|} \phi + \frac{1}{en_e} \nabla_{\|} p_e$, $p_e = n_e T_e \rightarrow$ solve for $J_{\|}$

- Sub above into continuity, use quasineutrality $n_e \simeq n_i$ ($\lambda_D \ll \ell$)
Compare to zonally averaged 2D DNS

1D resembles simplified version of DNS. One key difference: 3-field model equivalent to taking stationary “best-fit” spectrum. Some system memory lost
Reynolds stress: intensity scaling

- Whereas learned $\Gamma$ is essentially $\propto \varepsilon$, $\Pi$ scaling with $\varepsilon$ is nontrivial.
- Learned exponent is 1 for small intensity, close to zero for large intensity.
- Jibes with intuition from strong turbulence theory.

**Figure** Reynolds stress dependence on gradients at fixed $\varepsilon, U, U''$. 
Drift-wave/zonal flow system

- Drift-wave turbulence features complex interaction between mean density profile, ZF, and turbulence.
- Dynamics controlled by cross-correlations between fluctuating quantities (turbulent fluxes).
- Difficult to calculate, requires successive, often dubious approximations to make progress.

Figure Feedback loop illustrating interaction of mean fields in DW turbulence.
Reynolds stress: hyperviscosity

Hyperviscous term, crucial for stability, has small coefficient. Sensitive test of method.

Figure \( U'' \) level curves of Reynolds stress as function of \( U \), at fixed \( \varepsilon, U', N' \)

Figure \( \varepsilon \) level curves of Reynolds stress as function of \( U'' \), at fixed \( U, U', N'' \)
Cousin models

- Compare to bistable models for subcritical transition to fluid turbulence [Barkley et al., 2015, Pomeau, 2015].
- Compare to [Gil and Sornette, 1996] model for sandpile avalanches

\[
\begin{align*}
\partial_t S &= \gamma \left( |\partial_x h|/g_c - 1 \right) S + \beta S^2 - S^3 + \partial_x (D_S S \partial_x S) \\
\partial_t h &= \partial_x (D_h S \partial_x h).
\end{align*}
\]

- \( S \leftrightarrow I, \ h \leftrightarrow p \)
- Weak gradient coupling limit \( D_p \ll D_I \Rightarrow \) our model
- Strong gradient coupling limit: \( I \) slaved to \( p \). \( \partial_x p \propto I^{-1} \Rightarrow \) linear term is \( c - \gamma I \), where \( c \) is a constant which depends on BCs. Bistable again!
Consider spreading of turbulence from lin. unstable to lin. stable zone

Simple model: \( \gamma_1 = \gamma_g > 0 \) for \( x < 0 \),
\( \gamma_1 = -\gamma_d < 0 \) for \( x > 0 \)

Allow turbulent front to form in lefthand region and propagate

In Fisher model, penetration is weak: forms stationary, exponentially-decaying profile with
\( \lambda \sim \sqrt{D_0/\gamma_{nl}} \sim \Delta_c \). Dubiously consistent with observation
Avalanche threshold (details)

- Strategy: assume initial puff is symmetric, has single max $I_0$ and single lengthscale $L$
- Expand intensity curve about max to quadratic order, plug into dynamical equation, integrate over extent of cap
- Result: growth if

$$L > L_{\text{min}} = \sqrt{\frac{D(\alpha)I_0}{f(I_0) - \frac{1}{3}(I_0 - \alpha)f'(I_0)}} = \sqrt{\frac{3D\alpha I_0}{\gamma(I_0 - \alpha)((1 - 2\alpha)I_0 + \alpha)}}$$

- Power law $L_{\text{min}} \sim (I_0 - \alpha)^{-1/2}$
Evidence for subcriticality

- [Inagaki et al., 2013]: experiments demonstrate hysteresis between fluctuation intensity and driving gradient (no TB present). Suggests bistable S-curve relation?

- Turbulence subcritical in presence of strong perpendicular flow shear [Carreras et al., 1992, Barnes et al., 2011, van Wyk et al., 2016] or in the presence of magnetic shear [Biskamp and Walter, 1985, Drake et al., 1995]

- Profile corrugations [Waltz, 1985, Waltz, 2010] and phase space structures [Lesur and Diamond, 2013] can drive nonlinear instability

Figure: Hysteresis between intensity and gradient, flux and gradient
Closure theory

- How to go from dynamical equations

\[ \partial_t \phi_\alpha^{k} + i \omega_k \phi_\alpha^{k} = \frac{1}{2} \sum_{k'+k''=k} M^{\alpha,\beta,\gamma}_{k,k',k''} \phi_\beta^{k'} \phi_\gamma^{k''} \]

to equations for spectra \( \langle \phi^\alpha_k \phi^{\alpha'}_{-k} \rangle \)?

- Multiplying thru by \( \phi_\alpha^{k'} \) yields equation which involves third-order moments \( \langle \phi \phi \phi \rangle \), third-order moment equation involves fourth-order moments, etc.

- “Closure problem”: how to express higher-order moments in terms of lower-order moments and close system?

- DIA (Kraichnan): \( \langle \phi_k \phi_{k'} \phi_{k''} \rangle \simeq \langle \phi_k^{(c)} \phi_{k'} \phi_{k''} \rangle + \ldots \) where \( \phi_k^{(c)} \) coherent to direct beat \( \phi_{k'} \phi_{k''} \). Equiv. to 1-loop renormalization
Spectral equations

Weak turb. spectral equations for arbitrary number of scalar fields $\phi^\alpha$ (in eigenbasis) can be derived straightforwardly:

$$\partial_t C_k^{\alpha\alpha'} = \sum_{k'+k''=k} \sum_{\beta\gamma} \left| M_{k,k',k''}^{\alpha\beta\gamma} \right|^2 C_{k'}^{\beta\beta} C_{k''}^{\gamma\gamma} \delta(\omega_k^\alpha - \omega_{k'}^\beta - \omega_{k''}^\gamma) \delta_{\alpha\alpha'}$$

$$+ M_{k,k',k''}^{\alpha\beta\gamma} M_{k',k,-k''}^{\beta\alpha\gamma} C_k^{\alpha\alpha'} C_{k''}^{\gamma\gamma} \left( \pi \delta(\omega_k^\alpha - \omega_{k'}^\beta - \omega_{k''}^\gamma) + i\mathcal{P} \frac{1}{\omega_k^\alpha - \omega_{k'}^\beta - \omega_{k''}^\gamma} \right)$$

$$+ M_{k,k',k''}^{\alpha\beta\gamma*} M_{k',k,-k''}^{\beta\alpha\gamma*} C_k^{\alpha\alpha'} C_{k''}^{\gamma\gamma} \left( \pi \delta(\omega_k^{\alpha'} - \omega_{k'}^\beta - \omega_{k''}^\gamma) - i\mathcal{P} \frac{1}{\omega_k^{\alpha'} - \omega_{k'}^\beta - \omega_{k''}^\gamma} \right) \right].$$

where $\langle \phi_k^{\alpha} \phi_k^{\alpha'} \rangle = C^{\alpha\alpha'} \delta(k + k') e^{-i(\omega_k^\alpha - \omega_k^{\alpha'})t}$, $M_{kk'k''}^{\alpha\beta\gamma}$ are symmetrized nonlinear coupling coefficients. PV integrals vanish in case of real coupling coefficients and a single field, recover Sagdeev-Galeev result.
MHD snapshots at $\beta = 3000$ at $t = 400$

Figure $\nabla^2 \psi$

Figure A
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