# CHNS: A Case Study in Elastic Turbulence

**P.H. Diamond**
**U.C. San Diego**

**with: Xiang Fan; Luis Chacon, H. Li, LANL**

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Ackn: A. Pouquet, D.W. Hughes, S.M. Tobias,

F. Cattaneo, A.V. Gruzinov, D.R. Nelson

Inspiration

- R. Ruiz, D.R. Nelson; Phys. Rev. A (1981)
- P. Perlekar, et. al.; Phys. Rev. Lett. (2014)
- R. Pandit, et. al.; Phys. Fluids (2017)

# 2D CHNS and 2D MHD

- 2D CHNS Equations:

$$\partial_t \psi + \vec{v} \cdot \nabla \psi = D \nabla^2 (-\psi + \psi^3 - \xi^2 \nabla^2 \psi)$$

$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega$$

| |
|---|
| $-\psi$: Negative diffusion term |
| $\psi^3$: Self nonlinear term |
| $-\xi^2 \nabla^2 \psi$ : Hyper-diffusion term |

With $\vec{v} = \hat{z} \times \nabla \phi$, $\omega = \nabla^2 \phi$, $\vec{B}_\psi = \hat{z} \times \nabla \psi$, $j_\psi = \xi^2 \nabla^2 \psi$.

- 2D MHD Equations:

$$\partial_t A + \vec{v} \cdot \nabla A = \eta \nabla^2 A$$

$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{1}{\mu_0 \rho} \vec{B} \cdot \nabla \nabla^2 A + \nu \nabla^2 \omega$$

$A$: Simple diffusion term

With $\vec{v} = \hat{z} \times \nabla \phi$, $\omega = \nabla^2 \phi$, $\vec{B} = \hat{z} \times \nabla A$, $j = \frac{1}{\mu_0} \nabla^2 A$.

| | 2D MHD | 2D CHNS |
|---|---|---|
| Magnetic Potential | $A$ | $\psi$ |
| Magnetic Field | $\mathbf{B}$ | $\mathbf{B}_\psi$ |
| Current | $j$ | $j_\psi$ |
| Diffusivity | $\eta$ | $D$ |
| Interaction strength | $\frac{1}{\mu_0}$ | $\xi^2$ |

# Ideal Quadratic Conserved Quantities

- **2D MHD** $(\mathbf{R_M} \gg \mathbf{1})$

1. Energy

$$E = E^K + E^B = \int (\frac{v^2}{2} + \frac{B^2}{2\mu_0}) d^2x$$

2. Mean Square Magnetic Potential

$$H^A = \int A^2 \, d^2x$$

3. Cross Helicity

$$H^C = \int \vec{v} \cdot \vec{B} d^2x$$

- **2D CHNS** $(\boldsymbol{P_e} \gg \mathbf{1})$

1. Energy

$$E = E^K + E^B = \int (\frac{v^2}{2} + \frac{\xi^2 B_\psi^2}{2}) d^2x$$

2. Mean Square Concentration

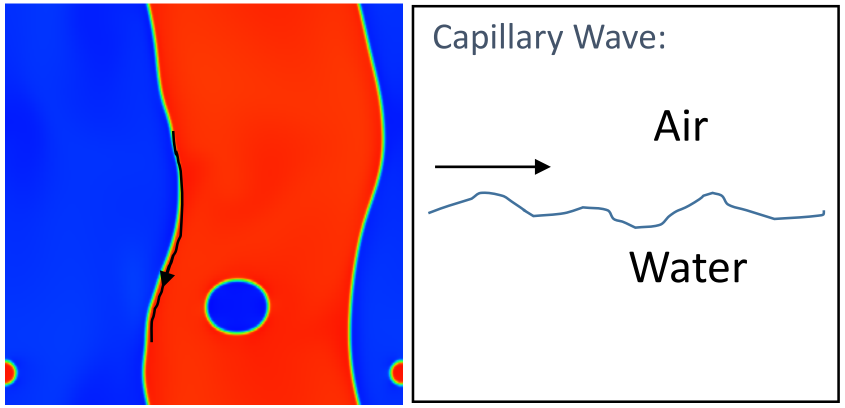
$$H^\psi = \int \psi^2 \, d^2x$$

3. Cross Helicity

$$H^C = \int \vec{v} \cdot \vec{B}_\psi \, d^2x$$

Dual cascade expected!

# Linear Wave




- CHNS supports linear "elastic" wave:

$$\omega(k) = \pm\sqrt{\frac{\xi^2}{\rho}}\left|\vec{k}\times\vec{B}_{\psi 0}\right| - \frac{1}{2}i(CD+\nu)k^2$$

Where $C \equiv [-1 - 6\psi_0\nabla^2\psi_0/k^2 - 6(\nabla\psi_0)^2/k^2 - 6\psi_0\nabla\psi_0\cdot i\mathbf{k}/k^2 + 3\psi_0^2 + \xi^2k^2]$

- Akin to capillary wave at phase interface. Propagates ***only*** along the interface of the two fluids, where $|\vec{B}_\psi| = |\nabla\psi| \neq 0$.
- Analogue of Alfven wave.

→ Important differences:
  - $\vec{B}_\psi$ in CHNS is large only in the interfacial regions.
  - Elastic wave activity does not fill space.

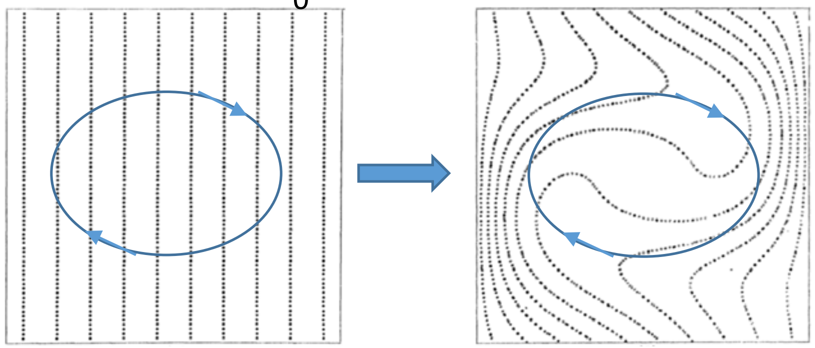
→ CHNS qualifies as an 'elastic fluid'.

# What of a Single Eddy?
# (Homogenization)

# Flux Expulsion

- Simplest dynamical problem in MHD (Weiss '66, et. seq.)
- Closely related to "PV Homogenization"

$B_0$

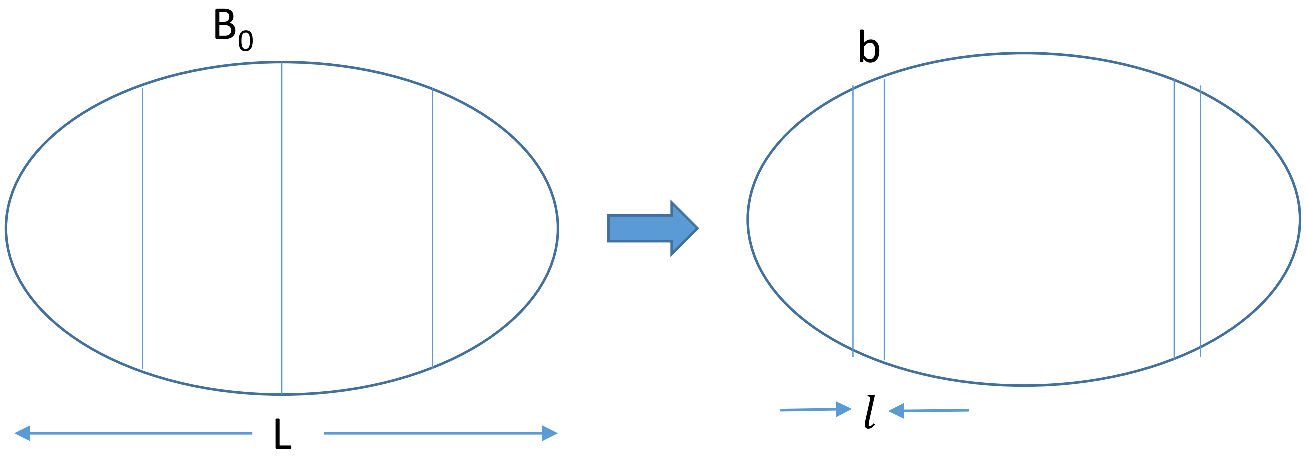


$Rm \sim vL/\eta \gg 1$

- Field wound-up, "expelled" from eddy
- For large Rm, field concentrated in boundary layer of eddy
- Ultimately, back-reaction asserts itself for sufficient $B_0$

  c.f. Gilbert et. al. '16; Mak et. al. '17

# How to Describe?




after n turns:
nl=L

- Flux conservation: $B_0L$~bl   Wind up: b=n$B_0$ (field stretched)
- Rate balance: wind-up ~ dissipation

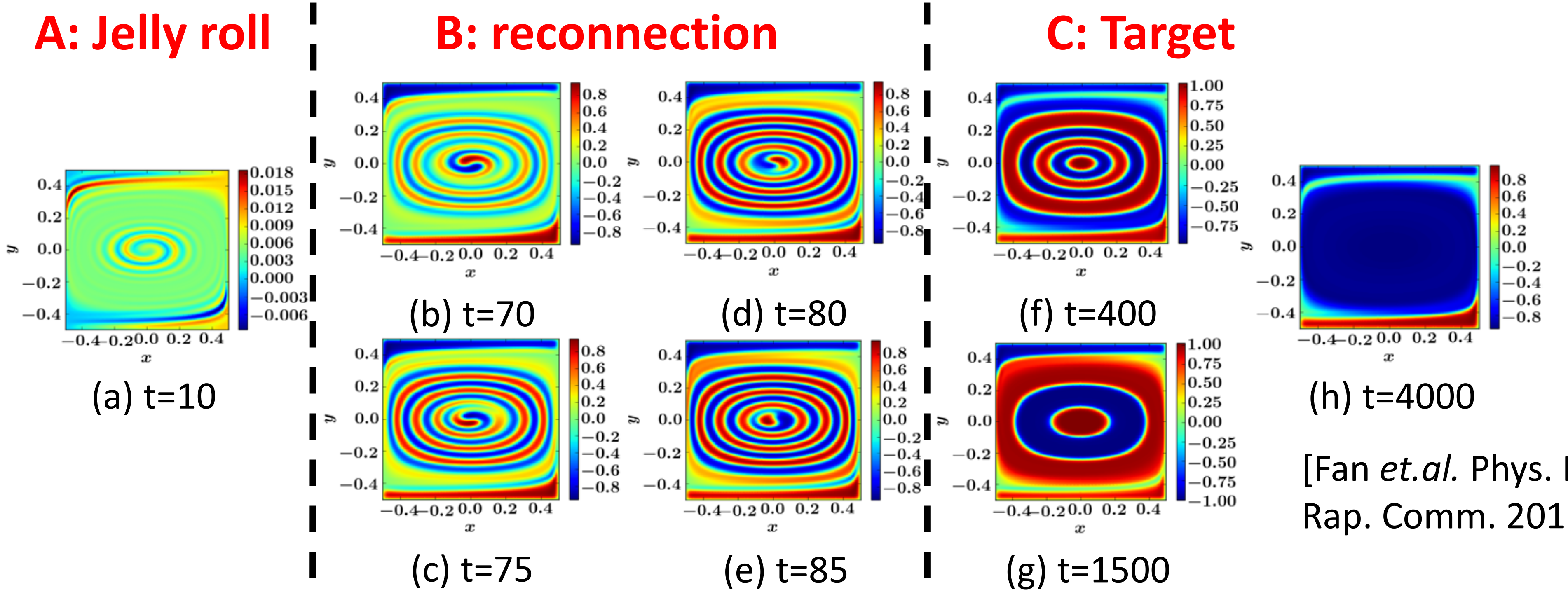
$$\frac{v}{L}B_0 \sim \frac{\eta}{l^2}b\ . \quad \tau_{expulsion} \sim \left(\frac{L}{v_0}\right)Rm^{1/3}.$$

$$l \sim \delta_{BL} \sim L/Rm^{1/3}\ . \quad b \sim Rm^{1/3}B_0\ .$$

N.B. differs from Sweet-Parker!

# Single Eddy Mixing -- Cahn-Hilliard

- 3 stages: (A) the *"jelly roll"* stage, (B) the *topological evolution* stage, and (C) the *target pattern* stage.
- $\psi$ ultimately homogenized on slow time scale, but metastable target patterns formed and coarsen.




[Fan *et.al.* Phys. Rev. E Rap. Comm. 2017]

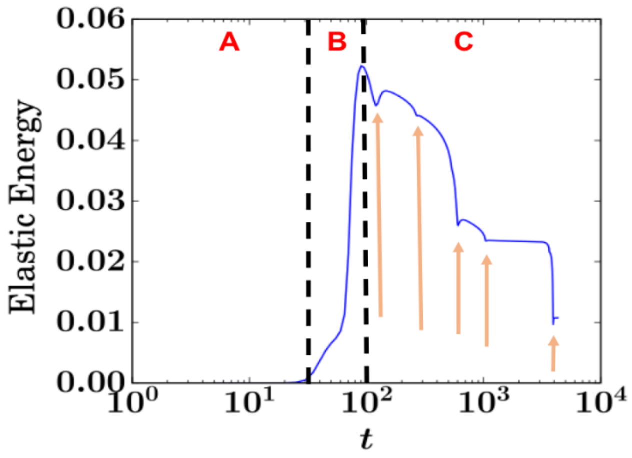
- Additional mixing time emerges.

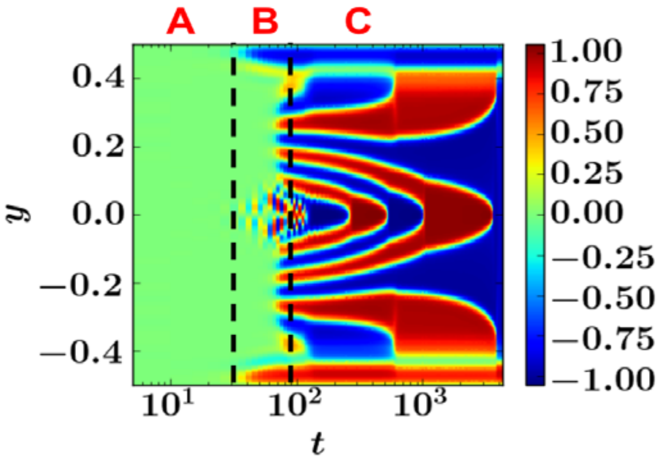
Note coarsening!

$$\tau_{mix} \sim \tau_0 \;/\; P_e^{-1/5}\; C_h^{-2/5}$$

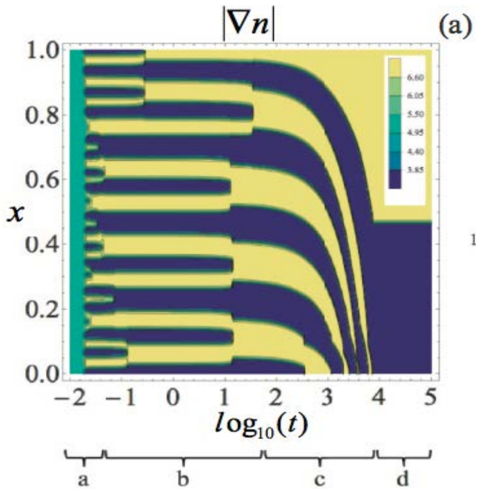
# Single Eddy Mixing – CH, cont'd

- The bands merge on a time scale long relative to eddy turnover time.
- The 3 stages are reflected in the elastic energy plot. Note t logarithmic.
- The target bands mergers are related to dips in target pattern stage.
- The band merger process is similar to the step merger in drift-ZF staircases.




Episodic relaxation-coarsening Cahn-Hilliard dynamics




[Ashourvan *et.al.* 2016]

- Contrast: Rhines and Young

2 steps: - Shear Dispersion $\sim \tau_0\, R_e^{1/3}$

- Viscous Mixing $\sim \tau_0\, R_e$

# Some Aspects of CHNS Turbulence

A Comparison and Contrast with 2D MHD

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# MHD Turbulence – Quick Primer

- (Weak magnetization / 2D)
- Enstrophy conservation broken
- Alfvenic in $B_{rms}$ field – "magneto-elastic" (E. Fermi '49)

$$\epsilon = \frac{\langle \tilde{v}^2 \rangle^2}{l^2} \frac{l}{B_{rms}} \implies E(k) = (\epsilon B_{rms})^{1/2} k^{-3/2}$$

- Dual cascade: Forward in energy; Inverse in $\langle A^2 \rangle \sim k^{-7/3}$

reduced transfer rate: Kraichnan

- What is dominant (A. Pouquet)?
  - conventional wisdom focuses on energy
  - yet $\langle A^2 \rangle$ conservation – freezing-in law!? 3D → $\langle \vec{A} \cdot \vec{B} \rangle$

→ Is the inverse cascade of $\langle A^2 \rangle$ the 'real' process, with energy dragged to small scale by fluid?

→ i.e. 'Pouquet Conjecture'

# Ideal Quadratic Conserved Quantities

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2. Mean Square Magnetic Potential

$$H^A = \int A^2 \, d^2x$$

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- **2D CHNS**

1. Energy

$$E = E^K + E^B = \int (\frac{v^2}{2} + \frac{\xi^2 B_\psi^2}{2}) d^2x$$

2. Mean Square Concentration

$$H^\psi = \int \psi^2 \, d^2x$$

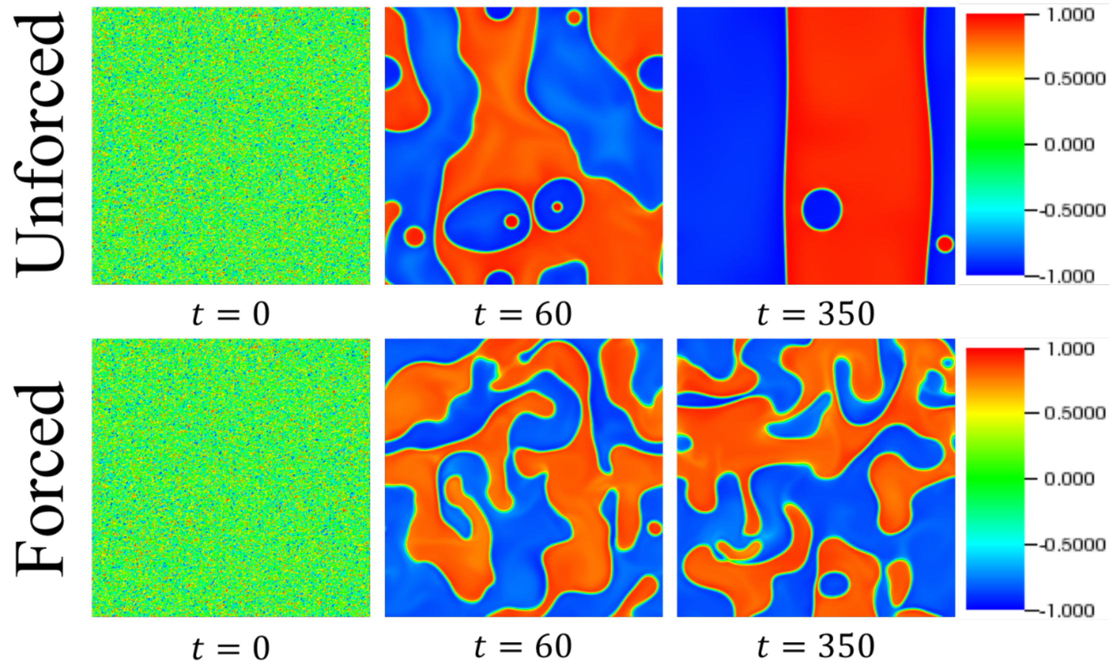
3. Cross Helicity

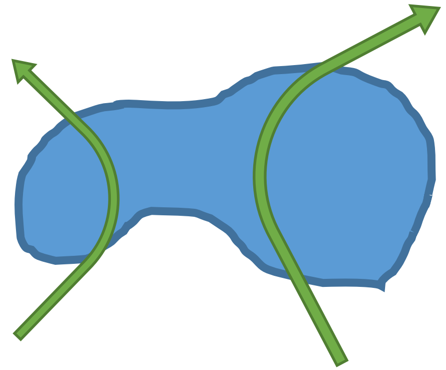
$$H^C = \int \vec{v} \cdot \vec{B}_\psi \, d^2x$$

Dual cascade expected!

# Scales, Ranges, Trends





How big is a raindrop?

- Turbulent straining vs capillarity.
- $\rho v^2$ vs $\sigma / l$.

[Hinze 1955]

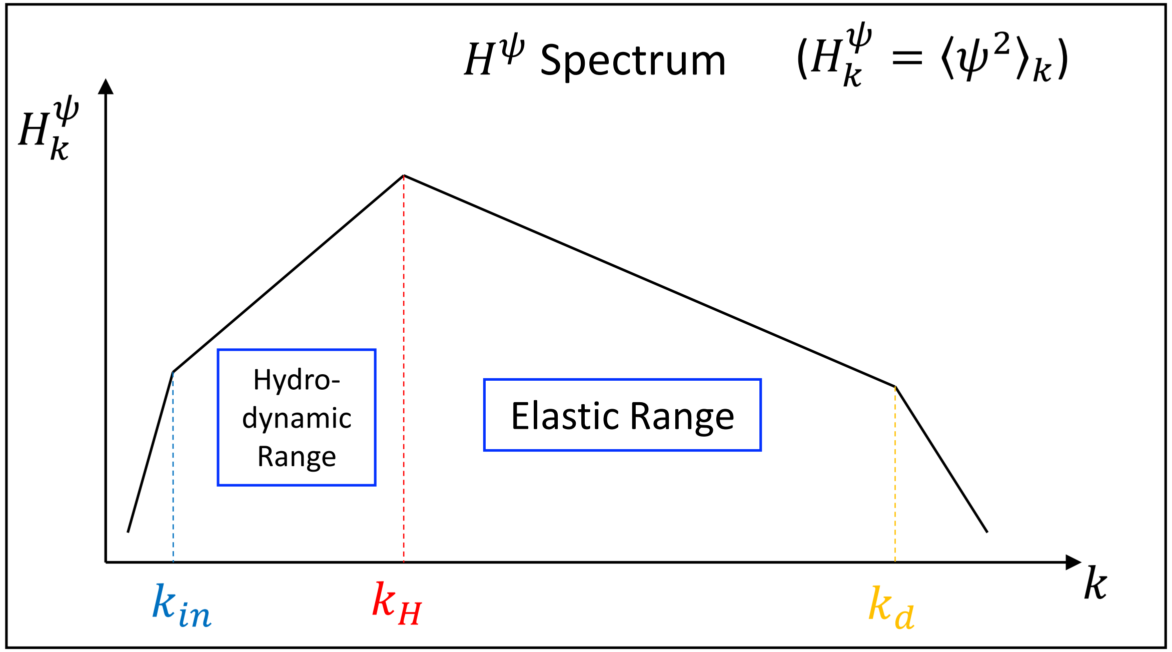
- ➢Fluid forcing → Fluid straining vs Blob coalescence
- ➢Straining vs coalescence is fundamental struggle of CHNS turbulence
- ➢Scale where turbulent straining ~ elastic restoring force (due surface tension): <u>*Hinze Scale*</u>

$$L_H \sim \left(\frac{\rho}{\xi}\right)^{-1/3} \epsilon_\Omega^{-2/9}$$

- ➢Like Ozmidov, Rhines, $\beta$ scales ... Hinze scale is <u>emergent</u>

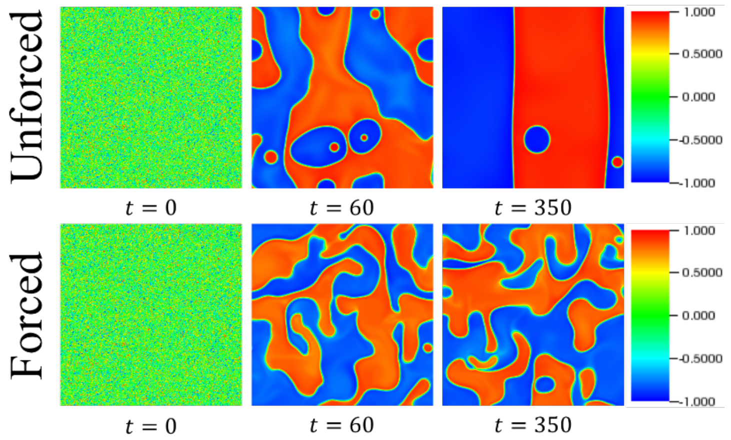
# Scales, Ranges, Trends

- Elastic range: $L_H > l > L_d$: where elastic effects matter.
- $L_H/L_d \sim (\frac{\rho}{\xi})^{-1/3} \nu^{-1/2} \epsilon_\Omega^{-1/18}$ → Extent of the elastic range
- $L_H \gg L_d$ required for large elastic range → case of interest



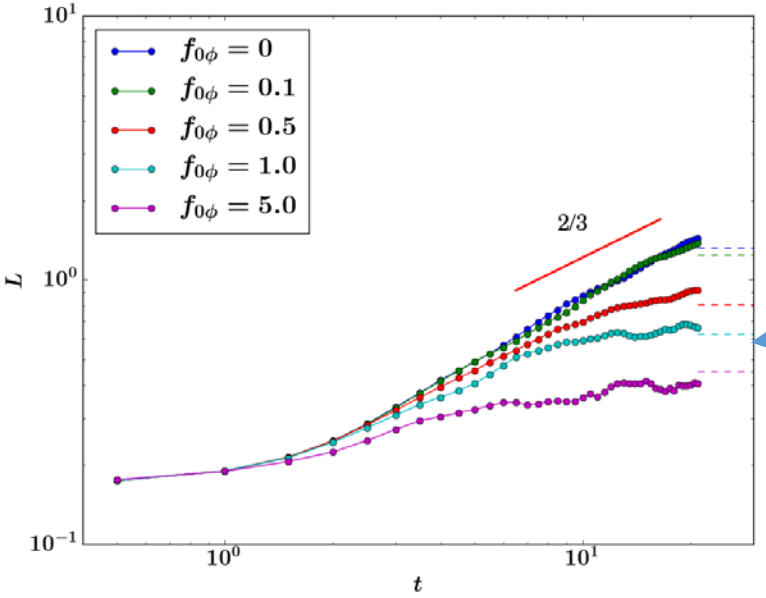

CHNS vs Elastic Turbulence

# Scales, Ranges, Trends




- Key elastic range physics: ***<u>Blob coalescence</u>***
- Unforced case: $L(t) \sim t^{2/3}$.

  (Derivation: $\vec{v} \cdot \nabla \vec{v} \sim \frac{\xi^2}{\rho} \nabla^2 \psi \nabla \psi \Rightarrow \frac{\dot{L}^2}{L} \sim \frac{\sigma}{\rho} \frac{1}{L^2}$)
- Forced case: blob coalescence arrested at Hinze scale $L_H$.

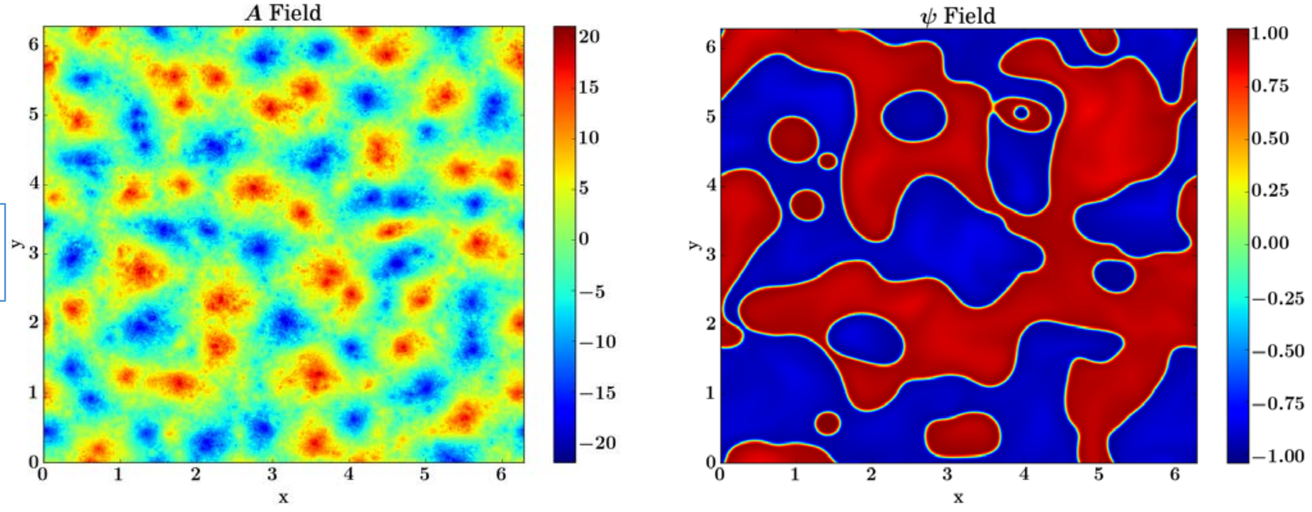



- $L(t) \sim t^{2/3}$ recovered
- Blob growth arrest observed
- Blob growth saturation scale tracks Hinze scale (dashed lines)

- Blob coalescence suggests <u>inverse cascade</u> <u>is</u> <u>fundamental here</u>.

# Cascades: Comparing the Systems

MHD




CHNS

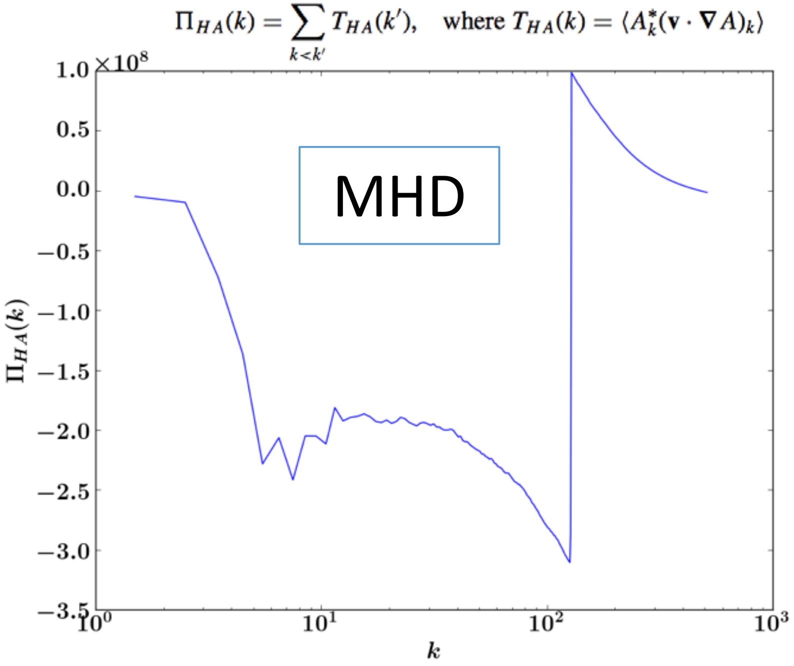
- Blob coalescence in the elastic range of CHNS is analogous to flux coalescence in 2D MHD.
- Suggests ***inverse cascade*** of $\langle \psi^2 \rangle$ in CHNS.
- Arrested by straining.

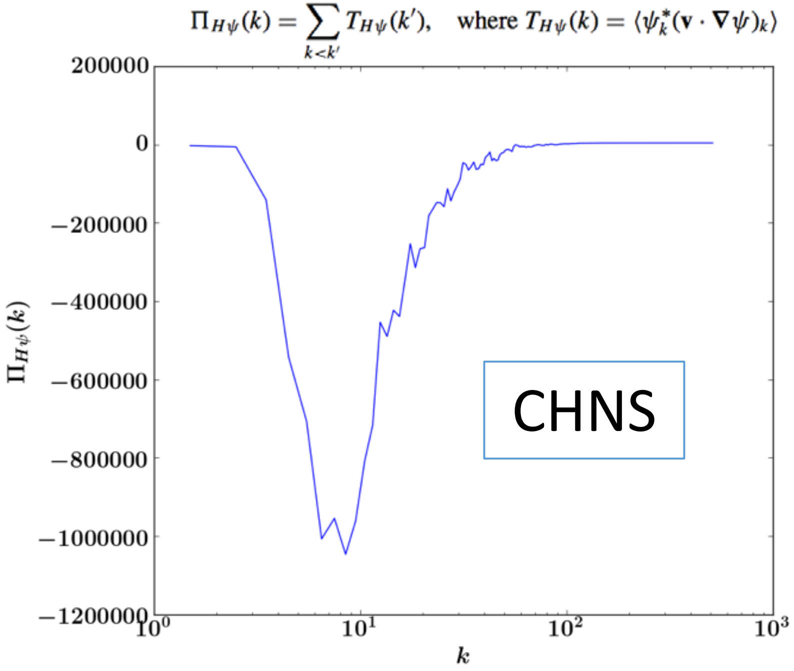
# Cascades - the Story (one might think)

- So, *dual cascade*:
  - ***Inverse*** cascade of $\langle\psi^2\rangle$
  - ***Forward*** cascade of $E$
- Inverse cascade of $\langle\psi^2\rangle$ is formal expression of blob coalescence process → generate larger scale structures till limited by straining
- Forward cascade of $E$ as usual, as elastic force should break enstrophy conservation
- Forward cascade of energy is analogous to counterpart in 2D MHD

# Cascades

➢Spectral flux of $\langle A^2 \rangle$: Spectral flux of $\langle \psi^2 \rangle$:



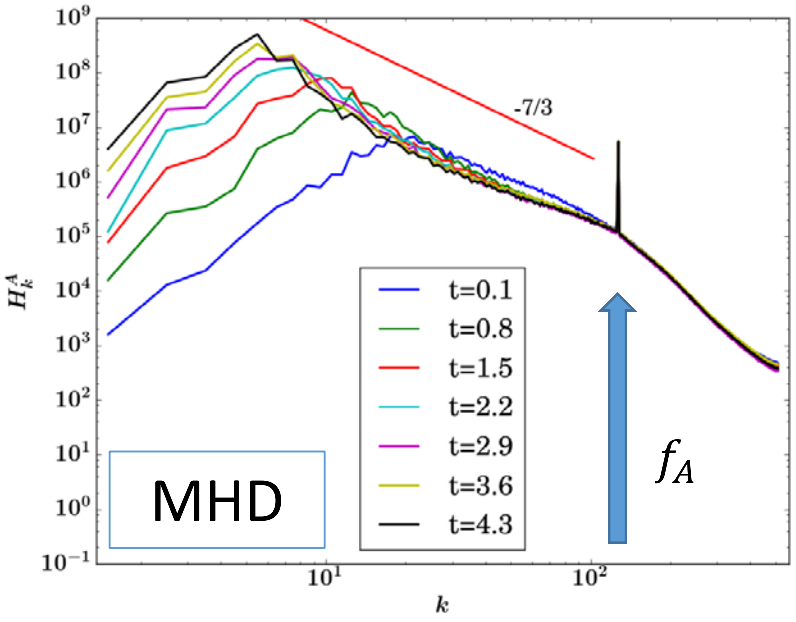

➢MHD: weak additional small scale forcing on $A$ drives inverse cascade

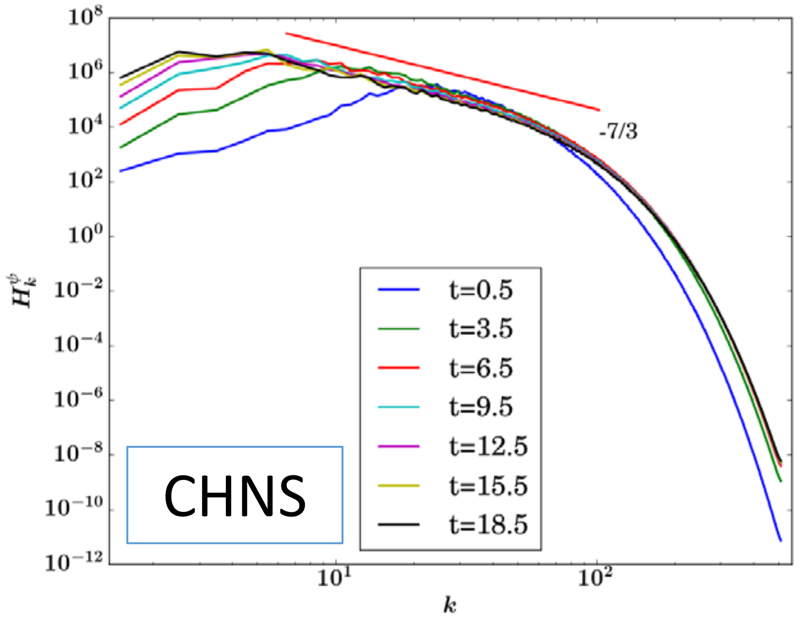
➢CHNS: $\psi$ is unforced → aggregates *naturally* ⟺ structure of free energy

➢Both fluxes ***negative*** → ***inverse*** cascades

# Power Laws

- $\langle A^2 \rangle$ spectrum: $\langle \psi^2 \rangle$ spectrum:




- Both systems exhibit $k^{-7/3}$ spectra.
- Inverse cascade of $\langle \psi^2 \rangle$ exhibits same power law scaling, so long as $L_H \gg L_d$, maintaining elastic range: Robust process.

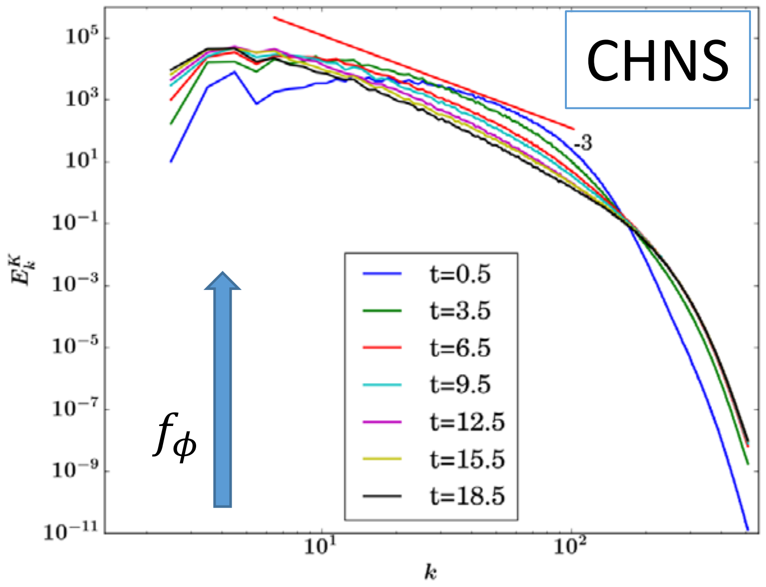
# Power Laws

- Derivation of -7/3 power law:
- For MHD, key assumptions:
  - Alfvenic equipartition ($\rho\langle v^2\rangle \sim \frac{1}{\mu_0}\langle B^2\rangle$ )
  - Constant mean square magnetic potential dissipation rate $\epsilon_{HA}$, so $\epsilon_{HA}\sim\frac{H^A}{\tau}\sim(H_k^A)^{\frac{3}{2}}k^{\frac{7}{2}}$.
- Similarly, assume the following for CHNS:
  - Elastic equipartition ($\rho\langle v^2\rangle \sim \xi^2\langle B_\psi^2\rangle$)
  - Constant mean square magnetic potential dissipation rate $\epsilon_{H\psi}$, so $\epsilon_{H\psi}\sim\frac{H^\psi}{\tau}\sim(H_k^\psi)^{\frac{3}{2}}k^{\frac{7}{2}}$.

# More Power Laws

- Kinetic energy spectrum (Surprise!):
- 2D CHNS: $E_k^K \sim k^{-3}$;
- 2D MHD: $E_k^K \sim k^{-3/2}$.

!




- The -3 power law:
  - Closer to enstrophy cascade range scaling, in 2D Hydro turbulence.
  - Remarkable departure from expected -3/2 for MHD. ***Why?***
- Why does CHNS ←→ MHD correspondence hold well for $\langle \psi^2 \rangle_k \sim \langle A^2 \rangle_k \sim k^{-7/3}$, yet break down drastically for energy???
- ***What physics*** underpins this surprise??

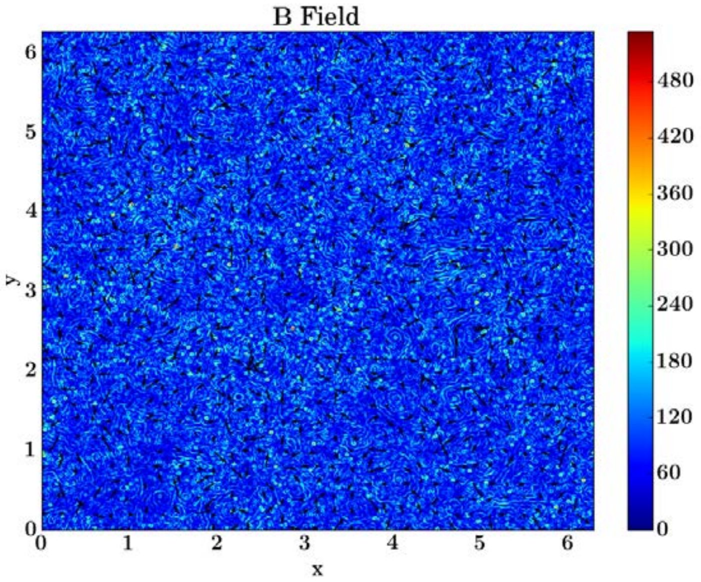
# Interface Packing Matters! – Pattern!

- Need to understand ***differences***, as well as similarities, between CHNS and MHD problems.

**2D MHD:**

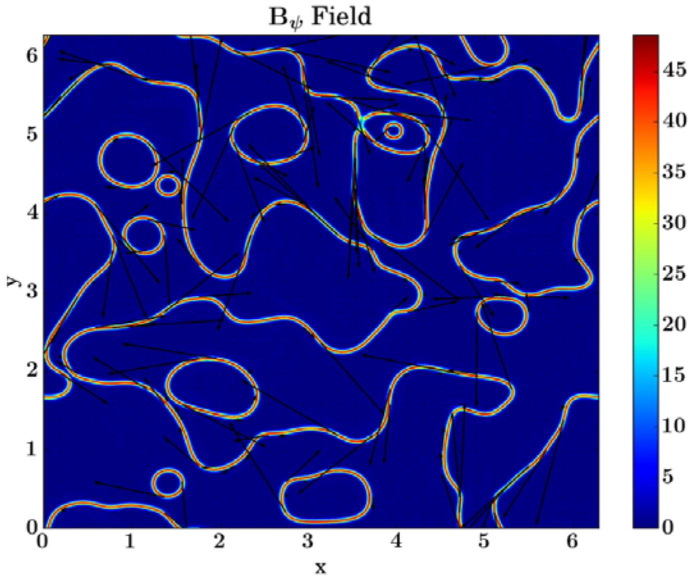
- Fields pervade system.

MHD




**2D CHNS:**

- Elastic back-reaction is limited to regions of density contrast i.e. $|\vec{B}_{\psi}| = |\nabla\psi| \neq 0$.
- As blobs coalesce, interfacial region diminished. 'Active region' of elasticity decays.



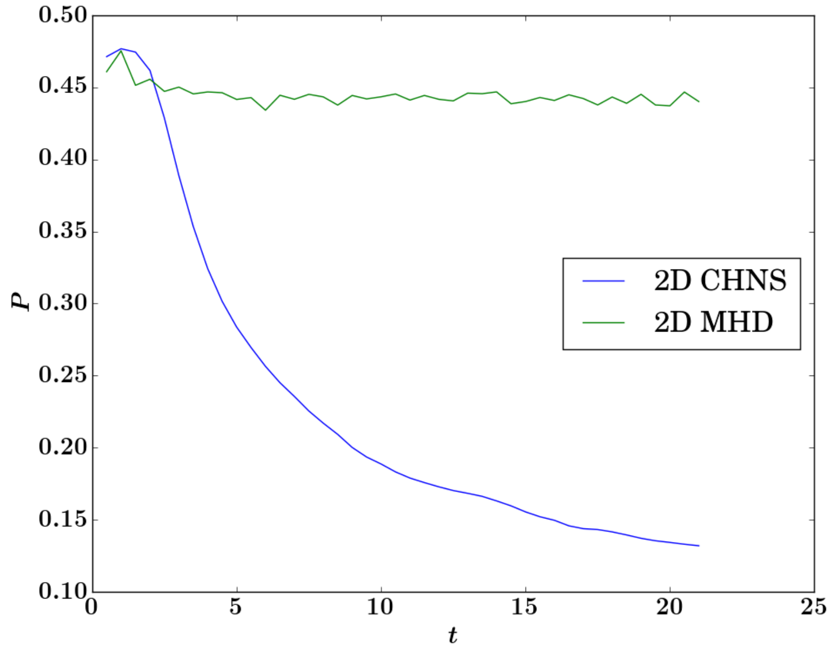

CHNS

# Interface Packing Matters!

- Define the ***interface packing fraction*** $P$:

$$P = \frac{\#\text{ of grid points where } |\vec{B}_\psi| > B_\psi^{rms}}{\#\text{ of total grid points}}$$

  - $P$ for CHNS decays;
  - $P$ for MHD stationary!




- $\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega$: small $P$ → local back reaction is weak.
- Weak back reaction → reduce to 2D hydro → k-spectra
- Blob coalescence coarsens interface network

# What Are the Lessons?

- Avoid power law tunnel vision!
- ***<u>Real space</u>*** realization of the flow is necessary to understand key dynamics. Track interfaces and packing fraction $P$.
- One player in dual cascade (i.e. $\langle\psi^2\rangle$) can modify or constrain the dynamics of the other (i.e. $E$).
- Against conventional wisdom, $\langle\psi^2\rangle$ inverse cascade due to blob coalescence is the robust nonlinear transfer process in CHNS turbulence.
- Begs more attention to magnetic helicity in 3D MHD.

# Looking Ahead

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- (Whacky) $\beta$ –plane CHNS (after $\beta$ –plane MHD)

  Experiment? (c.f. Tobias, et. al. '07)

  Point: Hinze vs. Rhines interplay

- Drag Reduction (slightly less whacky)

  ~ in pipe, a transport barrier problem

  - The question: CHNS vs Polymers (Oldroyd-B, et. seq.)
  - Polymers: Zimm damping, scale independent

    $\epsilon^{1/3} \,/\, l^{2/3} > w_z$ for activation

  - CHNS: hyper diffusion

    $\epsilon^{1/3} / \, l^{2/3} > Dh^2 \,/\, l^4$ active. Low Cahn # !

# Reading

Fan, P.D., Chacon:

- PRE Rap Comm 99, 041201 (2019)
  - → Active Scalar Transport 2D MHD
- PoP 25, 055702 (2018)
  - → Plasma/MHD Connection
- PRE Rap Comm 96, 041101 (2017)
  - → Single Eddy
- Phys Rev Fluids 1, 054403 (2016)
  - → Turbulence

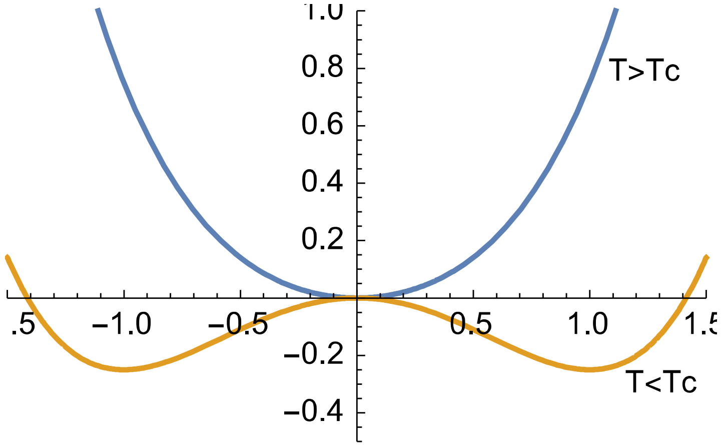
# Thank you!

# Back-Up

# A Brief Derivation of the CHNS Model

- ➢Second order phase transition → Landau Theory.
- ➢<u>Order parameter</u>: $\psi(\vec{r},t) \stackrel{\text{def}}{=} [\rho_A(\vec{r},t) - \rho_B(\vec{r},t)]/\rho$ → density contrast
- ➢<u>Free energy</u>:

$$\mathrm{F}(\psi) = \int d\vec{r}\Big(\underbrace{\frac{1}{2}C_1\psi^2 + \frac{1}{4}C_2\psi^4}_{\text{Phase Transition}} + \underbrace{\frac{\xi^2}{2}|\nabla\psi|^2}_{\text{Gradient Penalty}}\Big)$$




- ➢$C_1(T), C_2(T)$.
- ➢Isothermal $T < T_C$. Set $C_2 = -C_1 = 1$:

$$\mathrm{F}(\psi) = \int d\vec{r}\Big(-\frac{1}{2}\psi^2 + \frac{1}{4}\psi^4 + \frac{\xi^2}{2}|\nabla\psi|^2\Big)$$

# A Brief Derivation of the CHNS Model

- Continuity equation: $\frac{d\psi}{dt} + \nabla \cdot \vec{J} = 0$. Fick's Law: $\vec{J} = -D\nabla\mu$.
- Chemical potential: $\mu = \frac{\delta F(\psi)}{\delta\psi} = -\psi + \psi^3 - \xi^2\nabla^2\psi$.
- Combining above → Cahn Hilliard equation:

$$\boxed{\frac{d\psi}{dt} = D\nabla^2\mu = D\nabla^2(-\psi + \psi^3 - \xi^2\nabla^2\psi)}$$

- $d_t = \partial_t + \vec{v} \cdot \nabla\mu$: force in Navier-Stokes equation:

$$\boxed{\partial_t\vec{v} + \vec{v} \cdot \nabla\vec{v} = -\frac{\nabla p}{\rho} - \psi\nabla\mu + \nu\nabla^2\vec{v}}$$

- For incompressible fluid, $\nabla \cdot \vec{v} = 0$.

# Why 2D ?

- Electromagnetic Plasma Turbulence

    2D MHD + Alfven Wave in 3$^{rd}$ Direction

    = Reduced MHD / Strauss Eqns (after Rosenbluth, Kadomtsev)

- Zonal Flow Formation - akin 'Spinodal Decomposition of Momentum'

    - c.f. Manfroi – Young

- 'Blooby Turbulence' – Spatial Structure and Coalescence

CHNS touches on all, with many new twists