Climbing the Potential Vorticity Staircase: How Profile Modulations Nucleate Profile Structure and Transport Barriers

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Thanks for:

• Collaborations: M. Malkov, A. Ashourvan, Y. Kosuga, O.D. Gurcan, D.W. Hughes

• Discussions: G. Dif-Pradalier, Z.B. Guo, P.-C. Hsu, W.R. Young, J.-M. Kwon
Outline

A) A Primer on “Tokamak Plasma” Turbulence, Zonal Flows and Modulational Instability

I) Systems:
   – “Tokamak Plasma” Primer

II) Mesoscopic Patterns
   – Avalanches
   – Zonal Flows – via modulation of the gas of drift waves

B) Pattern competition – Enter the staircase!

C) The Basics: QG staircase
   – Model content
   – Results and FAQ’s
Outline, cont’d

D) The H.-W. staircase: profile structure and barrier formation
   – extending the model
   – profile formation
   – transport bifurcation

F) Lessons, Conclusions, Future
I) The System:
What is a Tokamak?

How does confinement work?

N.B. No programmatic advertising intended…
Magnetically confined plasma

- Nuclear fusion: option for generating large amounts of carbon-free energy
- Challenge: ignition -- reaction release more energy than the input energy
  
  Lawson criterion:

\[ n_i \tau_E T_i > 3 \times 10^{21} \text{m}^{-3} \text{s keV}. \]

  \rightarrow \text{confinement}
  
  \rightarrow \text{turbulent transport}

- Turbulence: instabilities and collective oscillations
  
  \rightarrow \text{lowest frequency modes dominate the transport}
  
  \rightarrow \text{drift wave}
Primer on Turbulence in Tokamaks I

- Strongly magnetized
  - Quasi 2D cells
  - Localized by $\mathbf{k} \cdot \mathbf{B} = 0$ (resonance)
- $\mathbf{V}_\perp = + \frac{c}{B} \mathbf{E} \times \mathbf{\hat{z}}$
- $\nabla T_e, \nabla T_i, \nabla n$ driven
- Akin to thermal Rossby wave, with: $g \rightarrow$ magnetic curvature
- Resembles wave turbulence, not high $Re$ Navier-Stokes turbulence
- $Re$ ill defined, "$Re" \leq 100$
- $K \sim \tilde{V} \tau_c / \Delta \sim 1 \rightarrow Kubo \# \approx 1$
- Broad dynamic range $\rightarrow$ multi-scale problem: $a, L_p, \Delta r_c, \rho_i, \Delta r_{ce}, \rho_e$
• Characteristic scale $\sim$ few $\rho_i \Rightarrow$ "mixing length"
• Characteristic velocity $v_d \sim \rho_* c_s$
• Transport scaling: $D \sim \rho v_d \sim \rho_* D_B \sim D_{GB}$, $D_B \sim \rho_s c_s$

i.e. Bigger is better! $\Rightarrow$ sets profile scale via heat balance
(Why ITER is enormous...)

• Reality: $D \sim \rho_*^\alpha D_B$, $\alpha < 1 \Rightarrow$ why?? – pattern competition?
• 2 Scales, $\rho_* \ll 1 \Rightarrow$ key contrast to familiar pipe flow
Geophysical fluids

- Phenomena: weather, waves, large scale atmospheric and oceanic circulations, water circulation, jets...

- Geophysical fluid dynamics (GFD): low frequency ($\omega < \Omega$)

  "We might say that the atmosphere is a musical instrument on which one can play many tunes. High notes are sound waves, low notes are long inertial waves, and nature is a musician more of the Beethoven than the Chopin type. He much prefers the low notes and only occasionally plays arpeggios in the treble and then only with a light hand." – J.G. Charney

- Geostrophic motion: balance between the Coriolis force and pressure gradient

  \[ R_0 = \frac{V}{(2\Omega L)} \ll 1 \]
  \[ \rightarrow \quad \nu = -\nabla \times \hat{z} / 2\Omega \]
  
  \[ P \leftrightarrow \text{stream function} \]
  \[ \rightarrow \quad \omega = \hat{z} \cdot (\nabla \times \nu) = \nabla^2 \psi \]
Kelvin’s theorem – unifying principle

- Kelvin’s circulation theorem for rotating system
  \[ \oint v \cdot dl = \int (\nabla \times v + 2\Omega) \cdot \hat{z} \, dS \equiv C \quad \dot{C} = 0 \]
  relative planetary

- Displacement on beta-plane
  \[ \dot{C} = 0 \rightarrow \frac{d}{dt} \nabla^2 \psi = -2\Omega \cos \theta \frac{d\theta}{dt} = -\beta v_y \]
  \[ \beta = 2\Omega \cos \theta_0 / R_\oplus \]

- Quasi-geostrophic eq
  \[ \frac{d}{dt} (\nabla^2 \psi + \beta y) = 0 \text{ PV conservation} \]
  \[ \rightarrow \text{Rossby wave} \]
  \[ t = 0 \]
  \[ t > 0 \]
  \[ \omega < 0 \]
  \[ \omega > 0 \]
  G. Vallis 06
Drift wave model – Fundamental prototype

• Hasegawa-Wakatani: simplest model incorporating instability

\[ V = \frac{C}{B} \hat{\zeta} \times \nabla \phi + V_{pol} \]

\[ J_\perp = n |e| V^i_{pol} \quad \eta J_\parallel = -\nabla_\parallel \phi + \nabla_\parallel P_e \]

\[ \nabla_\perp \cdot J_\perp + \nabla_\parallel J_\parallel = 0 \quad \Rightarrow \text{vorticity:} \quad \rho_s^2 \frac{d}{dt} \nabla^2 \phi = -D_\parallel \nabla^2_\parallel (\phi - n) + \nu \nabla^2 \nabla^2 \phi \]

\[ \frac{dn_e}{dt} + \nabla_\parallel J_\parallel = 0 \quad \Rightarrow \text{density:} \quad \frac{d}{dt} n = -D_\parallel \nabla^2_\parallel (\phi - n) + D_0 \nabla^2 n \]

\[ \Rightarrow \text{PV conservation in inviscid theory} \quad \frac{d}{dt} (n - \nabla^2 \phi) = 0 \]

\[ \Rightarrow \text{PV flux} = \text{particle flux} + \text{vorticity flux} \]

\[ \Rightarrow \text{zonal flow being a counterpart of particle flux} \]

• Hasegawa-Mima \( \left( D_\parallel k^2 / \omega \gg 1 \rightarrow n \sim \phi \right) \)

\[ \frac{d}{dt} (\phi - \rho_s^2 \nabla^2 \phi) + \nu_* \partial_y \phi = 0 \]
**PV conservation** \( \frac{dq}{dt} = 0 \)

<table>
<thead>
<tr>
<th>GFD: Quasi-geostrophic system</th>
<th>Plasma: Hasegawa-Wakatani system</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q = \nabla^2 \psi + \beta y )</td>
<td>( q = n - \nabla^2 \phi )</td>
</tr>
<tr>
<td>relative vorticity</td>
<td>density (guiding center)</td>
</tr>
<tr>
<td>planetary vorticity</td>
<td>ion vorticity (polarization)</td>
</tr>
<tr>
<td>Physics: ( \Delta y \to \Delta \left( \nabla^2 \psi \right) \to \text{ZF} )</td>
<td>Physics: ( \Delta r \to \Delta n \to \Delta \left( \nabla^2 \phi \right) \to \text{ZF} ! )</td>
</tr>
</tbody>
</table>

- **Charney-Haswrgawa-Mima equation**

  \[
  n = n_0 + \tilde{n} \\
  \tilde{n} \sim \frac{e\phi}{T}
  \]

  H-W \( \to \) H-M:  
  \[
  \frac{1}{\omega_{ci}} \frac{\partial}{\partial t} \left( \nabla^2 \phi - \rho_s^{-2} \phi \right) + \frac{1}{L_n} \frac{\partial}{\partial y} \phi + \frac{\rho_s}{L_n} J(\phi, \nabla^2 \phi) = 0
  \]

  Q-G:  
  \[
  \frac{\partial}{\partial t} \left( \nabla^2 \psi - L_d^{-2} \psi \right) + \beta \frac{\partial}{\partial x} \psi + J(\psi, \nabla^2 \psi) = 0
  \]
II) Mesoscopic Patterns in Tokamak Turbulence

→ Avalanches and ‘Non-locality’
→ Zonal Flows
"Truth is never pure and rarely simple" (Oscar Wilde)

Transport: Local or Non-local?

- 40 years of fusion plasma modeling
  - local, diffusive transport
    \[ Q = -n\chi(r)\nabla T, \quad \chi \leftrightarrow D_{GB} \]
- 1995 → increasing evidence for:
  - transport by avalanches, as in sand pile/SOCs
  - turbulence propagation and invasion fronts
  - “non-locality of transport”
    \[ Q = -\int \kappa(r, r')\nabla T(r')dr' \]
    \[ \kappa(r, r') \sim S_0 / [(r - r')^2 + \Delta^2] \]
- Physics:
  - Levy flights, SOC, turbulence fronts…
- Fusion:
  - gyro-Bohm breaking
    (ITER: significant $\rho_*$ extension)
  \[ \rightarrow \text{fundamentals of turbulent transport modeling}?? \]

Guilhem Dif-Pradalier et al. PRL 2009
• ‘Avalanches’ form! – flux drive + geometrical ‘pinning’

• Avalanching is a likely cause of ‘gyro-Bohm breaking’ → Intermittent Bursts

  ➜ localized cells self-organize to form transient, extended transport events

• Akin domino toppling:

• Pattern competition with shear flows!

Newman PoP96 (sandpile) (Autopower frequency spectrum of ‘flip’)

GK simulation also exhibits avalanching (Heat Flux Spectrum) (Idomura NF09)

Toppling front can penetrate beyond region of local stability
What regulates radial extent? ➔

Shear Flows ‘Natural’ to Tokamaks

• Zonal Flows Ubiquitous for:
  ~ 2D fluids / plasmas \( R_0 < 1 \)
  Rotation \( \tilde{\Omega} \), Magnetization \( \tilde{B}_0 \), Stratification

Ex: MFE devices, giant planets, stars…
Heuristics of Zonal Flows a): How Form?

Simple Example: Zonally Averaged Mid-Latitude Circulation

- classic GFD example: Rossby waves + Zonal flow (c.f. Vallis ’07, Held ’01)
- Key Physics:

Rossby Wave:

\[
\omega_k = -\frac{\beta k_x}{k_{\perp}^2}
\]

\[
v_{gy} = 2\beta \frac{k_x k_y}{(k_{\perp}^2)^2}, \quad \langle \tilde{v}_y \tilde{v}_x \rangle = \sum_k -k_x k_y |\tilde{\phi}_k|^2
\]

\[
\therefore v_{gy} v_{ph_y} < 0 \rightarrow \text{Backward wave!}
\]

→ Momentum convergence

at stirring location
...“the central result that a rapidly rotating flow, when stirred in a localized region, will converge angular momentum into this region.” (I. Held, '01)

- Outgoing waves $\Rightarrow$ incoming wave momentum flux

<table>
<thead>
<tr>
<th>Viscous damping</th>
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<tbody>
<tr>
<td>Source</td>
</tr>
<tr>
<td>Zonal shear layer formation</td>
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</table>

- Local Flow Direction (northern hemisphere):
  - Eastward in source region
  - Westward in sink region
  - Set by $\beta > 0$

- Some similarity to spinodal decomposition phenomena
  $\Rightarrow$ Both ‘negative diffusion’ phenomena
MFE perspective on Wave Transport in DW Turbulence

- localized source/instability drive intrinsic to drift wave structure

- outgoing wave energy flux → incoming wave momentum flux → counter flow spin-up!

- zonal flow layers form at excitation regions

\[
\nu_{gr} = -2 \rho_s^2 \frac{k_\theta k_r v_*}{(1 + k_\perp^2 \rho_s^2)^2}
\]

\[
\langle v_{rE} v_{\theta E} \rangle = -\frac{c^2}{B^2} |\phi_k|^2 k_r k_\theta < 0
\]

\[
x > 0 \implies \nu_{gr} > 0
\]

\[
v_* < 0, \ k_r k_\theta > 0
\]
Zonal Flows I

• What is a Zonal Flow?
  - $n = 0$ potential mode; $m = 0$ (ZFZF), with possible sideband (GAM)
  - toroidally, poloidally symmetric $ExB$ shear flow

• Why are Z.F.’s important?
  - Zonal flows are secondary (nonlinearly driven):
    • modes of minimal inertia (Hasegawa et. al.; Sagdeev, et. al. ‘78)
    • modes of minimal damping (Rosenbluth, Hinton ‘98)
    • drive zero transport ($n = 0$)
  - natural predators to feed off and retain energy released by gradient-driven microturbulence
Zonal Flows II

• Fundamental Idea:
  – Potential vorticity transport + 1 direction of translation symmetry
    → Zonal flow in magnetized plasma / QG fluid
  – Kelvin’s theorem is ultimate foundation

• G.C. ambipolarity breaking → polarization charge flux → Reynolds force
  – Polarization charge
    \[ \rho^2 \nabla^2 \phi = n_{i,GC}(\phi) - n_e(\phi) \]
    polarization length scale
    ion GC
    electron density

  – so \( \Gamma_{i,GC} \neq \Gamma_e \)
    \[ \rho^2 \left\langle \overline{v_{rE}} \nabla^2 \overline{\phi} \right\rangle \neq 0 \]
    ‘PV transport’
    polarization flux
    \( \rightarrow \) What sets cross-phase?

  – If 1 direction of symmetry (or near symmetry):
    \[ -\rho^2 \left\langle \overline{v_{rE}} \nabla^2 \overline{\phi} \right\rangle = -\partial_r \left\langle \overline{v_{rE}} \overline{v_{\perp E}} \right\rangle \] (Taylor, 1915)
    \[ -\partial_r \left\langle \overline{v_{rE}} \overline{v_{\perp E}} \right\rangle \]
    Reynolds force
    \( \rightarrow \) Flow
Zonal Flows Shear Eddys I

- Coherent shearing: (Kelvin, G.I. Taylor, Dupree’66, BDT’90)
  - radial scattering + \( \langle V_E \rangle' \) → hybrid decorrelation
  - \( k_r^2 D_\perp \rightarrow (k_\theta^2 \langle V_E \rangle^2 D_\perp / 3)^{1/3} = 1 / \tau_c \)
  - Akin shear dispersion
  - shaping, flux compression: Hahm, Burrell ’94

- Other shearing effects (linear):
  - spatial resonance dispersion:
  - differential response rotation → especially for kinetic curvature effects

→ N.B. Caveat: Modes can adjust to weaken effect of external shear
  (Carreras, et. al. ‘92; Scott ‘92)
Shearing II

- Zonal Shears: Wave kinetics (Zakharov et. al.; P.D. et. al. ‘98, et. seq.)
  Coherent interaction approach (L. Chen et. al.)

\[
\frac{dk_r}{dt} = -\partial(\omega + k_\theta V_E) / \partial r ; \quad V_E = \left\langle V_E \right\rangle + \widetilde{V}_E
\]

Mean shearing
\[ k_r = k_r^{(0)} - k_\theta V'_E \tau \]

Zonal
\[ \left\langle \delta k_r^2 \right\rangle = D_k \tau \]

Random shearing
\[ D_k = \sum_q k_\theta^2 \left| \widetilde{V}_{E,q} \right|^2 \tau_{k,q} \]

- Wave ray chaos (not shear RPA) underlies \( D_k \rightarrow \) induced diffusion
- Induces wave packet dispersion
- Applicable to ZFs and GAMs

Mean Field Wave Kinetics
\[
\frac{\partial N}{\partial t} + (\vec{V}_{gr} + \vec{V}) \cdot \nabla N - \frac{\partial}{\partial r} (\omega + k_\theta V_E) \cdot \frac{\partial N}{\partial k} = \gamma_k N - C\{N\}
\]

\[
\Rightarrow \frac{\partial}{\partial t} \left\langle N \right\rangle - \frac{\partial}{\partial k_r} D_k \frac{\partial}{\partial k_r} \left\langle N \right\rangle = \gamma_k \left\langle N \right\rangle - \left\langle C\{N\} \right\rangle
\]

Zonal shearing \( \rightarrow \) computed using modulational response
Shearing III

• Energetics: Books Balance for Reynolds Stress-Driven Flows!

• Fluctuation Energy Evolution – Z.F. shearing

\[
\int d\vec{k}\omega \left( \frac{\partial}{\partial t} \langle N \rangle - \frac{\partial}{\partial k_r} D_k \frac{\partial}{\partial k_r} \langle N \rangle \right) \Rightarrow \frac{\partial}{\partial t} \langle \varepsilon \rangle = -\int d\vec{k} V_{gr} (\vec{k}) D_k \frac{\partial}{\partial k_r} \langle N \rangle \quad V_{gr} = \frac{-2k_r k_\theta V_s \rho_s^2}{(1 + k_\perp^2 \rho_s^2)^2}
\]

Point: For \( \frac{d\langle \Omega \rangle}{dk_r} < 0 \), Z.F. shearing damps wave energy

• Fate of the Energy: Reynolds work on Zonal Flow

Modulational Instability

\[
\partial_t \delta V_\theta + \partial_r \left( \delta \langle \tilde{V}_r \tilde{V}_\theta \rangle \right) - \gamma \delta V_\theta
\]

\[
\delta \langle \tilde{V}_r \tilde{V}_\theta \rangle \sim \frac{k_r k_\theta \partial \Omega}{(1 + k_\perp^2 \rho_s^2)^2}
\]

N.B.: Wave decorrelation essential:

Equivalent to PV transport (c.f. Gurcan et. al. 2010)

Modulation \( \rightarrow \) inhomogeneity in PV mixing

• Bottom Line:

− Z.F. growth due to shearing of waves

− “Reynolds work” and “flow shearing” as relabeling \( \rightarrow \) books balance

− Z.F. damping emerges as critical; MNR ‘97
Approaches to Modulation

~ Weak, Wave Turbulence Problems

→ Quasi-particle, Wave Kinetics \( \Rightarrow \delta N \)

See: P.D. Itoh, Itoh, Hahm ‘05 PPCF

→ Envelope Theory, Generalized NLS \( \Rightarrow \psi \)


N.B.: Representation of PV mixing and its inhomogeneity is crucial
Feedback Loops

- Closing the loop of shearing and Reynolds work
- Spectral ‘Predator-Prey’ equations

Prey $\rightarrow$ Drift waves, $\langle N \rangle$

\[
\frac{\partial}{\partial t} \langle N \rangle - \frac{\partial}{\partial k_r} D_k \frac{\partial}{\partial k_r} \langle N \rangle = \gamma_k \langle N \rangle - \frac{\Delta \omega_k}{N_0} \langle N \rangle^2
\]

Predator $\rightarrow$ Zonal flow, $|\phi_q|^2$

\[
\frac{\partial}{\partial t} |\phi_q|^2 = \Gamma_q \left[ \frac{\partial \langle N \rangle}{\partial k_r} \right] |\phi_q|^2 - \gamma_d |\phi_q|^2 - \gamma_{NL}[|\phi_q|^2] |\phi_q|^2
\]
Feedback Loops II

- Recovering the ‘dual cascade’:
  - Prey → \langle N \rangle ∼ \langle \Omega \rangle ⇒ induced diffusion to high k_r \Rightarrow Analogous → forward potential enstrophy cascade; PV transport
  - Predator → |\phi_q|^2 ∼ \langle V^2_{E,\theta} \rangle \Rightarrow growth of n=0, m=0 Z.F. by turbulent Reynolds work ⇒ Analogous → inverse energy cascade

- Mean Field Predator-Prey Model
  (P.D. et. al. ’94, DI²H ‘05)

\[
\frac{\partial}{\partial t} N = \gamma N - \alpha V^2 N - \Delta \omega N^2
\]

\[
\frac{\partial}{\partial t} V^2 = \alpha NV^2 - \gamma_d V^2 - \gamma_{NL} (V^2) V^2
\]

System Status

<table>
<thead>
<tr>
<th>State</th>
<th>No flow</th>
<th>Flow (\alpha_2 = 0)</th>
<th>Flow (\alpha_2 \neq 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N ) (drift wave turbulence level)</td>
<td>( \frac{\gamma}{\Delta \omega} )</td>
<td>( \frac{\gamma_d}{\alpha} )</td>
<td>( \frac{\gamma_d + \alpha_2 \gamma \alpha^{-1}}{\alpha + \Delta \omega \alpha \alpha^{-1}} )</td>
</tr>
<tr>
<td>( V^2 ) (mean square flow)</td>
<td>0</td>
<td>( \frac{\gamma}{\alpha} - \frac{\Delta \omega \gamma d}{\alpha^2} )</td>
<td>( \frac{\gamma - \Delta \omega \gamma d \alpha^{-1}}{\alpha + \Delta \omega \alpha \alpha^{-1}} )</td>
</tr>
<tr>
<td>Drive/excitation mechanism</td>
<td>Linear growth</td>
<td>Linear growth</td>
<td>Linear growth Nonlinear damping of flow</td>
</tr>
<tr>
<td>Regulation/inhibition mechanism</td>
<td>Self-interaction of turbulence</td>
<td>Random shearing, self-interaction</td>
<td>Random shearing, self-interaction</td>
</tr>
<tr>
<td>Branching ratio ( \frac{V^2}{N} )</td>
<td>0</td>
<td>( \frac{\gamma - \Delta \omega \gamma d \alpha^{-1}}{\gamma_d} )</td>
<td>( \frac{\gamma_d + \alpha_2 \gamma \alpha^{-1}}{\gamma_d} )</td>
</tr>
<tr>
<td>Threshold (without noise)</td>
<td>( \gamma &gt; 0 )</td>
<td>( \gamma &gt; \Delta \omega \gamma d \alpha^{-1} )</td>
<td>( \gamma &gt; \Delta \omega \gamma d \alpha^{-1} )</td>
</tr>
</tbody>
</table>
IV) The Central Question: Secondary Pattern Selection ?!

• Two secondary structures suggested
  – Zonal flow → quasi-coherent, regulates transport via shearing
  – Avalanche → stochastic, induces extended transport events

• Both flux driven… by relaxation

• Nature of co-existence??

• Who wins? Does anybody win?
B) Pattern Competition:
Enter the Staircase....
Motivation: ExB staircase formation (1)

- ExB flows often observed to self-organize in magnetized plasmas
  eg. mean sheared flows, zonal flows, ...

- `ExB staircase’ is observed to form
  (G. Dif-Pradalier, P.D. et al. Phys. Rev. E. ’10)
  - flux driven, full f simulation
  - Quasi-regular pattern of shear layers and profile corrugations
  - Region of the extent $\Delta \gg \Delta_c$ interspersed by temp. corrugation/ExB jets
    $\rightarrow$ ExB staircases
  - so-named after the analogy to PV staircases and atmospheric jets
  - Step spacing $\rightarrow$ avalanche outer-scale
Basic Ideas:
Transport bifurcations and ‘negative diffusion’ phenomena
Transport Barrier Formation (Edge and Internal)

- Observation of ETB formation (L→H transition)
  - THE notable discovery in last 30 yrs of MFE research
  - Numerous extensions: ITB, I-mode, etc.
  - Mechanism: turbulence/transport suppression by ExB shear layers generated by turbulence

- Physics:
  - Spatio-temporal development of bifurcation front in evolving flux landscape
  - Cause of hysteresis, dynamics of back transition

- Fusion:
  - Pedestal width (along with MHD) → ITER ignition, performance
  - ITB control → AT mode
  - Hysteresis + back transition → ITER operation

J.W. Huges et al., PSFC/JA-05-35
Why Transport Bifurcation?  BDT ‘90, Hinton ‘91

• Sheared $V_{E \times B}$ flow quenches turbulence, transport $\Rightarrow$ intensity, phase correlations

• Gradient + electric field $\Rightarrow$ feedback loop (central concept)

\[ i.e. \; \vec{E} = \frac{\nabla P_i}{n q} - \vec{V} \times \vec{B} \Rightarrow V_E' = V_E' (\nabla T) \]

$\Rightarrow$ minimal model

\[ Q = -\frac{\chi (\nabla T) \nabla T}{\left[ 1 + \left( \frac{V_E'}{\omega_{eff}} \right)^2 \right]^n} - \chi_{neo} \nabla T \]

Residual collisional

Turbulent transport $+ \; \text{shear suppression}$

$n \equiv \text{quenching exponent}$
- Feedback:

\[ Q \uparrow \rightarrow \nabla T \uparrow \rightarrow V'_E \uparrow \rightarrow (\tilde{n}/n)^2, \chi_T \downarrow \]

\[ \Rightarrow \nabla T \uparrow \rightarrow \ldots \]

- Result:

1\textsuperscript{st} order transition (L\(\rightarrow\)H):

- Heat flux vs \(\nabla T_i\)

- T profiles
  a) L-mode
  b) H-mode
• S curve ➔ “negative diffusivity” i.e. $\delta Q / \delta \nabla T < 0$

• Transport bifurcations observed and intensively studied in MFE since 1982 yet:

➔ Little concern with staircases, but if now include modulated ZF feedback on transport?

➔ Key questions:

1) Is zonal flow pattern really a staircase? ➔ consequence of inhomogeneous PV mixing induced by modulation?

2) Might observed barriers form via step coalescence in staircases?
What is a staircase? – sequence of transport barriers

Cf Phillips’72:

\[
\frac{\partial \Gamma}{\partial \nabla u} < 0; \text{ flux dropping with increased gradient}
\]

\[
\Gamma_b = -D_b \nabla b, \quad Ri = g \nabla b / (\nu')^2
\]

Obvious similarity to transport bifurcation
In other words, via modulational instability:

Configuration instability of profile + turbulence intensity field

Buoyancy profile

Intensity field

Some resemblance to Langmuir turbulence
i.e. for Langmuir: caviton train

\[ \varepsilon \]

\[ \delta n \]

\[ \delta n / n \approx -\varepsilon \]

\( b \)

gradient

intensity

end state of profile corrugation from modulational instability ?!
• The physics: “Negative Diffusion” (BLY, ’98)

\[ \Gamma_b / \nabla b < 0 \]

“H-mode” like branch (i.e. residual collisional diffusion) is not input
- Usually no residual diffusion
- ‘branch’ upswing \( \rightarrow \) nonlinear processes (turbulence spreading)
- If significant molecular diffusion \( \rightarrow \) second branch via collisions

• Instability driven by local transport bifurcation

\[ \delta \Gamma_b / \delta \nabla b < 0 \]

\( \rightarrow \) ‘negative diffusion’

\( \rightarrow \) Feedback loop \( \Gamma_b \downarrow \rightarrow \nabla b \uparrow \rightarrow l \downarrow \rightarrow \Gamma_b \downarrow \)

Critical element: \( l \rightarrow \) mixing length
• OK: Is there a “simple model” encapsulating the ideas?
• Balmforth, Llewellyn-Smith, Young 1998 → staircase in stirred stably stratified turbulence
• Idea: 1D $K - \epsilon$ model, in lieu W.K.E.
  – turbulence energy; with production, dissipation spreading
  – Mean field evolution
  – Diffusion: $\nabla \cdot \mathbf{V} l_m \sim (\epsilon)^2 l_m \kappa$
  – $l_m \kappa$ → mixing length ?!
  – $\delta \Gamma / \delta \nabla b < 0$ enters via nonlinearity, gradient dependence of length scale
The model

• Mean Field:

\[ \partial_t b = \partial_z (D \partial_z b) \]

\[ D = e^{1/2} l \]

\[ 1/l^2 = 1/l_f^2 + 1/l_{oz}^2 \]

\[ e = \langle \tilde{V}^2 \rangle \]

N.B. \( \int [e - zb] = 0 \) (energy balance)

• Fluctuations:

\[ \partial_t e = \partial_z D \partial_z e - le\tilde{\omega} \partial_z b - \frac{e^2}{l} + F \]

 forcing \( F \sim \sqrt{e} (u_0^2 - e) \)

 spreading

 production \( g \langle \tilde{V} \delta \rho \rangle \)

 \( 3 \)

 dissipation

 Ozmidov scale
• What is $l_{mix}$?

$$1/l^2 = 1/l_f^2 + 1/l_{oz}$$

$l_{oz} \sim$ Ozmidov scale

$\sim$ balance of buoyancy production vs. dissipation

i.e. $\tilde{V}^3/l \sim g\langle \tilde{V} \delta b \rangle$

$$\delta b \sim (\tilde{V}/(\tilde{V}/l))\partial b/\partial z$$

$\Rightarrow 1/l_{oz} \approx (b_z/e)^{1/2}$

or $V(l)/l \sim N \Rightarrow l_{oz}$

$\Rightarrow$ smallest "stratified" scale

$\Rightarrow$ necessary feedback loop

System mixes at steady state on scale of energy balance

N.B.: $b_z \uparrow$, $e \downarrow \Rightarrow l \downarrow$

$e \approx \langle \tilde{V}^2 \rangle$ energy
• **A Few Results**

Plot of $b_z$ (solid) and $e$ (dotted) at early time. Buoyancy flux is dashed $\Rightarrow$ near constant in core

Later time $\Rightarrow$ more akin expected “staircase pattern”. Some condensation into larger scale structures has occurred.
C) Basics: QG Staircase
Staircase in QG Turbulence: A Model

- PV staircases observed in nature, and in the unnatural
- Formulate ‘minimal’ dynamical model ?! (n.b. Dritschel-McIntyre 2008 does not address dynamics)

Observe:
- 1D adequate: for ZF need ‘inhomogeneous PV mixing’ + 1 direction of symmetry. Expect ZF staircase
- Best formulate intensity dynamics in terms potential enstrophy $\epsilon = \langle \tilde{q}^2 \rangle$

Length? : $\Gamma_q \frac{\partial \langle q \rangle}{\partial y} \sim \tilde{q}^3$ (production-dissipation balance)

$\Rightarrow l \sim \langle \tilde{q}^2 \rangle^{\frac{1}{2}} / \frac{\partial \langle q \rangle}{\partial y} \sim l_{Rhines}$ (i.e. $\omega_{Rossby} \sim k \tilde{v}$)

- Rhines scale is natural length $\rightarrow$ ‘memory’ of scale
\textbf{Model:} \quad \Gamma_q = \langle \tilde{v}_y \tilde{q} \rangle = -D \partial_y \langle q \rangle / \partial y \text{ is fundamental quantity (PV flux)}

→ Mean: \( \partial_t \langle q \rangle = \partial_y D \partial_y \langle q \rangle \)

→ Potential Enstrophy density: \( \partial_t \varepsilon - \partial_y D \partial_y \varepsilon = D \left( \partial_y \langle q \rangle \right)^2 - \varepsilon^2 + F \)

Where:

\[
\frac{1}{l^2} = \frac{1}{l_f^2} + \frac{1}{l_{Rh}^2}
\]

\( D \sim l^2 \sqrt{\varepsilon} \) (dimensional)

\( l_{Rh}^2 = \varepsilon / \left( \partial_y \langle q \rangle \right)^2 \)

\( D_{spr} \approx D_{PV} \)

→ \( D \rightarrow PV \) mixing

\( l_{Rh} (\nabla q) \) ensures inhomogeneity
Alternative Perspective:

• Note: \[ l^2 = \frac{1}{1 + 1/l^2_{Rh}} \rightarrow \frac{1}{1 + \langle q \rangle^2 / \epsilon} \quad (l_f \sim 1) \]

• Reminiscent of weak turbulence perspective:

\[ D = D_{pv} = \sum_k \frac{\langle \tilde{V}^2 \rangle \Delta \omega_k}{\omega_k^2 + \Delta \omega_k^2} \]

\[ \omega_k = -k_x \langle q \rangle' / k^2 \]

\[ \Delta \omega_k \approx k \tilde{V}_k \]

Ala’ Dupree’67:

\[ D_{pv} \approx \frac{1}{k^2} \left( \sum_k k^2 \langle \tilde{V}^2 \rangle_k - \frac{k_x^2 \langle \langle q \rangle' \rangle^2}{(k^2)^2} \right)^{1/2} \]

Steeper \( \langle q \rangle' \) quenches diffusion \( \rightarrow \) mixing reduced via PV gradient feedback
\[ D_{pv} \approx \frac{l_0^2}{\varepsilon^2} \frac{1}{1 + \frac{l_0^2}{\varepsilon} \langle q \rangle'^2} \]

- \( \omega \) vs \( \Delta \omega \) dependence gives \( D_{pv} \) roll-over with steepening
- Rhines scale appears naturally, in feedback strength
- Recovers effectively same model

Physics:

1. “Rossby wave elasticity’ (MM) \( \rightarrow \) steeper \( \langle q \rangle' \) \( \rightarrow \) stronger memory (i.e. more ‘waves’ vs turbulence)
2. Distinct from shear suppression \( \rightarrow \) interesting to dis-entangle
Aside

• What of wave momentum? Austauch ansatz

  Debatable (McIntyre) - but \( l_m \kappa (?) \ldots \)

• PV mixing \( \leftrightarrow D \partial_y \langle q \rangle \)

  So \( \rightarrow \langle \tilde{V} \tilde{q} \rangle \rightarrow \partial_y \langle \tilde{V}_y \tilde{V}_x \rangle \rightarrow \text{R.S.} \)

• But:

  \( \text{R.S.} \leftrightarrow \langle k_x k_y \rangle \leftrightarrow V_{gy} E \)

  ➔ Feedback:

  \[ \langle q \rangle' \uparrow \rightarrow l \downarrow \rightarrow \epsilon \downarrow \rightarrow D \downarrow \]

  (Production)

- Equivalent!
- Formulate in terms mean, Pseudomomentum?
* - Red herring for barriers \( \rightarrow l_m \kappa \) quenched
Results:
- Analysis of QG Model Dynamics
- FAQ
• Re-scaled system

\[ Q_t = \partial_y \frac{\varepsilon^{1/2}}{(1 + Q_y^2/\varepsilon)^\kappa} Q_y + DQ_{yy} \quad \text{for mean} \]

\[ \varepsilon_t = \partial_y \frac{\varepsilon^{1/2}}{(1 + Q_y^2/\varepsilon)^\kappa} Q_y + L^2 \left\{ \frac{Q_y^2}{(1 + Q_y^2/\varepsilon)^\kappa} - \frac{\varepsilon}{\varepsilon_0} + 1 \right\} \varepsilon^{1/2} + DQ_{yy} \]

• Note:

  – Quenching exponent usually \( \kappa = 2 \) for saturated modulational instability

  – Potential enstrophy conserved to forcing, dissipation, boundary

  – System size \( L \) ➞ strength of drive ↔ boundary condition effects!
Structure of RHS: $\epsilon$ equation

→ Bistability evident

→ $Q_y^2$ vs $\epsilon_0$ dependencies define range
Basic Results

Weak Drive

$\rightarrow$ 1 step staircase

$\rightarrow$ $L^2$ increased

$\rightarrow$ Turbulence forces asymmetry
Mergers Occur

→ Surface plot $Q(t, x)$ for Dirichlet
→ 12 → 7, then persist till 2 layer disappear into wall
→ Further mergers at boundary
Characterization

- FWHM $\rightarrow$ jumps/layer
  - FWHM Step
  - FWHM Ped

widths

step

jump
corner
Illustrating the merger sequence

- $\epsilon$
- $Q_y$  top
- $Q$
- $\Gamma_q$  bottom

Note later staircase mergers induce strong flux episodes!
The Hasegawa-Wakatani Staircase:
Profile Structure:
From Mesoscopics \(\rightarrow\) Macroscopics
Extending the Model

Reduced system of evolution Eqs. is obtained from HW system for DW turbulence.

Variables:

\[ u = \partial_x V_y \quad \text{Zonal shearing field} \]

Reduced density:

\[ \log(N/N_0) = n(x,t) + \hat{n}(x,y,t) \]

Vorticity:

\[ \rho_s^2 \nabla^2 (e \varphi / T_e) = u(x,t) + \hat{u}(x,y,t) \]

Potential Vorticity (PV):

\[ q = n - u, \quad \text{Turbulent Potential Enstrophy (PE):} \quad \varepsilon = \frac{1}{2} \langle (\hat{n} - \hat{u})^2 \rangle \]

Mean field equations:

**Density**

\[ \partial_t n = -\partial_x \Gamma_n + \partial_x [D_c \partial_x n], \quad \Gamma_n = \langle \hat{\psi}_x \hat{n} \rangle = -D_n \partial_x n \]

**Vorticity**

\[ \partial_t u = -\partial_x \Pi_u + \partial_x [\mu_c \partial_x u], \quad \Pi_u = \langle \hat{\psi}_y \hat{u} \rangle = \langle \chi - D_n \partial_x n \rangle - \chi \hat{\phi}_x u \]

Reflect instability

Taylor ID: \[ \Pi_u = \langle \hat{\psi}_x \hat{u} \rangle = \partial_x \langle \hat{\psi}_x \hat{\psi}_x \rangle \]

Residual vort. flux

Turb. viscosity

Turbulence evolution:

\[ \partial_t \varepsilon = \partial_x [D_\varepsilon \partial_x \varepsilon] - (\Gamma_n - \Gamma_u) \partial_x (n - u) - \varepsilon_c^{-1} \varepsilon^{3/2} + P \]

Turbulence spreading

Internal production

dissipation

External production \( \sim \gamma \varepsilon \)

Two fluxes \( \Gamma_n, \Gamma_u \) set model

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What is new in this model?

- In this model PE conservation is a central feature.
- Mixing of Potential Vorticity (PV) is the fundamental effect regulating the interaction between turbulence and mean fields.
- We use dimensional arguments to obtain functional forms for the turbulent diffusion coefficients. From the QL relation for HW system we obtain

\[ D_n \approx l^2 \frac{\varepsilon}{\alpha} \quad \chi \approx c^2 l^2 \frac{\varepsilon}{\sqrt{\alpha^2 + a_u u^2}} \]

- Inhomogeneous mixing of PV results in the sharpening of density and vorticity gradients in some regions and weakening them in other regions, leading to shear lattice and density staircase formation.

Jet sharpening in stratosphere, resulting from inhomogeneous mixing of PV. (McIntyre 1986)

\[ PV \quad Q = \nabla^2 \psi + \beta y \]

Relative vorticity

Planetary vorticity
Snapshots of evolving profiles at t=1 (non-dimensional time)

Initial conditions: \( n = g_0(1 - x), \; u = 0, \; \varepsilon = \varepsilon_0 \)

Boundary conditions: \( n(0, t) = g_0, \; n(1, t) = 0; \; u(0, 1; t) = 0; \; \partial_x \varepsilon(0, 1; t) = 0 \)

Structures:
- **Staircase in density profile:**
  - jumps → regions of steepening
  - steps → regions of flattening
- **At the jump locations, turbulent PE is suppressed.**
- **At the jump locations, vorticity gradient is positive**

![Graphs and diagrams illustrating the staircase structures in density profile and vorticity gradient.](image-url)
Mergers Occur

Nonlinear features develop from linear instabilities

\[ \varepsilon(x = 0,1) = 0 \quad \text{u}(x,t) \]

Local profile reorganization: Steps and jumps merge (continues up to times \( t \sim O(10) \))

Merger between steps

Merger between jumps
Shear layer propagation

- Shear pattern detaches and delocalizes from its initial position of formation.
- Mesoscale shear lattice moves in the up-gradient direction. Shear layers condense and disappear at $x=0$.
- Shear lattice propagation takes place over much longer times. From $t \sim O(10)$ to $t \sim (10^4)$.

- Barriers in density profile move upward in an “Escalator-like” motion.

⇒ Macroscopic Profile Re-structuring
Time evolution of profiles

(a) Fast merger of micro-scale SC. Formation of meso-SC.
(b) Meso-SC coalesce to barriers
(c) Barriers propagate along gradient, condense at boundaries
(d) Macro-scale stationary profile
Macroscopics: Flux driven evolution

We add an external particle flux drive to the density Eq., use its amplitude $\Gamma_0$ as a control parameter to study:

- What is the mean profile structure emerging from this dynamics?
- Variation of the macroscopic steady state profiles with $\Gamma_0$ (shearing, density, turbulence, and flux).
- Transport bifurcation of the steady state (macroscopic)
- Particle flux-density gradient landscape.

\[
\partial_t n = -\partial_x \Gamma - \partial_x \Gamma_{dr} (x,t) \quad \Rightarrow \text{Write source as } \nabla \cdot \Pi_{ex}
\]

- External particle flux (drive)
  \[
  \Gamma_{dr} (x,t) = \Gamma_0 (t) \exp[-x/\Delta_{dr}]
  \]

- Internal particle flux (turb. + col.)
  \[
  \Gamma = -[D_n (\varepsilon, \partial_x q) + D_{col}] \partial_x n
  \]
Transition to Enhanced Confinement can occur

Steady state solution for the system undergoes a transport bifurcation as the flux drive amplitude $\Gamma_0$ is raised above a threshold $\Gamma_{th}$

$$\Gamma_1 < \Gamma_{th} < \Gamma_2$$

$\Gamma_0 = \Gamma_1 \rightarrow$ Normal Conf. (NC)
$\Gamma_0 = \Gamma_2 \rightarrow$ Enhanced Conf. (EC)

With NC to EC transition we observe:

- Rise in density level
- Drop in turb. PE and turb. particle flux beyond the barrier position
- Enhancement and sign reversal of vorticity (shearing field)
Hysteresis evident in the flux-gradient relation

In one sim. run, from initially flat density profile, $\Gamma_0$ is adiabatically raised and lowered back down again.

Forward Transition:
Abrupt transition from NC to EC (from A to B). During the transition the system is not in quasi-steady state.

From B to C:
We have continuous control of the barrier position. Barrier moves to the right with lowering the density gradient.

Backward Transition:
Abrupt transition from EC to NC (from C to D). Barrier moves rapidly to the right boundary and disappears. System is not in quasi-steady.

$$\langle \Gamma \rangle = \int_0^1 \Gamma(x) \, dx$$

$$\langle -\partial_x n \rangle = \int_0^1 [-\partial_x n(x, t)] \, dx$$
Role of Turbulence Spreading

- Large turbulence spreading wipes out features on smaller spatial scales in the mean field profiles, resulting in the formation of smaller number density and vorticity jumps.

\[ \partial_t \varepsilon = \beta \partial_x \left[ \left( l^2 \varepsilon^{1/2} \right) \partial_x \varepsilon \right] + ... \]

Initial condition dependence

- Solutions are not sensitive to initial value of turbulent PE.
- Initial density gradient is the parameter influencing the subsequent evolution in the system.
- At lower viscosity more steps form.
- Width of density jumps grows with the initial density gradient.
E) Conclusions and Lessons
→ Towards a Better Model
Lessons

• A) Staircases happen
  – Staircase is ‘natural upshot’ of modulation in bistable/multi-stable system
  – Bistability is a consequence of mixing scale dependence on gradients, intensity ↔ define feedback process
  – Bistability effectively locks in inhomogeneous PV mixing required for zonal flow formation
  – Mergers result from accommodation between boundary condition, drive(L), initial secondary instability
  – Staircase is natural extension of quasi-linear modulational instability/predator-prey model
Lessons

• B) Staircases are Dynamic
  – Mergers occur
  – Jumps/steps migrate. B.C.’s, drive all essential.
  – Condensation of mesoscale staircase jumps into macroscopic
    transport barriers occurs. ➔ Route to barrier transition by global
    profile corrugation evolution vs usual picture of local dynamics
  – Global $1^{st}$ order transition, with macroscopic hysteresis occurs
  – Flux drive + B.C. effectively constrain system states.
Status of Theory

• N.B.: Alternative mechanism via jam formation due flux-gradient time delay \( \rightarrow \) see Kosuga, P.D., Gurcan; 2012, 2013

• a) Elegant, systematic WTT/Envelope methods miss elements of feedback, bistability

  b) \( K - \epsilon \) genre models crude, though elucidate much

• Some type of synthesis needed

• Distribution of dynamic, nonlinear scales appear desirable

• Total PV conservation demonstrated utility and leverage.
• Are staircase models:
  – Natural solution to “predator-prey” problem domains via decomposition (akin spiondal)?
  – Natural reduced DOF models of profile evolution?
  – Realization of ‘non-local’ dynamics in transport?
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