Momentum and Heat Transport in MHD Turbulence in Presence of Stochastic Magnetic Fields

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This work is supported by the U.S. Department of Energy under award number DE-FG02-04ER54738

Changchun Chen Defense, May 4th 2022
Outline

• Introduction:
  a. Importance of momentum Transport in quasi-2D astrophysical system and tokamak.
  b. Zonal flow formation/suppression, PV mixing.
  c. Comparison—β-plane and edge plasma physics.

• My research work/Results:
  a. **Stochastic-field induced** effect on momentum transport in the solar tachocline—β-plane MHD and the Rochester& Rosenbluth’s proposal for describing stochastic field.
  c. Stochastic-field-induced effect on Parallel momentum and ion heat transport (Chen et al. PPCF **64**, 015006, 2022).

• Results:
  a. **β-plane MHD**: The momentum transport problem of the solar tachocline is due to the highly disordered magnetic field decorrelation effect.
  b. Stochastic fields can form a fractal, elastic network.
  c. The stochastic-field induced dephasing at the mean field intensity lower than that for the fully Alfvénization.
• Results:
  d. **Poloidal momentum transport in stochastic B-field**: Suppression of PV diffusivity and the shear-eddy tilting feedback loop.
  e. Power threshold increment for L-H transition. Intrinsic Rotation in presence of stochastic fields.
  f. **Parallel and ion heat transport in stochastic B-field**: In strong turbulence regime, the mean flow is driven by stochastic-turbulent scattering.
  g. We calculate the **explicit form** of the stochastic-field-induced transports which have different mechanisms in presence of strong/weak electrostatic turbulence.

• Future Work:
  a. Neutrals and Drift-Rossby-Alfvén turbulence: Ambipolar Diffusion
  b. Staircase and Mixing length in presence of stochastic fields
Momentum Transport in Astrophysical system and in tokamaks

Solar Tachocline

The solar tachocline inward spreading problem.

Jovian Atmosphere

Formation and evolution of the Rossby flow and the Great Red Spot.

Tokamaks

Reynolds stress suppression during L-H transition in presence of RMPs.

Momentum transport in presence of magnetic fields is a generic problem.
The Solar Tachocline and Disordered B Fields

- The solar tachocline (Weak mean magnetization):
  a. A layer between the convective and radiative zone.
  b. It is strongly stratified/Pancake-like structures.
    —> Incompressible rotating fluid in 2D layers: $\beta$-plane model
  c. Zonal Flow and Rossby Waves — as in the Jovian Atmosphere.
  d. Large magnetic perturbation $|\tilde{B}| \gg B_0$.
  e. Meridional cells forms tachocline but will make it spread inward.

- Where does stochastic fields come from?

The drift-Rossby wave for the tachocline is **Quasi-2D**

The stochastic magnetic field has been “pumped” from the convection zone into the stably stratified region.
Where does \textbf{stochastic fields} come from? Resonant magnetic perturbation (RMP) is externally imposed, which raises L-H transition power threshold.

- ELMs are quasi-periodic relaxation events occurring at edge pedestal in H-mode plasma.
- ELMs can damage wall components of a fusion device.

3D geometry with $k \cdot B$ resonance leads to \textbf{quasi-2D} drift-Alfvén wave in presence of stochastic field.
**PV mixing and Zonal Flow Formation**

- What is Potential Vorticity (PV)?
  PV is a generalized vorticity. It is conserved along the fluid (conserved phase space density).

- Momentum transport and flow formation are determined by **inhomogeneous PV mixing**.

- How does zonal flow evolves?

**Taylor Identity:**
\[ \langle \tilde{u}_y \tilde{\zeta} \rangle = -\frac{\partial}{\partial y} \langle \tilde{u}_y \tilde{u}_x \rangle \]

\[ PV \text{ flux} \quad \text{Reynolds force} \]

**Evol. of zonal flow:**
\[ \frac{\partial}{\partial t} \langle u_x \rangle = \langle \tilde{u}_y \tilde{\zeta} \rangle = -\frac{\partial}{\partial y} \langle \tilde{u}_y \tilde{u}_x \rangle. \]

\[ PV \equiv \zeta \equiv \nabla \times v \text{ (pure 2D fluid)} \]

\[ PV \equiv \zeta + 2 \Omega \sin \phi_0 + \beta y \text{ (on the \( \beta \)-plane)} \]

\[ PV \equiv (1 - \rho_s^2 \nabla^2) \frac{|e| \phi}{T} + \frac{X}{L_m} \text{ (Hasegawa-Mima eq. for tokamak)} \]
PV mixing and Zonal Flow Formation

- Non-local wave-wave interaction:
  The resultant wave (zonal flow mode) has wavenumber much smaller than that of the other two waves.

Strongly nonlinear processes like PV mixing and wave breaking yield turbulent PV flux, and form a large-scale zonal flow.
Turbulent mixing, instabilities, and turbulence lead to inhomogeneity, which acts as a free energy source awakens turbulence. Instabilities/turbulence further feed into zonal flow, which, in turn, 'feeds' into the zonal flow. The predator-prey model highlights how turbulence acts as the prey, growing from the free energy source (i.e., environmental nutrients), while the zonal flow (i.e., the predator) is 'fed' upon the turbulence.
## Physical System Summary

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<td>ExB Drift waves</td>
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<td>$Ro \ll 1$</td>
<td>$Re \approx 10 - 100$</td>
<td>$Rm = \frac{L \nu_A}{\eta} = 10^5 - 10^7$</td>
<td>$E \times B$ shear flow (poloidal)</td>
<td>$Ku_{fluid} &lt; 1$</td>
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**Zonal band (toroidal)**

**ExB shear flow (poloidal)**

\[ Ku_f \equiv \frac{\delta}{\Delta} \sim \frac{\bar{v}}{\tau_ac} \sim \frac{\tau_ac}{\tau_{eddy}} \approx 1, \]
Momentum transport in the solar tachocline—β-plane MHD
My Research — The Solar Tachocline

- Setup:
  x—toroidal (zonal)
  y—poloidal (latitudinal)
  z—radial
  \( \beta \)—Rossby parameter \( \propto \) rotation

- The existence of the tachocline is inferred from the helioseismology, but there is no direct observational evidence.
  
  a. Spiegel & Zahn (1992)— Spreading of the tachocline is opposed by turbulent viscous diffusion of momentum in latitude.
  b. Gough & McIntyre (1998)— Spreading of the tachocline is opposed by a hypothetical fossil field in the radiational zone.

These two models ignore the “likely” strong stochasticity of the tachocline magnetic field.

"At the heart of this argument is the role of the fast turbulent processes in redistributing angular momentum on a long timescale."

— (Tobias et al. 2007)
How we describe the stochastic magnetic field at the Tachocline

- Fluid Kubo number:
  \[ Ku_f \equiv \frac{\delta_l}{\Delta_{\perp}} \sim \frac{\bar{\nu} \tau_{ac}}{\Delta_{\perp}} \sim \frac{\tau_{ac}}{\tau_{edd}} \approx 1, \]

- Magnetic Kubo number:
  \[ Ku_{mag} \equiv \frac{\delta_l}{\Delta_{edd}} = \frac{l_{ac} |\bar{B}|}{\Delta_{edd} B_0} \]

- Zel’dovich, 1983 proposed a physical picture of the stochastic fields:

  The large-scale magnetic field is distorted by the small-scale fields. The system thus is the ‘soup’ of cells threaded by sinews of open field line.

  A weak mean field— might lead to a large magnetic Kubo number, since \(|\bar{B}^2|/B_0^2 \gg 1|.

  Auto correlation time
  Eddy turnover time
  \[ Ku = \begin{cases} < 1, & \text{Quasi-linear theory} \\ > 1, & \text{Quasi-linear theory fails} \end{cases} \]
How we describe the stochastic magnetic field Tachocline

- The system is strongly nonlinear and simple quasi-linear method **fails**.

  A “frontal assault” on calculating PV transport in an ensemble of tangled magnetic fields is a daunting task.

**Goal:** Find an analytical model beyond quasi-linear (QL) theory that can describe the stochastic-field-induced effect.

- How to describe the stochastic fields? Rechester & Rosenbluth (1978) suggested replacing the “full” problem with one where waves, instabilities, and transport are studied in the presence of an ensemble of prescribed, static, stochastic fields.

  ![Mean field diagram](image)

- Assumptions for stochastic field:
  1. Amplitudes of random fields distributed statistically.
  2. Auto-correlation length of fields is small so that \( k u_{mag} \) is small.

\[
\left( \text{delta correlation } l_{ac} \rightarrow 0, \text{ such that } K u_{mag} \equiv \frac{\bar{u} \tau_{ac}}{\Delta_{eddy}} = \frac{l_{ac} |B|}{\Delta B_0} < 1, \text{ even } \frac{B}{B_0} > 1. \right)
\]
β-plane MHD Turbulence—Order of Scale

- Two-average method:

1. \[ F = \int dR^2 \int dB_r \cdot P(B_r) F \]

2. \[ \langle \rangle = \frac{1}{L} \int dx \frac{1}{T} \int dt \text{ Ensemble average over the zonal scales} \]

- Two main equation:

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) \zeta &- \beta \frac{\partial \psi}{\partial x} = - \frac{(B \cdot \nabla)(\nabla^2 A)}{\mu_0 \rho} + \nu (\nabla \times \nabla^2 u) \\
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) A &= B_0 \frac{\partial \psi}{\partial x} + \eta \nabla^2 A,
\end{align*}
\]

\[ l_{RM} = \sqrt{\frac{v_{x,A}}{\beta}} \text{ separates the Rossby and eddy regime} \]

\[
\begin{align*}
A &= A_1 + \tilde{A} + A_{st} \\
B &= B_1 + \tilde{B} + B_{st} \\
J &= 0 + \tilde{J} + J_{st} \\
\psi &= \langle \psi \rangle + \tilde{\psi} \\
u &= \langle u \rangle + \tilde{u} \\
\zeta &= \langle \zeta \rangle + \tilde{\zeta}
\end{align*}
\]
The Solar Tachocline Results

Reynolds stress suppression when mean field is weak (before the system is fully Alfvénized).

\[ \Gamma = \langle \tilde{u}_y \zeta \rangle = - \frac{\partial}{\partial y} \langle \tilde{u}_y \tilde{u}_x \rangle \]

\[ = - \sum_k |\tilde{u}_{y,k}|^2 \frac{\nu k^2 + \frac{B_{y,k}^2}{\mu_0 \rho k^2}}{\omega^2 + \left( \nu k^2 + \frac{B_{y,k}^2}{\mu_0 \rho k^2} \right)^2} \left( \frac{\partial}{\partial y} \tilde{\zeta} + \beta \right) \]

Reynolds stress dephasing \rightarrow PV flux suppression

Random magnetic fields have an effect on both the PV flux and the magnetic drag: Two effects!

\[ \frac{\partial}{\partial t} \langle u_x \rangle = \langle \Gamma \rangle - \frac{1}{\eta \mu_0 \rho} \langle B_{st,y}^2 \rangle \langle u_x \rangle + \nu \nabla^2 \langle u_x \rangle \]

Stochastic fields dephase the Reynolds stress and hence suppress the zonal flow.

(Chen et al., ApJ 892, 24 (2020))
The Solar Tachocline Results

- Mutli-scale dephasing:

\[
\Gamma = - \sum_k |\tilde{u}_{y,k}|^2 \left( \frac{\nu k^2 + \left( \frac{B_0^2 k_x^2}{\mu_0 \rho} \right) \eta k^2 + \frac{B_{st,y}^2 k^2}{\mu_0 \rho \eta k^2}}{\left( \omega - \left( \frac{B_0^2 k_x^2}{\mu_0 \rho} \right) \omega + \eta^2 k^4 \right)^2} + \left( \nu k^2 + \left( \frac{B_0^2 k_x^2}{\mu_0 \rho} \right) \eta k^2 + \frac{B_{st,y}^2 k^2}{\mu_0 \rho \eta k^2} \right) \right)^2 \left( \frac{\partial}{\partial y} \bar{\eta} + \beta \right)
\]

The large- and small-scale magnetic fields have a synergistic effect on the cross-phase in the Reynolds stress.

- Dispersion relation of the Rossby-Alfvén wave with stochastic fields:

\[
\left( \omega - \omega_R + \frac{iB_{st,y}^2 k^2}{\mu_0 \rho \eta k^2} + i\nu k^2 \right) \left( \omega + i\eta k^2 \right) = \frac{B_{0,x}^2 k_x^2}{\mu_0 \rho}
\]

Drag+dissipation effect: This implies that the tangled fields and fluids define a resisto-elastic medium.

\[
\text{dissipation constant} = \frac{B_{st}^2 k^2 / \mu_0 \rho}{\eta k^2}
\]

(Chen et al., ApJ 892, 24 (2020))
β-plane MHD Turbulence—Results

- Stochastic magnetic fields form a resisto-elastic medium:

\[ \omega^2 + i(\alpha + \eta k^2)\omega - \left(\frac{B_{st}^2 k_x^2}{\mu_0 \rho} + \frac{B_{0}^2 k_y^2}{\mu_0 \rho}\right) = 0 \]

Two contributions: \( \overline{B_{st}^2} > B_0^2 \)

- We obtain the Dimensionless parameters

The transition parameter \( \lambda = 1 \), where the waves is critically damped and successfully predict the transition line.

\[ \lambda = \left| \frac{\omega_{im}}{\omega_{re}} \right| \approx \frac{\eta k^2 \omega_A^2}{\omega_R^3}, \text{ in the limit } \omega_R \gg \omega_A \]

(Chen et al., ApJ 892, 24 (2020))
β-plane MHD Turbulence—Conclusions

- Reynolds stress will undergo decoherence at levels of field intensities well below that of Alfvénization (where Maxwell stress balances the Reynolds stress).

\[
\frac{\partial}{\partial t} \langle u_x \rangle = \langle \Gamma \rangle - \frac{1}{\eta \mu_0 \rho} \langle B_{st,x}^2 \rangle \langle u_x \rangle + \nu \nabla^2 \langle u_x \rangle
\]

- Turbulent momentum transport in the tachocline is suppressed by the enhanced memory of stochastically induced elasticity.

Both Spiegel & Zahn (1992) and Gough McIntyre (1998) models for the solar Tachocline are not correct. These two models both ignore strong stochastic fields of the tachocline. The truth here is ‘neither pure nor simple’ (apologies to Oscar Wilde).

Future Work

- This network can be fractal (multi-scale) and intermittent
  a. packing fractional factor: \( B_{st}^2 \rightarrow pB_{st}^2 \)
  b. “fractons” (Alexander & Orbach 1982).
Momentum Transport at the edge of fusion devices—Drift-wave turbulence

Part I: Poloidal Reynolds Stress Dephasing
Why we study stochastic fields in fusion device?

- ELMs are quasi-periodic relaxation events occurring at edge pedestal in H-mode plasma.
- ELMs can damage wall components of a fusion device.

Shear Flow → Quench Turbulence → Pedestal Formation → Edge-Localized Mode (ELM)

Edge-localized mode (ELM) → Edge gradient ↑ → Peeling-balloonning mode → ELM Burst

Suppress (by inducing magnetic perturbation)

Boundary Control: Resonant Magnetic Perturbation (RMP)
Stochastic field effect is important for boundary control

Shear Flow → Edge-Localized Mode (ELM) → ELM Burst

Boundary Control: Resonant Magnetic Perturbation (RMP)

Suppress (by inducing magnetic perturbation)

Stochasticity: Overlapping of Magnetic islands.

Trade off: RMPs controls gradients and mitigates ELM, but raise the power threshold.

Key Questions:

How RMPs influence the Reynolds stress and hence suppress the zonal flow?
How stochastic fields increase the power threshold of L-H transition?

We examine the physics of stochastic fields interaction with zonal flow near the edge.

(Chen et al., PoP 28, 042301 (2021))
Experimental Results with RMP for L–H Transition — fluctuations

DIII-D Experimental results: RMPs lower the Reynolds stress and increase the power threshold of L-H transition.

This section: Mean poloidal (zonal flow) in stochastic fields.

\[ \langle E_r \rangle = \frac{\nabla \langle p_i \rangle}{ne} - \langle \mathbf{u} \rangle \times \langle \mathbf{B} \rangle \]

Next section: mean toroidal flow in stochastic fields.
Model

1. Cartesian coordinate: strong mean field $B_0$ is in $z$ direction (3D).
2. Rechester & Rosenbluth (1978): waves, instabilities, and transport are studied in the presence of external excited, static, stochastic fields.
3. $k \cdot B = 0$ (or $k_\parallel = 0$) resonant at rational surface in third direction —
   $$\omega \rightarrow \omega \pm v_A k_z,$$
   and Kubo number: $Ku_{mag} = \frac{l_{ac} | \tilde{B} |}{\Delta B_0} < 1$.
4. Four-field equations —
   (a) Potential vorticity equation — vorticity — $\nabla^2 \psi \equiv \zeta$
   (b) Induction equation — $A$, $J$
   (c) Pressure equation — $p$
   (d) Parallel flow equation — $u_z$

We use mean field approximation:

$$\zeta = \langle \zeta \rangle + \tilde{\zeta},$$

Perturbations produced by turbulences

where $\langle \rangle = \frac{1}{L} \int dx \frac{1}{T} \int dt$ 

$\langle \zeta \rangle = \frac{\partial v_{E \times B}}{\partial x}$ (E × B shear)

ensemble average over the zonal scales

We define rms of normalized stochastic field $b \equiv \sqrt{(B_{st}/B_0)^2}$

(Chen et al.,PoP 28, 042301 (2021))

Magnetic islands overlapping forms stochastic

Well beyond HM model
When does stochastic field effect becomes significant?

We consider timescales: (Chen et al., PoP 28, 042301 (2021))

\[
D \equiv v_A D_M = v_A \sum_k \pi \delta(k_z) b_k^2 \propto B_{st}^2
\]

Stochastic field decoherence beats the self-decoherence.

Perturbations propagate ultimately in \( \perp \) (along stochastic fields) \( \rightarrow \) characteristic velocity \( (v_A) \) emerges from the calculation of \( \nabla \cdot J = 0 \)
Dimensionless Parameters

How ‘stochastic’ is magnetic field?

Alfvénic Dispersion

\[ \frac{v_A}{L_{||}} \]

(excited by drift-Alfvénic coupling)

Stochastic broadening

\[ Dk^2 \]

Two dimensionless Parameters:

1. \( l_{ac} \approx Rq \)
   \[ e \equiv \frac{L_n}{R} \sim 10^{-2} \]
   \[ \beta \approx 10^{-2} - 10^{-3} \]
   \[ \rho_* \equiv \frac{\rho}{L_n} \sim 10^{-2} - 10^{-3} \]

2. Broadening parameter
   \[ \alpha \equiv \frac{b^2}{\rho_*^2 \sqrt{\beta} \, \epsilon} \]

\( K_{u_{mag}} \) (Magnetic Kubo number)

\[ \equiv \frac{\text{stochastic field scattering length}}{\text{perpendicular magnetic fluctuation size}} = \frac{l_{ac} b}{\Delta_{eddy}} \leq 1, \]

(for a \( b \) given)

\[ Dk^2 > \Delta \omega \]

Criterion for stochastic fields effect important to L-H transition.

\[ b^2 \equiv \left( \frac{\delta B_r}{B_0} \right)^2 > \sqrt{\beta} \rho_*^2 \epsilon \sim 10^{-8} \]

\[ \alpha = 1: \]

stochastic broadening = natural linewidth

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PhD Defense
May 4th 2022

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Decoherence of eddy tilting feedback

Snell’s law:
\[
\frac{d}{dt}k_x = -\frac{\partial(\omega_0 + u_y k_y)}{\partial x} = -k_y \frac{\partial u_y}{\partial x}
\]

Gives an non-zero \( \langle k_x k_y \rangle \)
\[\rightarrow \langle \tilde{u}_x \tilde{u}_y \rangle \propto \langle k_x k_y \rangle \]
shear flow

Self-feedback loop:
The \( E \times B \) shear generates the \( \langle \tilde{k}_x \tilde{k}_y \rangle \) correlation and hence support the non-zero Reynolds stress.
\[
\langle \tilde{u}_x \tilde{u}_y \rangle \simeq \sum_k \frac{|\phi_k|^2}{B_0^2} (k_y^2 \partial u_y \partial x \tau_c)
\]
The Reynolds stress modifies the shear via momentum transport.

Shear flow reinforce the self-tilting.

Dispersion relation of drift-Alfvén coupling
\[
\omega^2 - \omega_D \omega - k_\parallel^2 v_A^2 = 0
\]

Stochastic Fields Effect
\[
\begin{align*}
\omega &= \omega_D + \delta \omega \\
k_\parallel &= k_\parallel^{(0)} + b_\perp \cdot k_\perp \\
(\omega_D + \delta \omega)^2 - \omega_D (\omega_D + \delta \omega) - (k_\parallel + b \cdot k_\perp)^2 v_A^2 &= 0
\end{align*}
\]

eigen-frequency shift
\[
\delta \omega \simeq \frac{v_A^2}{\omega_D} (2k_\parallel \cdot b_\perp + (b \cdot k_\perp)^2)
\]

\( \omega_D \) (drift wave turbulence frequency) \( \equiv \frac{k_y \rho_s C_s}{L_n} \)
Decoherence of eddy tilting feedback

Expectation frequency:

\[ \langle \delta \omega \rangle \simeq \frac{v_A^2}{\omega_0} (2k_b \cdot k + (b \cdot k) \perp)^2 \]

\[ \omega = \omega_D + \delta \omega \]

\[ \langle \omega \rangle \simeq \omega_D + \frac{1}{2} \frac{v_A^2}{\omega_D} b^2 k^2 \perp \]

Ensemble average of eigen-frequency shift

\[ \langle \delta \omega \rangle \simeq \frac{v_A^2}{\omega_0} \langle (b \cdot k) \perp \rangle = \frac{1}{2} \frac{v_A^2}{\omega_0} b^2 k^2 \perp \]

Snell's law:

\[ \frac{d}{dt} k_x = - \frac{\omega_k}{\omega_x} \]

\[ = - k_y \frac{\partial u_y}{\partial x} - \frac{1}{2} \frac{v_A^2 k^2 \perp}{\omega_D} \frac{\partial b^2}{\partial x} \]

\[ \langle \tilde{u}_x \tilde{u}_y \rangle \simeq \sum_k \left| \tilde{\phi}_k \right|^2 \left( k_y^2 \frac{\partial u_y}{\partial x} \tau_c + \frac{1}{2} k_y^2 \frac{v_A^2 k^2 \perp}{\omega_D} \frac{\partial b^2}{\partial x} \tau_c \right) \]

Due to the Ensemble average eigen-frequency shift

Self-feedback loop is broken by \( b^2 \):

Stochastic dephasing

Stochastic fields (random ensemble of elastic loops) act as elastic loops and resist the tilting of eddies.

→ change the cross-phase btw \( \tilde{u}_x \) and \( \tilde{u}_y \).

(Chen et al., PoP 28, 042301 (2021))
Decoherence of eddy tilting feedback

Stochastic fields interfere with shear-tilting feedback loop.
Results—Suppression of PV diffusivity

The ensemble average Reynolds force $\frac{\partial}{\partial x} \langle \tilde{u}_x \tilde{u}_y \rangle$:

$PV \text{ flux } = \langle \tilde{u}_x \tilde{\xi} \rangle = \frac{\partial}{\partial x} \langle \tilde{u}_x \tilde{u}_y \rangle = - D_{PV} \frac{\partial}{\partial x} \langle \xi \rangle + F_{res} k \frac{\partial}{\partial x} \langle p \rangle$

Taylor Identity: $\langle \tilde{u}_x \tilde{\xi} \rangle = \frac{\partial}{\partial x} \langle \tilde{u}_x \tilde{u}_y \rangle$

PV diffusivity $D_{PV} = \sum_{k\omega} |\tilde{u}_{x,k\omega}|^2 \frac{v_A b^2 l_{ac} k^2}{\bar{\omega}^2 + (v_A b^2 l_{ac} k^2)^2}$

Residual Stress $\langle \xi \rangle = \frac{\partial}{\partial x} (E \times B \text{ shear})$

Curvature $\langle \xi \rangle = \frac{\partial}{\partial x} \langle \tilde{\zeta} \rangle$

PV transport will be suppressed by stochastic fields via decoherence.

Zonal flow acceleration $\frac{\partial}{\partial t} \langle u_y \rangle = D_{PV} \frac{\partial}{\partial x} \langle \xi \rangle - F_{res} k \frac{\partial}{\partial x} \langle p \rangle$

Zonal flow acceleration is slowed down by the stochastic field.

This stochastic dephasing effect is insensitive to turbulent modes, e.g. ITG, TEM,...etc.
Results — Increment of $P_{\text{LH}}$

Stochastic field stress dephasing effect requires: $\Delta \omega \leq k_{\perp}^2 D$ (where $D = D_{MVA}$).

This gives Broadening parameter $(\alpha)$: $\alpha \equiv \frac{b^2}{\sqrt{\beta \rho^2}} q > 1$

$\alpha$ quantifies the strength of stochastic dephasing.

Macroscopic Impact

Kim-Diamond Model

(Kim & Diamond, PoP 10, 1698 (2003))

This reduce model for the L-H transition is useful for testing trends in power threshold increment induced by stochastic fields.

Predator: zonal flow
Prey: turbulence

We expect stochastic fields to raise L-H transition thresholds.
Results — Increment of $P_{LH}$

The threshold increase due to stochastic dephasing effect is seen in turbulence intensity, zonal flow, and pressure gradient.

$$\alpha \equiv \frac{b^2}{\sqrt{\beta \rho_\epsilon^2}} q = 0.0, 0.2, 0.4, 0.6, 0.8..., 2.0$$

(Chen et al., PoP 28, 042301 (2021))
Results — Increment of $P_{LH}$

Stochastic field stress dephasing effect requires: $\Delta \omega \leq k^2 D$

Iterative Threshold Increase

The threshold increase linearly, in proportional to $\alpha$. This is due to stochastic dephasing effect.

Broadening parameter

$\alpha \equiv \frac{b^2 q}{\sqrt{\beta \rho^* \epsilon}}$

$\alpha$ quantifies the strength of stochastic dephasing.
Conclusions

- **Dephasing effect** caused by stochastic fields quenches poloidal Reynolds stress (e.g. $\Delta \omega < Dk^2_\perp$). Here, $D = v_A D_M$.
- Stochastic fields interfere with shear-tilting feedback loop, and prevent the production of zonal flow.
- $b^2$ shift L-H threshold to higher power, in proportional to $\alpha \equiv \frac{b^2}{\sqrt{\beta \rho^2_\ast}} \frac{q}{\epsilon}$. ITER has smaller $\rho_\ast$, leading to a higher $\alpha$.

Future Works

- We study the scale corrugation of staircases in presence of stochastic fields.
- Detailed calculations for symmetry breaking of toroidal residual stress.
Takeaways for Experimentalists

- Reynolds stress suppression due to stochastic dephasing → generation of zonal flow is suppressed. Zonal intensity stays the same but damping occurs due to the stochastic dephasing.

- Stochastic fields broadening effect can be parameterized by $\alpha$.

- $b^2$ shift L-H threshold to higher power, in proportional to $\alpha \equiv \frac{b^2}{\sqrt{\beta \rho^2}} \frac{q}{\epsilon}$.

- $\alpha \propto \frac{1}{\rho^2}$, $\rho^2_*$ is small → $\alpha \uparrow$ (pessimistic)

- Our results predicts the power threshold of L-H transition increases linearly as stochastic magnetic field intensity increases.

(L. Schmitz et al, NF 59 126010 (2019))
Momentum Transport at the edge of fusion devices—Drift-wave turbulence

Part II: Ion heat and Parallel Momentum Transport
 Ion Heat and Parallel Momentum Transport

Intrinsic rotation and external neutral beam injection are common in experiments in fusion devices.

- Dephasing effect caused by stochastic fields quenches poloidal Reynolds stress—the mean \( E \times B \) shear is suppressed by this effect. However, observe

\[
\frac{\partial}{\partial t} \langle u_{EB} \rangle = D_{PV} \frac{\partial}{\partial x} \langle \zeta \rangle - \frac{F_{res}}{\lambda} \frac{\partial}{\partial x} \langle p \rangle
\]

\[
\langle E_r \rangle = \frac{\nabla \langle p_i \rangle}{n e} - \langle u \rangle \times \langle B \rangle
\]

We examine the physics of stochastic fields interaction with the ion pressure and the parallel flow.

Understanding the physics of ion heat and parallel momentum transport is critical to control the instabilities in fusion devices.
Coexistence of Stochastic Field and Turbulence

Magnetic islands overlapping forms stochastic fields

Strong electrostatic turbulence

Both stochastic field and turbulence enter the cross-phase $\langle \tilde{b} \tilde{p} \rangle$, $\langle \tilde{b} \tilde{u}_\parallel \rangle$, and hence enter the dephasing mechanism.

In strong turbulence regime, we have faster turbulent scattering timescale:

$$\frac{1}{\tau_{c,k}} \equiv k^2_\perp D_T \gg 1$$

Key question:

How does stochastic fields influence on the response of parallel flow and pressure in strong turbulence regime (faster turbulent scattering timescale)?

We analyze the dephasing effect of stochastic field in strong and weak electrostatic turbulence—how they together drives transport.

(Chen et al., PPCF, accepted (2021))
Why we study the Kinetic Stress?

Experimental Result of Madison Symmetric Torus (MST)

(MST Experimental results: demonstrated the similarity of the kinetic stress to the parallel flow.)

**Macroscopic** parallel flow dynamics.

**Microscopic** effect measured from the fluctuations of the pressure and the stochastic field.

Nonlinear momentum transport

(Ding et al., PRL 110, 065008 (2013))
Physical Picture of Pressure Response

- We start with the parallel acceleration and pressure equation:

\[
\frac{\partial}{\partial t}u_z + (\mathbf{u} \cdot \nabla)u_z = \nabla_z p
\]

\[
\frac{\partial}{\partial t}p + (\mathbf{u} \cdot \nabla)p = -\gamma p(\nabla_z \cdot \mathbf{u})
\]

\[
\nabla \rightarrow \nabla_z + b_\perp \nabla_\perp
\]

Turbulent fluid diffusivity:

\[
D_T \equiv \sum_k |\tilde{u}_{\perp,k}|^2 \tau_{ac}
\]

- Local pressure excess \( (\tilde{b}_x \partial_x \langle p \rangle) \) caused by magnetic perturbation is balanced by:

\[
0 + (\tilde{u}_\perp \cdot \nabla_\perp)\tilde{u}_z = -\frac{1}{\rho} \frac{\partial}{\partial z} \langle p \rangle - \frac{1}{\rho} \tilde{b}_x \frac{\partial}{\partial x} \langle p \rangle
\]

\[
\text{pressure excess } \quad \frac{1}{\rho} \tilde{b}_x \partial_x \langle p \rangle = D_T \nabla_z^2 \tilde{u}_z - \frac{1}{\rho} \nabla_z \tilde{p}
\]

(a) (Strong turbulence regime) ... by parallel flow perturbation, which is damped by turbulent viscosity.

(b) (Weak turbulence regime) ... by parallel pressure gradient.

\[\text{Finn et al., PoP 4, 1152 (1992)}\]
Physical Picture of Pressure Response

Local \( \text{pressure excess} (\tilde{b}_r \partial_r \langle p \rangle) \) caused by magnetic perturbation is balanced by: \( (u_\parallel \text{ response in the same way}) \)

... by parallel flow perturbation, which is damped by turbulent viscosity.

1. Strong Turbulence: \( \tilde{b}_r \partial_r \langle p \rangle \approx D_T \nabla^2 \tilde{u}_z \)

Mean Toroidal Magnetic Fields

Distorted Toroidal Magnetic Fields

Flow particles

\( \uparrow \): Parallel Speed

\( \tilde{b}_x \)

\( \tilde{u}_z(x) \)

Rate of turbulent (i.e. viscous) mixing \( D_T / l_\perp^2 \) > other rate: turbulent viscosity will dissipate the parallel flow.

Only strong turbulent cases are relevant!

2. Weak Turbulence: \( \tilde{b}_r \partial_r \langle p \rangle \approx -\nabla_z \tilde{p} \) \((\mathbf{B} \cdot \nabla p = 0)\).

Mean Toroidal Magnetic Fields

Distorted Toroidal Magnetic Fields

\( \tilde{p}(z) \)

Rate of sound propagation \( c_s / l_\parallel \) > other rate: pressure gradient builds up parallelly.

Finn et al., PoP 4, 1152 (1992)

Chang-Chun Samantha Chen

PhD Defense

May 4th 2022
Mean field equation for parallel flow and the pressure equation:
\[
\frac{\partial}{\partial t} \langle u_z \rangle + \frac{\partial}{\partial x} \langle \tilde{u}_x \tilde{u}_z \rangle = -\frac{1}{\rho} \frac{\partial}{\partial x} \langle \tilde{b}_x \tilde{p} \rangle \equiv -\frac{1}{\rho} K, \text{ where the kinetic stress } K \equiv \frac{1}{\rho} \langle \tilde{b}_x \tilde{p} \rangle
\]
\[
\frac{\partial}{\partial t} \langle p \rangle + \frac{\partial}{\partial x} \langle \tilde{u}_x \tilde{p} \rangle = -\rho c_s^2 \frac{\partial}{\partial x} \langle \tilde{b}_x \tilde{u}_z \rangle \equiv -\frac{\partial}{\partial x} H, \text{ where the compressible heat flux } H \equiv \rho c_s^2 \langle \tilde{b}_x \tilde{u}_z \rangle
\]
(or, ion heat density flux)

Perturbed equation with Riemann variables \( f_\pm \equiv \tilde{u}_{z,k\omega} \pm \frac{p_{k\omega}}{\rho c_s} \):

The propagator \( 1/(k_y^4 D_T^2 + k_z^2 c_s^2) \) contains the turbulent mixing \( (k_y^4 D_T^2) \) and the magnetic shear effect \( (k_z^2 c_s^2) \).

**Magnetic shear Effect**

\[
k_z^2 c_s^2 = \left(\frac{k_y x}{L_s}\right)^2 c_s^2
\]

\[
\sum_{k,k_z} = \int dm \int dy \int dx \frac{|m|}{q} q' = r_0 \int dk_y \int dx \frac{|k_y|}{q} \hat{s}
\]

1. \( \langle \tilde{b}_x \tilde{p} \rangle = \int dk_y \frac{k_y r_0 q'}{q^2} \int dx \frac{|\tilde{b}_{x,k}|^2}{1 + (x/\chi)^2} \left( -\rho c_s^2 \frac{\partial}{\partial x} \langle u_z \rangle \right) \)

2. \[ + \int dk_y \frac{k_y r_0 q'}{q^2} \int dx |\tilde{b}_{x,k}|^2 \left( \frac{1}{(k_y^2 D_T^2 + k_z^2 c_s^2)} \left( i k_z c_s^2 \frac{\partial}{\partial x} \langle p \rangle \right) \right) \]
We consider length scales:

\[ x_s \]

**Acoustic width**

**Spatial width of \( |\tilde{b}_{x,k}|^2 \)**

**Acoustic width**

\[ x_s \]

\[ W_k \]

\[ x_s \]

**Weak turbulence regime**

**Strong turbulence regime**

\[ x_s \equiv \frac{L_s}{k_y c_s \tau_{c,k}} \] is the **acoustic width**—\( x_s \) defines the location where the **parallel acoustic streaming rate** = **decorrelation rate**.

\( x_s \) is analogous to ion Landau resonant point.

Dimensionless parameter

\[ \lambda \equiv \frac{x_s}{W_k} \]

defines the competition between the stochastic-field and turbulent effect.
Results—Strong Turbulence Regime

In strong turbulence \((k_z^2 D_T \gg k_z c_s \text{ or } \lambda > 1)\):

\[
K \equiv \frac{1}{\rho} \langle \tilde{b}_x \tilde{p} \rangle \simeq - D_{st} \frac{\partial}{\partial x} \langle u_z \rangle , \text{ where}
\]

\[
D_{st} = D_{st}(x) = \sum_{k, k_z} \left| \tilde{b}_{x,k} \right|^2 \rho c_s^2 \frac{k^2 D_T}{k_z^2}
\]

Turbulent fluid diffusivity \(D_T \equiv \sum_k |\tilde{\eta}_{\perp,k}|^2 \tau_{ac}\)

\(D_{st} : \text{the hybrid turbulent diffusivity—explain how the kinetic stress is scattered by stochastic B fields and turbulence.}\)

\[
H \equiv \rho c_s^2 \langle \tilde{b}_x \tilde{u}_z \rangle \simeq - D_{st} \frac{\partial}{\partial x} \langle p \rangle
\]

The pressure gradient in presence of tilted B lines balances with the hybrid turbulent diffusion.
Consider electron density evolution:

\[
\frac{\partial \langle n_e \rangle}{\partial t} - \frac{\partial}{\partial x} \langle b_x J_{z,e} \rangle = 0
\]

Ampère's Law
\[
- \nabla^2 A_z = \mu_0 (J_{z,e} + J_{z,i})
\]

\[
\frac{\partial \langle n_e \rangle}{\partial t} = - \frac{1}{\mu_0 |e|} \frac{\partial}{\partial x} \langle b_x \nabla^2 A_{z,e} \rangle - n_0 \frac{\partial}{\partial x} \langle b_x u_{z,i} \rangle
\]

Electron particle flux
\[
\frac{\partial \langle n_e \rangle}{\partial t} = - \frac{\partial}{\partial x} \Gamma_{e,s}
\]

\[
\Gamma_{e,s} = \frac{\partial}{\partial x} \left( B_0 \frac{\partial}{\partial x} \langle b_x b_y \rangle + n_0 \frac{\partial}{\partial x} D_M \frac{\partial}{\partial x} \langle u_{z,i} \rangle \right)
\]

Total current contribution
Ion current contribution

\[
\langle b_x u_{z} \rangle = - D_M \frac{\partial}{\partial x} \langle u_{z} \rangle
\]

\[
D_M \equiv \sum_{k_x,k_z} |\tilde{b}_{x,k}|^2 \tau_{d,k} \epsilon_s
\]

Dispersal timescale of an acoustic wave packet along the stochastic magnetic field

Stochastic lines and parallel ion flow gradient drives a net electron particle flux, in addition to the Maxwell force contribution.
Conclusions

- We calculate the **explicit form** of the stochastic-field-induced transports—kinetic stress $K$ and the compressive energy flux $H$—have different mechanisms in presence of strong/weak electrostatic turbulence.

- In practice, **only strong turbulent cases** ($k^2_D T \gg k_s c_s$ or $\lambda > 1$) are relevant. We found mean parallel flow and mean pressure are driven via the **hybrid diffusivity** that involves effect of stochastic field and turbulent scattering:

  \[
  D_{st} = D_{st}(x) = \sum_{k_x k_z} |\tilde{b}_{x,k}|^2 \frac{c_s^2}{k^2_D T} \tag{1}
  \]

  (Chen et al. PPCF 64, 015006, 2022)

**Future Works**

- Magnetic drift—effect of stochastic field and turbulence upon geodesic acoustic modes.

- One should include the effect of $\langle \tilde{b} \tilde{\phi} \rangle \neq 0$ in the future. (Cao & Diamond PPCF 64 035016, 2022)

- Relevant problems: cosmic ray acceleration and propagation.
Summary of My Research


a. \(\beta\)-plane MHD: We obtain the momentum transport problem of the solar tachocline is due to the highly disordered magnetic field decorrelation effect.

b. We found Stochastic fields form a fractal, elastic network.

c. We show the stochastic-field induced dephasing at the mean field intensity lower than that for the fully Alfvénization.

(Chen et al. PoP 28, 042301, 2021)

d. Poloidal momentum transport in stochastic B-field: We obtain the suppression of PV diffusivity and the shear-eddy tilting feedback loop.

e. Calculate power threshold increment for L-H transition. Intrinsic Rotation in presence of stochastic fields.

(Chen et al. PPCF 64, 015006, 2022)

f. Parallel and ion heat transport in stochastic B-field: We found that in strong turbulence regime, the mean flow is driven by stochastic-turbulent scattering.

g. We calculate the explicit form of the stochastic-field-induced transports which have different mechanisms in presence of strong/weak electrostatic turbulence.
Future Work

Staircase and Mixing length in presence of stochastic fields
Fate of Spatial structure of zonal flow?

We are interested in zonal flow width in presence of stochastic fields.

The mixing length ($l_{mix}$) depends on two scales:

- Driving scale: $l_0$
- Rhines scale: $l_{RH} = \frac{\sqrt{\epsilon}}{|\partial_x q|}$

mixing scale: $l_{mix} = \frac{l_0}{(1 + l_0^2 / l_{RH}^2)^{1/2}}$
Main effect of diffusivity $D_n$ and $\chi$

For $\alpha_{DW}$ (a measurement of the resistive diffusion rate in the parallel direction) $> 1$ in H-W regime:

Density diffusivity:
$$D_n \approx \frac{l_{mix}^2 \epsilon}{\alpha_{DW}}$$

Resistive diffusion rate:
$$\alpha_{DW} = \frac{k_{\parallel}^2 v_{the}^2}{\nu}$$

Stochastic Fields Effect

$$k_{\parallel} = k \cdot \hat{b}_0 \simeq \frac{1}{Rq} + b_{\perp} \cdot \frac{1}{Rq} + \frac{b_{\perp}}{l_{mix}}$$

$$D_n \approx \frac{l_{mix}^2 \epsilon v/v_{the}^2}{\left(\frac{1}{Rq}\right)^2 + \left(\frac{b_{\perp}}{l_{mix}}\right)^2}$$

Same for $\chi$ (or $D_{PV}$ in this case).

Competition btw $\frac{1}{Rq}$ v.s. $\frac{b_{\perp}}{l_{mix}}$ gives

$$Ku_{mag} = bRq/l_{mix}$$

$Ku_{mag} = Ku_{mag}(l_{mix})$

The mixing length is not likely affected by $b^2$.

A change of scale selection or staircase corrugation requires $Ku_{mag} \geq 1$.  

PhD Defense  
May 4th 2022
Future Work

Neutrals and Drift-Rossby-Alfvén turbulence: Ambipolar Diffusion
Important Time Scales

- We consider rates:

\[ \gamma = \frac{(\alpha \rho_i \omega_{\text{eff}})^2}{\omega + i n_{\text{eff}} k^2 \cdot \tau_{\text{eddy}}} \cdot \frac{1}{\omega_{ci}} \]

- A simplified coupling parameter:

\[ \tilde{\alpha} \equiv \frac{\tau_{\text{eddy}}}{\tau_{\text{coll}}} = \frac{l_{\perp} \rho_{\text{tot}} \alpha}{\tilde{u}} \]

- Dynamics of neutrals and ions:

**Weak coupling:**
Neutrals and ions not well-coupled → Two fluid.

**Strong coupling:**
Neutrals couple to ions and behaves as one MHD fluid.
Ambipolar Conclusions

- We study Non-trivial neutral effect on DW-ZF turbulence: Ambipolar diffusion
- We derive the key parameter $\gamma$ for the Drift-Alfvén + Neutral effect:
  $$\gamma = \frac{(\alpha \rho_i) \omega_{\text{eff}}^2}{\omega + i n_{\text{eff}} k^2} \cdot \tau_{\text{eddy}} \cdot \frac{1}{\omega_{ci}}$$
- We study the Drift-Alfvén wave with neutrals and found—
  In strong coupling regime: one MHD fluid.
  In weak coupling regime: two fluid.
- Modified Zel’dovich Theorem. We derive the key parameter that regulates Maxwell and Reynolds stress competition:
  $$\langle \widetilde{B}^2 \rangle \propto B_0^2 / \eta \text{ (original)}$$
  $$\langle \widetilde{B}^2 \rangle \propto B_0 / A^{1/2} \text{ (ambipolar effect)}$$

Future Works

- Calculate $\widetilde{B}$ evolution for arbitrary $\gamma \rightarrow$ Pouquet + neutrals (clarify asymptotic regime).
- Study the physics of neutral entrainment.
Thank you!
Derivation of Magnetic Diffusivity

Vorticity equation: \( \frac{\partial}{\partial t} \mathbf{u} \cdot \nabla \nabla^2 \phi - v_A (\cdot \nabla || + b_{st,\perp} \cdot \nabla \perp) J_|| = 0 \)

\[
\begin{align*}
\text{0th order: } & v_A \frac{\partial}{\partial z} I_{0,z} = 0 \\
\text{1st order: } & (\frac{\partial}{\partial t} - \langle u_y \rangle \frac{\partial}{\partial y}) \nabla^2 \tilde{\phi} - v_A (\nabla || + b_{st,\perp} \cdot \nabla \perp) \tilde{J}_|| = 0
\end{align*}
\]

Curly bracket : \{ \} = \int_{-\infty}^{+\infty} d\tau
\[
\{ \frac{i}{-b_{st,\perp} k_\perp} \} = \int_{-\infty}^{+\infty} d\tau \{ e^{i b_{st,\perp} k_\perp \int_0^{\tau} d\tau'} \} = \int_{-\infty}^{+\infty} d\tau e^{-\frac{k_D M}{v_A} k_T \tau}
\]

\( dl \) is along magnetic fields

Characteristic velocity of \( b_{st,\perp} \) (parallel wave packet transit timescale)

\[
D \equiv v_A D_M = v_A \sum_k \frac{B^2_{st,k}}{B_0^2} \pi \delta(k_z) \propto \frac{1}{v_A} \frac{1}{v_A} v_A B^2_{st}
\]

Diffusivity \( D \) is independent of \( B_0 \).
Fate of Spatial structure of zonal flow?

We are interested in zonal flow width in presence of stochastic fields.

Zonal flow width is related to corrugation length.
Layering Structure—Mixing Length Model

A mixing length model for layering:

- Reduce evolution equations (based on H-W model).
- Energy and Potential entropy (PE) conserved.

\[
\begin{align*}
\text{Density: } & \frac{\partial}{\partial t} \langle n \rangle = \frac{\partial}{\partial x} \left( D_n \frac{\partial \langle n \rangle}{\partial x} \right) + D_c \frac{\partial^2 \langle n \rangle}{\partial x^2} \\
\text{Potential Vorticity: } & \frac{\partial}{\partial t} \langle \zeta \rangle = \frac{\partial}{\partial x} \left( (D_n - \chi) \frac{\partial \langle n \rangle}{\partial x} \right) + \chi \frac{\partial^2 \langle \zeta \rangle}{\partial x^2} + \mu_c \frac{\partial^2 \langle \zeta \rangle}{\partial x^2} \\
\text{Turbulent potential Enstrophy: } & \frac{\partial}{\partial t} \epsilon = \frac{\partial}{\partial x} \left( D_c \frac{\partial \epsilon}{\partial x} \right) + \chi \left( \frac{\partial (n - \zeta)}{\partial x} \right)^2 - \epsilon c^{-1/2} c^{3/2} + P
\end{align*}
\]

\[
\begin{align*}
\langle n \rangle : & \text{ density} \\
\langle \zeta \rangle : & \text{ potential vorticity} \\
\epsilon : & \text{ turbulent PE} \quad \epsilon \equiv (\delta n - \delta \zeta)^2/2 \\
D_n : & \text{ turbulent particle diffusivity} \\
\chi : & \text{ turbulent vorticity} \\
\mu_c : & \text{ mean-turb PE} \\
P : & \text{ production}
\end{align*}
\]

Density corrugation forms staircase-like structure.
Scale Selection

The mixing length \( l_{mix} \) depends on two scales:

- **Driving scale:** \( l_0 \)
- **Rhines scale:** \( l_{RH} = \frac{\sqrt{\epsilon}}{|\partial_x q|} \)

### Mixing Scale

\[
\text{mixing scale: } l_{mix} = \frac{l_0}{(1 + l_0^2 (\partial_x q)^2 / \epsilon)^{\kappa/2}} = \frac{l_0}{(1 + l_0^2 / l_{RH}^2)^{\kappa/2}}
\]

\[
\begin{align*}
\text{Strong mixing } (l_{RH} > l_0) : & \quad l_{mix} \approx l_0 \text{ (Weak mean PV gradient)} \\
\text{Weak mixing } (l_0 > l_{RH}) : & \quad l_{mix} \approx l_0^{1-\kappa l_{RH}^2} \text{ (Strong PV gradient)}
\end{align*}
\]

\( l_{mix} \) (hybrid length scale) sets the scale of zonal flow.

What is the effect of stochastic fields on staircases?
Main effect of diffusivity $D_n$ and $\chi$

For $\alpha_{DW}$ (a measurement of the resistive diffusion rate in the parallel direction) > 1 in H-W regime:

Density diffusivity:

$D_n \approx \frac{l_{mix}^2 \epsilon}{\alpha_{DW}}$

Resistive diffusion rate:

$\alpha_{DW} = \frac{k_\parallel^2 v_{the}^2}{\nu}$

Stochastic Fields Effect

$k_\parallel = k \cdot \hat{b}_0 \approx \frac{1}{Rq} + \frac{b_\perp}{Rq} \cdot \frac{1}{l_{mix}} + \frac{b_\perp}{l_{mix}}$

$D_n \approx \frac{l_{mix}^2 \epsilon v / v_{the}^2}{(\frac{1}{Rq})^2 + (\frac{b}{l_{mix}})^2}$

Same for $\chi$ (or $D_{PV}$ in this case).

Competition btw $\frac{1}{Rq}$ v.s. $\frac{b_\perp}{l_{mix}}$ gives

$Ku_{mag} = bRq/l_{mix}$

$Ku_{mag} = Ku_{mag}(l_{mix})$

The mixing length is not likely affected by $b^2$.

A change of scale selection or staircase corrugation requires $Ku_{mag} \geq 1$. 
Why we study neutrals in fusion devices?

Neutrals influx

Charge exchange damping or friction

Predator prey model:
- Predator: zonal flow
- Prey: turbulence

Increase the flow shear threshold

Conventional wisdom:
1. Neutrals damp the poloidal (zonal) flow, increase fluctuation level.
2. Increase effective fluctuation energy.

The neutral density is important in studies of L-H transition power threshold in fusion device.

Non-trivial:
2. Entrainment of neutral particles.

(Carreras et al., PoP 3, 4106 (1996))
Drift–Rossby Waves and Zonal Flow

(Simplest possible model)

• Evolution of zonal flow is from the competition btw the Reynolds and Maxwell Stress:

\[
\frac{\partial}{\partial t}\langle u_y \rangle = -\frac{\partial}{\partial x}\left(\frac{\langle u_x u_y \rangle}{\mu_0 \rho} - \frac{\langle B_x B_y \rangle}{\eta}\right)
\]

Zonal flow evolution:

- Reynolds stress
- Maxwell stress

Induction equation:

\[
\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u}_i \times \vec{B}) + \eta \nabla^2 \vec{B}
\]

η: increase the zonal flow 

\[\frac{B_0^2}{\eta}\rightarrow\text{regulates Mag. And Reynolds stress competition}\]

→ Zel’dovich Theorem

What if we add neutrals? Neutrals will enter this competition and have a production on zonal flow.
Multi-Fluid Model

- Equation of motion of three species on $\beta$-plane (rel. to planetary system):
  \[\rho_n \left( \frac{\partial}{\partial t} + u_n \cdot \nabla \right) u_n = -\nabla p_n^* - \rho_n \nu_{ni} (u_n - u_i) - \rho_n \nu_{ne} (u_n - u_e) - 2\rho_n \Omega \times u_n + \mu_n \nabla^2 u_n\]
  \[\text{Coriolis force}\]

- Ion and neutral field equation:
  \[\rho_i \left( \frac{\partial}{\partial t} + u_i \cdot \nabla \right) u_i = -\nabla p_i^* - \rho_i \nu_{in} (u_i - u_n) - \rho_i \nu_{ie} (u_i - u_e) + ne(E + u_e \times B) - 2\rho_i \Omega \times u_i + \mu_i \nabla^2 u_i\]

- Strength of coupling:
  \[\rho_i \nu_{in} = \rho_n \nu_{ni} \equiv \alpha \rho_i \rho_n\]

- Ion and neutral field equation:
  \[\rho_i \left( \frac{\partial}{\partial t} + u_i \cdot \nabla \right) u_i = -\nabla p_i^* + \frac{1}{\mu_0} (B \cdot \nabla) B + \rho_i \nu_{in} (u_n - u_i) - 2\rho_i \Omega \times u_i + \mu_i \nabla^2 u_i\]

- In high coupling regime, ion has force balance:
  \[\mathbf{J} \times \mathbf{B} + \text{drag} = 0\]
  
  So the neutrals will also feel the $\mathbf{J} \times \mathbf{B}$ force.

Ignore coupling btw e-i and e-n $m_e \nu_{en} \ll m_i \nu_{in}$
Electromagnetism: Ambipolar Diffusion

Key question:

How does the \textbf{ambipolar diffusion} (non-linear B-diffusivity from neutral effect) affect the zonal flow? What’s the effect on magnetic perturbation $\tilde{B}$.

Ohm’s Law:

$$E = \eta J - u_i \times B - \frac{m_e \nu_n}{e} (u_i - u_n),$$

JxB due to the ion-neutral coupling:

$$u_{i,\text{drag}} = u_i - u_n = \frac{J \times B}{\alpha \rho_i \rho_n}$$

B field frozen into the neutrals, but now the magnetic diffusion is \textbf{ambipolar diffusion $\eta_{am}$}.
Linear Modes of Coupled System

- In the linear theory, we obtain scalar equations:

Vorticity equation:
\[
\begin{align*}
\rho_i \left( \frac{\partial}{\partial t} + \langle u_i \rangle \cdot \nabla \right) \tilde{\zeta}_i &= - \rho_i \tilde{\mu}_{i,y} \frac{\partial}{\partial y} \langle \zeta_i \rangle + B_{0,x} \frac{\partial \tilde{J}_i}{\partial x} + \tilde{B}_y \frac{\partial \langle J_i \rangle}{\partial y} + \alpha \rho_i \rho_n (\tilde{\zeta}_n - \tilde{\zeta}_i) + \mu_i \nabla^2 \tilde{\zeta}_i,
\end{align*}
\]

Induction equation:
\[
\begin{align*}
\rho_n \left( \frac{\partial}{\partial t} + \langle u_n \rangle \cdot \nabla \right) \tilde{\zeta}_n &= - \rho_n \tilde{\mu}_{n,y} \frac{\partial}{\partial y} \langle \zeta_n \rangle + \alpha \rho_i \rho_n (\tilde{\zeta}_i - \tilde{\zeta}_n) + \mu_n \nabla^2 \tilde{\zeta}_n
\end{align*}
\]

\[
\tilde{\zeta}_{n,k} = \frac{D_n}{D} = \frac{S_i \rho_i (-i \omega_i + \nu_i k^2) + \alpha \rho_i \rho_n (S_1 + S_2)}{\rho_i \rho_n (-i \omega_i + \nu_i k^2) (-i \omega_n + \nu_n k^2) + \alpha \rho_i \rho_n (\rho_i (-i \omega_i + \nu_i k^2) + \rho_n (-i \omega_n + \nu_n k^2)) + \frac{i \alpha \rho_n}{\mu_0} \frac{B_0^2 k_i}{\omega + i \eta_{eff} k^2}}.
\]

- We obtain a critical dimensionless parameter:
\[
\gamma = \frac{(\alpha \rho_i) \omega_{eff}^2}{\omega + i \eta_{eff} k^2} \cdot \tau_{eddy} \cdot \frac{1}{\omega_{ci}}
\]

\[
\omega_{eff}^2 = \frac{B_0^2 k_i^2}{\omega + i \eta_{eff} k^2}
\]

\[
\left\{
\begin{array}{ll}
\gamma \ll 1 & \text{weak coupling regime} \\
\gamma \gg 1 & \text{strong coupling regime}
\end{array}
\right.
\]
Important Time Scales

- We consider rates:
  \[ \gamma = \frac{\alpha \rho_i \omega_{eff}^2}{\omega + i n_{eff}k^2} \cdot \frac{1}{\tau_{eddy}} \]

- A simplified coupling parameter:
  \[ \tilde{\alpha} \equiv \frac{\tau_{eddy}}{\tau_{coll}} = \frac{l_{\perp} \rho_{tot} \alpha}{\tilde{u}} \]

- Dynamics of neutrals and ions:
  Weak coupling:
  Neutrals and ions not well-coupled \( \rightarrow \) Two fluid.
  Strong coupling:
  Neutrals couple to ions and behaves as one MHD fluid.

Neutrals couple to ions and behaves as one MHD fluid.
• The induction equation, and consider strong coupling regime: $\alpha \rho_i \gg 1/\tau_{eddy}$.

JxB-drag balance on ion: $-\mathbf{J} \times \mathbf{B} = f_{d,i}$

$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u}_i \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}.
$\n
$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u}_n \times \mathbf{B}) + \nabla \times \left( \frac{\mathbf{J} \times \mathbf{B}}{\alpha \rho_i \rho_n} \times \mathbf{B} \right) + \eta \nabla^2 \mathbf{B},$

• The magnetic potential equation becomes

$\eta \rightarrow \eta + \eta_{am}$

$\frac{\partial A_z}{\partial t} + (\mathbf{u}_n \cdot \nabla) A_z = \frac{1}{\mu_0 \alpha \rho_i \rho_n} \nabla \cdot \left[ \eta_{am} \cdot \nabla A_z \right] + \eta \nabla^2 A_z + C$

$= \frac{1}{\mu_0 \alpha \rho_i \rho_n} \left( \frac{\partial}{\partial x} \nabla^2 A_z + \frac{\partial}{\partial y} \right) \left( \begin{array}{c} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \\ \mathcal{I} \end{array} \right) A_z + \eta \nabla^2 A_z + C$

Resistivity correction due to the ion-neutral drag

Ambipolar diffusion tensor for $A$:

$\eta_{am} = \left( \begin{array}{c} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \end{array} \right) = \left( \begin{array}{c} \frac{1}{2}(B_y^2 - B_x^2) \\ -B_x B_y \\ -B_x B_y \\ \frac{1}{2}(B_x^2 - B_y^2) \end{array} \right)$

Additional terms:

$C = \frac{1}{\mu_0 \alpha \rho_i \rho_n} \left[ -\frac{B^2}{2} \nabla^2 A_z + B_x^2 \frac{\partial^2}{\partial x^2} A_z + B_y^2 \frac{\partial^2}{\partial y^2} A_z + 2B_x B_y \frac{\partial^2}{\partial x \partial y} A_z \right]$
Ions and neutrals evolve separately but have a weak mutual drag on each other.

\[
(\omega - \omega_R + ik^2)(\omega + i\eta_{\text{eff}}k^2) = \frac{\rho_i}{\rho_{\text{tot}}} \frac{B_0^2k_x^2}{\omega + i\eta_{\text{eff}}k^2} \eta_{\text{eff},k} \equiv \eta + \frac{B_0^2}{\mu_0 \alpha \rho_i \rho_n}
\]

In strong collision regime, the ions and neutrals are strongly coupled and behave like single MHD fluid.

By solving the linear equations for ions, neutrals, and magnetic potential, we have

- **Strong coupling regime** $\gamma >> 1$ ($1/\tau_{\text{eddy}} \ll \alpha \rho_i \ll \omega_{\text{eff}} \ll \omega_{\text{ci}}$):

\[
(\omega - \omega_R + ik^2)(\omega + i\eta_{\text{eff}}k^2) = \frac{\rho_i}{\rho_{\text{tot}}} \frac{B_0^2k_x^2}{\omega + i\eta_{\text{eff}}k^2} \eta_{\text{eff},k} \equiv \eta + \frac{B_0^2}{\mu_0 \alpha \rho_i \rho_n}
\]

- **Weak coupling regime** $\gamma << 1$ ($\alpha \rho_i \ll 1/\tau_{\text{eddy}} \ll \omega_{\text{eff}} \ll \omega_{\text{ci}}$):

\[
(\omega_i - \omega_R + ik^2 + i\alpha \rho_n)(\omega_n - \omega_R + ik^2 + i\alpha \rho_i) = -\alpha^2 \rho_i \rho_n
\]

Ions and neutrals evolve separately but have a weak mutual drag on each other.

But contains **ion-neutral coupling effect**.

The results show that in strong collision regime, the ions and neutrals are strongly coupled and behave like a single MHD fluid. In weak coupling regime, the ion-neutral coupling effect is present, and the equations for ions and neutrals evolve separately but have a weak mutual drag on each other.

(But contains ion-neutral coupling effect.)
How to Determine Maxwell Stress?

- Zeldovich Theorem:

\[ Rm \simeq \frac{\langle \widetilde{B}^2 \rangle}{B_0^2} = \frac{\langle \widetilde{u}^2 \rangle \tau_c}{\eta} \]

\[ \langle \widetilde{B}^2 \rangle = \langle \widetilde{u}^2 \rangle \tau_c \frac{B_0^2}{\eta} \]

\[ B_0^2/\eta \] is a regulator in the competition btw Reynolds and Maxwell stress competition.

Magnetic diffusion \( \eta \) damps the pert. magnetic field \( \widetilde{B} \).

Evolution of zonal flow

\[ \frac{\partial \langle u_y \rangle}{\partial t} = - \frac{\partial}{\partial x} \left( \langle \widetilde{u}_x \widetilde{u}_y \rangle - \frac{\langle \widetilde{B}_x \widetilde{B}_y \rangle}{\mu_0 \rho} \right) \]

Mean diffusivity \( \eta \) damps while \( B_0^2 \) increase the Maxwell stress \( \langle \widetilde{B}_x \widetilde{B}_y \rangle \). And hence \( B_0^2/\eta \) regulates the growth of zonal flow.

What is the effect of neutrals on the Zel’dovich Theorem?
Modified Zel’dovich Theorem

• Without neutrals, magnetic stress is proportional to $B_0^2/\eta$:
  \[
  \langle \vec{B}^2 \rangle = \langle \vec{u}^2 \rangle \tau_c \frac{B_0^2}{\eta} \propto \frac{B_0^2}{\eta}
  \]

• Now, consider the neutral-ion drag effect:
  \[
  \eta \rightarrow \eta + \frac{\eta_{am}}{R_m \rightarrow R_m + R_{am}}
  \]
  \[
  R_{m,\text{tot}} = R_m + R_{am} = \frac{\langle \vec{B}^2 \rangle}{B_0^2} \frac{\langle \vec{B}^2 \rangle}{\eta \mu_0 \alpha \rho_i \rho_n} \frac{B_0^2}{\langle \vec{B}^2 \rangle} + \frac{\langle \vec{B}^2 \rangle}{R_m \rightarrow R_{am}} \frac{\langle \vec{B}^2 \rangle}{B_0^2}
  \]

Assumption:
  \[
  \langle \vec{B}^4 \rangle \simeq \langle \vec{B}^2 \rangle \langle \vec{B}^2 \rangle
  \]

\[
\eta \nabla^2 \vec{B} \text{ v.s. } \nabla \times (\vec{J} \times \vec{B})
\]

• In the limit where ambipolar diffusion dominated ($R_m \ll R_{am}$) we have:
  \[
  R_{m,\text{tot}} \simeq R_{am} = \frac{\langle \vec{B}^2 \rangle}{\eta \mu_0 \alpha \rho_i \rho_n} \frac{B_0^2}{\langle \vec{B}^2 \rangle} = \frac{\langle \vec{u}^2 \rangle \tau_c}{\eta} \eta 
  \]

Maxwell Stress intensity:
  \[
  \langle \vec{B}^2 \rangle = \left( \langle \vec{u}^2 \rangle \tau_c \mu_0 \rho_i \frac{B_0^2}{A} \right)^{1/2} \propto \frac{B_0}{A^{1/2}}
  \]

<table>
<thead>
<tr>
<th>Change in Scaling</th>
<th>Maxwell Stress Intensity</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Neutrals</td>
<td>$\langle \vec{B}^2 \rangle \propto \frac{B_0^2}{\eta}$</td>
</tr>
<tr>
<td>Ambipolar diffusion effect</td>
<td>$\langle \vec{B}^2 \rangle \propto \frac{B_0}{A^{1/2}}$</td>
</tr>
</tbody>
</table>

...alters the competition

The magnetic stress effect on zonal flow generation with neutrals is less sensitive to $B_0$. 
Conclusions

- We study Non-trivial neutral effect on DW-ZF turbulence: Ambipolar diffusion

- We derive the **key parameter** $\gamma$ for the Drift-Alfvén + Neutral effect:

\[
\gamma = \frac{(\alpha\rho_i)\omega_{\text{eff}}^2}{\omega + i\eta_{\text{eff}}k^2 \cdot \tau_{\text{eddy}}} \cdot \frac{1}{\omega_{ci}}
\]

- We study the Drift-Alfvén wave with neutrals and found—
  - In strong coupling regime: **one MHD fluid**.
  - In weak coupling regime: **two fluid**.

- **Modified Zel’dovich Theorem.** We derive the key parameter that **regulates** Maxwell and Reynolds stress competition:

\[
\begin{align*}
\langle \vec{B}^2 \rangle & \propto B_0^2/\eta \quad \text{(original)} \\
\langle \vec{B}^2 \rangle & \propto B_0/A^{1/2} \quad \text{(ambipolar effect)}
\end{align*}
\]

Future Works

- Calculate $\vec{B}$ evolution for arbitrary $\gamma \rightarrow$ Pouquet + neutrals (clarify asymptotic regime).

- Study the physics of neutral entrainment.