A Reduced Model of ExB and PV Staircase Formation and Dynamics

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Outline

• Basic Ideas: transport bifurcation and ‘negative diffusion’ phenomena
  – Inhomogeneous mixing: times and lengths
  – Feedback loops
  – Simple example

• Inhomogeneous mixing in space:
  – Staircase models in QG and drift wave system
  – Models, background
  – QG staircase: model and results
  – Aside: on PV mixing
  – Model for Hasegawa-Wakatani system – 2 field, and intro

• Some Lessons and Conclusions
I) Basic Ideas:
Transport bifurcations and ‘negative diffusion’ phenomena
Transport Barrier Formation (Edge and Internal)

- Observation of ETB formation (L→H transition)
  - THE notable discovery in last 30 yrs of MFE research
  - Numerous extensions: ITB, I-mode, etc.
  - Mechanism: turbulence/transport suppression by ExB shear layers generated by turbulence

- Physics:
  - Spatio-temporal development of bifurcation front in evolving flux landscape
  - Cause of hysteresis, dynamics of back transition

- Fusion:
  - Pedestal width (along with MHD) → ITER ignition, performance
  - ITB control → AT mode
  - Hysteresis + back transition → ITER operation
Why Transport Bifurcation?  BDT ‘90, Hinton ‘91

• Sheared $V_{E \times B}$ flow quenches turbulence, transport $\Rightarrow$ intensity, phase correlations

• Gradient + electric field $\Rightarrow$ feedback loop

i.e. $\vec{E} = \frac{\nabla P_i}{nq} - \vec{V} \times \vec{B} \Rightarrow V'_E = V'_E(\nabla T)$

$\Rightarrow$ minimal model  $Q = - \frac{\chi(\nabla T) \nabla T}{\left[ 1 + \left( \frac{V'_E}{\omega_{eff}} \right)^2 \right]^n} - \chi_{neq} \nabla T$

$\n \equiv$ quenching exponent
• Feedback:

\[ Q \uparrow \rightarrow \nabla T \uparrow \rightarrow V'_E \uparrow \rightarrow (\tilde{n}/n)^2, \chi_T \downarrow \]

\[ \rightarrow \nabla T \uparrow \rightarrow \ldots \]

• Result:

1\textsuperscript{st} order transition (L→H):

![Heat flux vs \(\nabla T_i\)]

![T profiles]

a) L-mode
b) H-mode
• S curve $\Rightarrow$ “negative diffusivity” i.e. $\frac{\delta Q}{\delta \nabla T} < 0$

• Transport bifurcations observed and intensively studied in MFE since 1982 yet:

→ Little concern with staircases

→ Key questions:

1) Might observed barriers form via step coalescence in staircases?

2) Is zonal flow pattern really a staircase (see GDP)?
Staircase in Fluids

• What is a staircase? – sequence of transport barriers

• Cf Phillips’72:

\[ \delta \Gamma_b / \delta Ri < 0 ; \text{ flux dropping with increased gradient} \]

\[ \Gamma_b = -D_b \nabla b, Ri = g \nabla b / (\nu')^2 \]

• Obvious similarity to transport bifurcation
In other words:

Gradient $b$

Intensity $I$

Some resemblance to Langmuir turbulence
i.e. for Langmuir: caviton train

$\delta n / n \approx -\varepsilon$

Configuration instability of profile + turbulence intensity field

Buoyancy profile

Intensity field

$\varepsilon$

$\delta n$

$\varepsilon$

$\delta n$

End state of modulational instability !?
• The physics: Negative Diffusion (BLY, ‘98)

• Instability driven by local transport bifurcation

  ➔ • \( \delta \Gamma_b / \delta \nabla b < 0 \)  
  ➔ ‘negative diffusion’  
  ➔ Feedback loop \( \Gamma_b \downarrow \rightarrow \nabla b \uparrow \rightarrow I \downarrow \rightarrow \Gamma_b \downarrow \)  
  ➔ Critical element: \( l \rightarrow \) mixing length

“H-mode” like branch
(i.e. residual collisional diffusion)
is not input
- Usually no residual diffusion
- ‘branch’ upswing \( \rightarrow \) nonlinear processes (turbulence spreading)
- If significant molecular diffusion \( \rightarrow \) second branch
The Critical Element: **Mixing Length**

- Sets range of inhomogeneous mixing

$$\frac{1}{l^2} = \frac{1}{l_0^2} + \frac{1}{l_{oz}^2}$$

- $l_{oz} \sim$ Ozmidov scale, smallest ‘stratified scale’

\[ \leftrightarrow \text{ balance of buoyancy and production} \]

- $\frac{1}{l_{oz}} \approx \left( \frac{b_z}{e} \right)^{\frac{1}{2}} \Rightarrow b_z$ dependence is crucial for inhomogeneous mixing

- Feedback loop: $b_z \uparrow \rightarrow e \downarrow \rightarrow l \downarrow$
• **A Few Results** → $\nabla \rho$ staircases

Plot of $b_z$ (solid) and $e$ (dotted) at early time. Buoyancy flux is dashed → near constant in core.

Later time → more akin expected "staircase pattern". Some condensation into larger scale structures has occurred.
II) Inhomogeneous Mixing in Space: Staircase Models in QG and Drift Wave Systems
**Drift wave model – Fundamental prototype**

- **Hasegawa-Wakatani**: simplest model incorporating instability

\[
V = \frac{c}{B} \hat{z} \times \nabla \phi + V_{\text{pol}}
\]

\[
J_\perp = n |e| V'_\text{pol} \quad \eta J_\parallel = -\nabla \phi + \nabla \cdot \mathbf{p}_e
\]

\[
\nabla_\perp \cdot J_\perp + \nabla_\parallel J_\parallel = 0 \quad \rightarrow \text{vorticity: } \rho_s^2 \frac{d}{dt} \nabla^2 \phi = -D_\parallel \nabla^2 (\phi - n) + v \nabla \cdot (\nabla \nabla \phi)
\]

\[
\frac{dn_e}{dt} + \frac{\nabla_\parallel J_\parallel}{-n_0 |e|} = 0 \quad \rightarrow \text{density: } \frac{dn}{dt} = -D_\parallel \nabla^2 (\phi - n) + D_0 \nabla^2 n
\]

\[ \rightarrow \text{PV conservation in inviscid theory } \frac{d}{dt} (n - \nabla^2 \phi) = 0 \]

\[ \rightarrow \text{PV flux = particle flux + vorticity flux} \]

\[ \rightarrow \text{zonal flow being a counterpart of particle flux} \]

- **Hasegawa-Mima** (\( D_\parallel k_\parallel^2 / \omega >> 1 \rightarrow n \sim \phi \))

\[
\frac{d}{dt} \left( \phi - \rho_s^2 \nabla^2 \phi \right) + v_\star \partial_y \phi = 0
\]

\[
\text{QL: } \frac{\partial}{\partial t} \langle n \rangle = -\frac{\partial}{\partial r} \langle \tilde{v} \cdot \tilde{n} \rangle
\]

\[
\rightarrow \frac{\partial}{\partial t} \langle \nabla^2 \phi \rangle = -\frac{\partial}{\partial r} \langle \tilde{v} \cdot \nabla^2 \phi \rangle = -\frac{\partial^2}{\partial r^2} \langle \tilde{v} \cdot \tilde{v}_b \rangle
\]
**Key:** PV conservation \( dq/dt = 0 \)

<table>
<thead>
<tr>
<th>GFD: Quasi-geostrophic system</th>
<th>Plasma: Hasegawa-Wakatani system</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q = \nabla^2 \psi + \beta y )</td>
<td>( q = n - \nabla^2 \phi )</td>
</tr>
<tr>
<td>( \tilde{n} \sim \frac{e\phi}{T} )</td>
<td>( \phi ) density (guiding center) ( \tilde{n} \sim \frac{e\phi}{T} )</td>
</tr>
<tr>
<td>Physics: ( \Delta y \to \Delta(\nabla^2 \psi) )</td>
<td>Physics: ( \Delta r \to \Delta n \to \Delta(\nabla^2 \phi) ) ZF!</td>
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**Charney-Haswaga-Mima equation**

\[
\begin{align*}
 n &= n_0 + \tilde{n} \\
 \tilde{n} &\sim \frac{e\phi}{T} \\
 H-W &\to H-M: \quad \frac{1}{\omega_{ci}} \frac{\partial}{\partial t} \left( \nabla^2 \phi - \rho_s^{-2} \phi \right) - \frac{1}{L_n} \frac{\partial}{\partial y} \phi + \frac{\rho_s}{L_n} J(\phi, \nabla^2 \phi) = 0 \\
 Q-G: \quad &\frac{\partial}{\partial t} \left( \nabla^2 \psi - L_d^{-2} \psi \right) + \beta \frac{\partial}{\partial x} \psi + J(\psi, \nabla^2 \psi) = 0
\end{align*}
\]
Staircase in QG Turbulence: A Model

- PV staircases observed in nature, and in the unnatural
- Formulate ‘minimal’ dynamical model?! (n.b. Dritschel-McIntyre 2008 does not address dynamics)

Observe:

- 1D adequate: for ZF need ‘inhomogeneous PV mixing’ + 1 direction of symmetry. Expect ZF staircase
- Best formulate intensity dynamics in terms potential enstrophy $\epsilon = \langle \tilde{q}^2 \rangle$
- Length? : $\Gamma_q \partial \langle q \rangle / \partial y \sim \tilde{q}^3$ (production-dissipation balance)
- $\rightarrow l \sim \langle \tilde{q}^2 \rangle^{1/2} / \partial \langle q \rangle / \partial y \sim l_{Riines}$ (i.e. $\omega_{Rossby} \sim k \tilde{v}$)
**Model:** \( \Gamma_q = \langle \tilde{v}_y q \rangle = -D \partial \langle q \rangle / \partial y \) is fundamental quantity

\( \Rightarrow \) **Mean:** \( \partial_t \langle q \rangle = \partial_y D \partial_y \langle q \rangle \)

\( \Rightarrow \) **Potential Enstrophy density:** \( \partial_t \epsilon - \partial_y D \partial_y \epsilon = D \left( \partial_y \langle q \rangle \right)^2 - \frac{\epsilon^2}{2} + F \)

Where:

\[
\frac{1}{l^2} = \frac{1}{l_f^2} + \frac{1}{l_{Rh}^2}
\]

\( D \sim l^2 \sqrt{\epsilon} \) (dimensional)

\[
\partial_t \left( \frac{\langle q \rangle^2}{2} + \epsilon \right) = 0, \text{ to forcing, dissipation}
\]

\[
l_{Rh}^2 = \epsilon / \left( \partial_y \langle q \rangle \right)^2 \]

\( D_{spr} \approx D_{PV} \)
Alternative Perspective:

• Note: \[ l^2 = \frac{1}{1 + 1/l_{R_h}^2} \rightarrow \frac{1}{1 + \langle q \rangle^2 / \epsilon} \quad (l_f \sim 1) \]

• Reminiscent of weak turbulence perspective:

\[ D = D_{pv} = \sum_{\vec{k}} \frac{\langle \tilde{V}^2 \rangle \Delta \omega_{\vec{k}}}{\omega_{\vec{k}}^2 + \Delta \omega_{\vec{k}}^2} \]

\[ \omega_{\vec{k}} = -k_x \langle q \rangle / k^2 \]

\[ \Delta \omega_{\vec{k}} \approx k \tilde{V}_{\vec{k}} \]

Ala’ Dupree’67:

\[ D_{pv} \approx \frac{1}{k^2} \left( \sum_{\vec{k}} k^2 \langle \tilde{V}^2 \rangle_{\vec{k}} - \frac{k^2 (\langle q \rangle')^2}{\langle k^2 \rangle^2} \right)^{1/2} \]

Steeper \langle q \rangle’ quenches diffusion \( \rightarrow \) barrier via PV gradient feedback
\[ D_{pv} \approx \frac{l_0^2 \epsilon^{-2}}{1 + \frac{l_0^2}{\epsilon} (\langle q \rangle')^2} \]

- \( \omega \) vs \( \Delta \omega \) dependence gives \( D_{pv} \) roll-over with steepening
- Rhines scale appears naturally, in feedback strength
- Recovers effectively same model

Physics:

1. "Rossby wave elasticity" (MM) \( \rightarrow \) steeper \( \langle q \rangle' \) \( \rightarrow \) stronger memory (i.e. more 'waves' vs turbulence)
2. Distinct from shear suppression \( \rightarrow \) interesting to dis-entangle
Aside

- What of wave momentum? Austauch ansatz
  Debatable (McIntyre) - but \( l_m \delta \) (?)...

- PV mixing \( \leftrightarrow D \partial_y \langle q \rangle \)
  So \( \rightarrow \langle \tilde{V} \tilde{q} \rangle \rightarrow \partial_y \langle \tilde{V}_y \tilde{V}_x \rangle \rightarrow \text{R.S.} \)

- But:
  \[ \text{R.S.} \leftrightarrow \langle k_x k_y \rangle \leftrightarrow V_{gy}E \]

\[ \langle q \rangle' \uparrow \rightarrow l \downarrow \rightarrow \epsilon \downarrow \rightarrow D \downarrow \]

(Production)

- Equivalent!
- Formulate in terms mean, Pseudomomentum?
* - Red herring for barriers \( \rightarrow l_m \delta \) quenched
Numerical Results:
Analysis of QG Model
• Re-scaled system

\[ Q_t = \partial_y \frac{\varepsilon^{1/2}}{(1 + Q_y^2 \varepsilon)^\kappa} Q_y + D_{neo} Q_{yy} \quad \text{for mean} \]

\[ \varepsilon_t = \partial_y \frac{\varepsilon^{1/2}}{(1 + Q_y^2 / \varepsilon)^\kappa} Q_y + L^2 \left\{ \frac{Q_y^2}{(1 + Q_y^2 / \varepsilon)^\kappa} - \frac{\varepsilon}{\varepsilon_0} + 1 \right\} \varepsilon^{1/2} \text{ for P.E.} \]

• Note:

  – Quenching exponent \( \kappa = 2 \) for saturated modulational instability

  – Potential enstrophy conserved to forcing, dissipation, boundary

  – System size \( L \) \( \rightarrow \) strength of drive
• Weak Drive $\Rightarrow$ 1 step staircase

- 1 step staircase forms
- Small scales not evident
- Dirichlet B.C.’s

\[ \text{Initial } \Delta Q \]
- Increased Drive ➔ Multi-step structure

- Multiple steps
- Steps move
- Some hint of step condensation at foot of Q profile

* - End state: barrier on LHS, step on RHS

⇒ Suggestive of barrier formation by staircase condensation
• $\nabla Q$ plot reveals structure and scales involved
  
  - FW HM max, min capture width of steep gradient region
  - Step width - minimum
• Mergers occur

- Same drive as before
- Staircase smooths
• $\nabla Q$ plot of Mergers

- Can see region of peak $\nabla Q$ expanding
- Coalescence of steps occurs
- Some evidence for “bubble competition” behavior
Mergers for yet stronger drive

- Mergers form broad barrier region at foot of Q profile
- Extended mean barrier emerging from step condensation
- $\nabla Q$ evolution in condensation process → same scale
- Note broadening of high $\nabla Q$ region near boundary
• Staircase Barrier Structure vs Drive

- $L^2 \uparrow \Rightarrow$ increasing drive
- FW HM max increases with $L^2$, so
- Width of barrier expands as $L^2$ increases
• Staircase Step Structure vs Drive

- Step width decreases and \( \sim \) saturated as \( L^2 \) increases
- Min, max converge
• More interesting model...

– From Hasegawa-Wakatani:

\[
\frac{d}{dt} \nabla^2 \phi = D_\parallel \nabla_\parallel^2 (n - \phi) + \nu_0 \nabla^2 \nabla^2 \phi
\]

\[
\frac{d}{dt} n = D_\parallel \nabla_\parallel^2 (n - \phi) + D_0 \nabla^2 n
\]

\[
\frac{\partial}{\partial t} + \vec{V} \cdot \nabla = \frac{d}{dt}; \quad \frac{k_\parallel^2 D_\parallel}{\omega} > 1; \quad \nu_0 > D_0
\]

– Evident that mean-field dynamics controlled by:

– \( \Gamma_n = \langle \vec{v}_r \vec{n} \rangle \rightarrow \) particle flux

– \( \Gamma_u = \langle \vec{v}_r \nabla^2 \vec{\phi} \rangle \rightarrow \) vorticity flux

\[\text{Relation of } \nabla n \text{ corrugations and shear layers}\]
• **K-ε Model**

\[
d_t n + \partial_x \Gamma_n = D_0 \partial_x^2 n
\]

\[
d_t u + \partial_x \Gamma_u = \nu_0 \partial_x^2 u
\]

\[
\varepsilon = \text{Pot Enstr} = \langle (\tilde{n} - \nabla^2 \phi)^2 \rangle
\]

\[
d_t \varepsilon + \partial_x \Gamma_\varepsilon = -(\Gamma_n - \Gamma_u)(\partial_x n - \partial_x u) - \varepsilon^2 + f
\]

- Total P.E. conserved, manifestly
- \( \Gamma_\varepsilon = \langle \nu_r \varepsilon \rangle \rightarrow \text{spreading flux} \)
- Forcing as linear stage irrelevant

\[
u = \nabla^2 \phi
\]
• Fluxes $\Gamma_n, \Gamma_u$

  – Could proceed as before ➔ PV mixing with feedback for steepened $\nabla q$

  – i.e. $\Gamma_n = -D_T \partial_x n$

  – $\Gamma_u = -D_T \partial_x u$

  – $D_T \sim l_m^2 \kappa (\varepsilon)^{1/2}$, with $1/l_m \kappa = 1/l_0^2 + 1/l_{Rh}^2$

  – $l_m^2 \kappa = l_0^2 \varepsilon / [\varepsilon + l_0^2 (\partial_r (n - u))^2]$
• Fluxes $\Gamma_n, \Gamma_u$

  – Could proceed as before $\Rightarrow$ PV mixing with feedback for steepened $\nabla q$

  – i.e. $\Gamma_n = -D_T \partial_x n$

  – $\Gamma_u = -D_T \partial_x u$

  – $D_T \sim l^2_{m, \kappa} (\varepsilon)^{1/2}$, with $1/l_{m, \kappa} = 1/l_0^2 + 1/l_{Rh}^2$

  – $l^2_{m, \kappa} = l_0^2 \varepsilon /[\varepsilon + l_0^2 (\partial_r (n - u))^2]$

  $\Rightarrow$ Feedback by $\nabla q$ steepening and reduced $D_T$ etc

  $\Rightarrow$ Barrier structure?!
• More interesting: As CDW turbulence is wave turbulence, use mean field/QL theory as guide to model construction

• For QL theory, see Ashourvan, P.D., Gurcan (2015)

• Simplified:

(a) \[ \Gamma_n \approx -D_n \partial_x n \]

\[ D_n = -\langle \tilde{v}_r^2 \rangle \tau_c, \quad \tau_c^{-1} = \langle k_{||}^2 D_{||} \rangle \]

• Key: electron response laminar

• Neglected weak particle pinch
(b) $\Gamma_u = -\chi_y \nabla u + \Pi^{\text{res i l}}$

$$\chi_y \approx \langle \tilde{v}_r^2 \rangle (\gamma_k / (\omega - k_\theta v_\theta)^2) \rightarrow \langle \tilde{v}_r^2 \rangle / |u|$$

$$\Pi^{\text{res i l}} \approx \Gamma_u / n_0 - \chi_y v_d \quad v_d = -\partial_r n$$

And $\langle \tilde{v}_r^2 \rangle \sim l_m^2 \kappa \varepsilon$

N.B.: In QLT, $D_n \neq \chi_y$

• Interesting to note varied roles of:
  
  – Transport coefficients $D_n, \chi$
  
  – Non-diffusive stress
  
  – Length scale, suppression exponent
  
  – Intensity dependence
• Studies so far:
  
  - $D_n = D_u$ with \( \nabla q \) feedback as in QG via \( l_m \kappa \)
    
    \( \kappa = 1, 2 \)
  
  - QL model with \( l_m \kappa (\nabla q) \)
    
    \( \kappa = 1, 2 \)
  
• \( \rightarrow \) these constitute perhaps the simplest cases conceivable...
• \( l_{m,k}^{-2} = l_0^{-2} + l_{Rh}^{-2} \), \( D_n = D_u \rightarrow \text{mixing} \)

• Barrier and irregular staircase form

• Shear layer self-organizes near boundary
• $l_{m}^{-2} = l_{0}^{-2} + l_{Rh}^{-2}$, $D_n = D_u \to$ mixing

$\kappa = 2$

• Density and vorticity staircase form
• Regular in structure
• Condensation to large steps, barrier forms
• Quasilinear with $l_m \kappa$ feedback

\[ D_n \neq \chi_y, \quad \Pi_{\text{resil}} \neq 0 \]

$\kappa = 1$

• Single barrier.....
• Quasilinear with $l_m \dot{k}$ feedback

$$D_n \neq \chi_y, \quad \Pi_{resil} \neq 0$$

$$\kappa = 2$$

• Density staircase forms and condenses to single edge transport barrier
• Quasilinear with $l_m i \kappa$ feedback

$$D_n \neq \chi_y, \quad \Pi_{\text{resid}} \neq 0$$

$$\kappa = 2$$
• Process of mergers
• What did we learn?

  – Absolutely simplest model recovers staircase

  – Boundary shear layer forms spontaneously

  – * Mergers and propagation down density gradient form macroscopic edge transport barrier from mesoscopic staircase steps!

  – $l_m \xi$ (gradient) feedback seems essential
Discussion

• “Negative diffusion” / clustering instability common to Phillips, QG and DW transport bifurcation and Jam mechanisms:
  – $\delta \Gamma_b / \delta \nabla b < 0 \rightarrow \Gamma_b$ nonlinearity
  – $\delta \Gamma_q / \delta \nabla q < 0 \rightarrow \Gamma_q(\nabla q)$ nonlinearly

• Key elements are:
  – Inhomogeneity in mixing: length scale $l_m \kappa$, and its $\nabla q$ dependence, $\tau_d$, etc
  – Feedback loop structure

* • Evidence of step coalescence to form larger scale barriers $\rightarrow$ pragmatic interest
Areas for further study:

- Structure of mixing representation, form of mixing scales
  \[ l_m, \kappa, \tau_d \]
- Non-diffusive flux contributions, form
- Further study of multiple field systems, i.e. H-W:
  \[ \langle n \rangle, \langle \nabla^2 \phi \rangle, \varepsilon \]
- Role of residual transport, spreading
- Step coalescence
- Shear vs PV gradient feedback in QG systems
Final Observation:

Staircases are becoming crowded...

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