Elastic Turbulence:

A Look at Some Simple Systems

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Outline

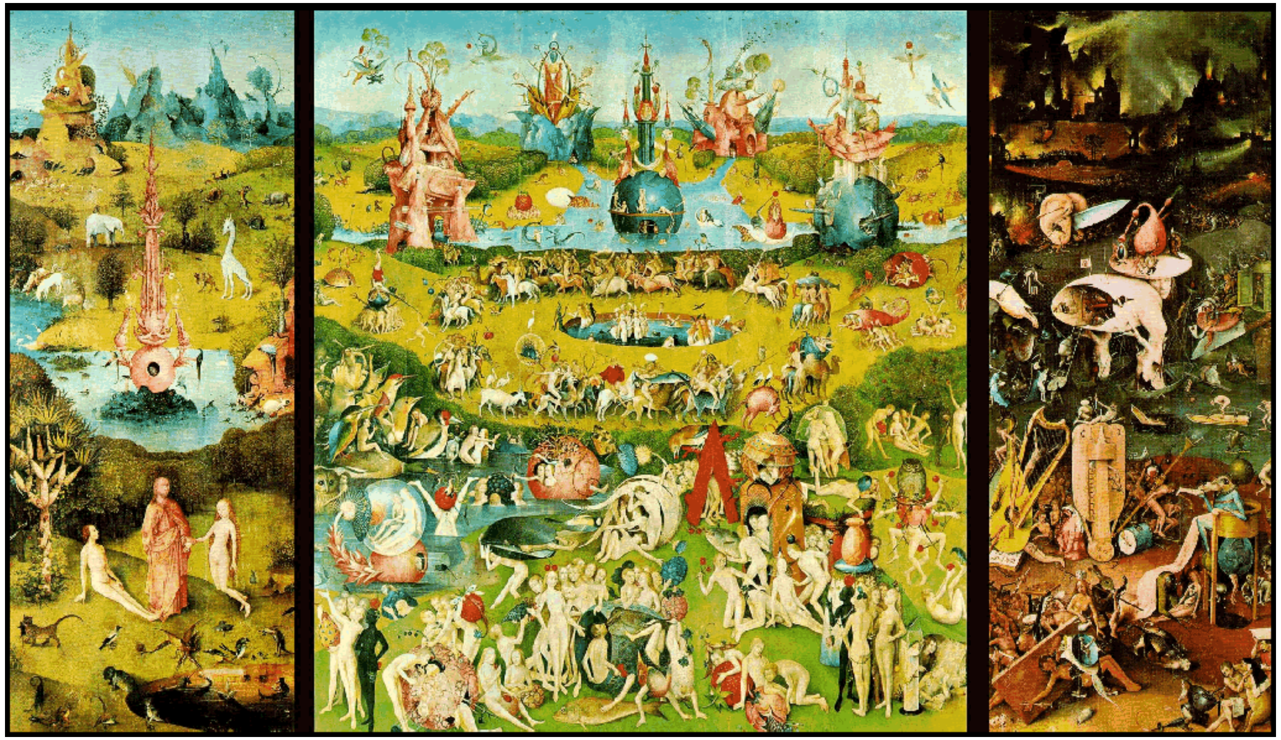
- What is Turbulence?
- What and Why of Elastic Fluids, and CHNS, in particular

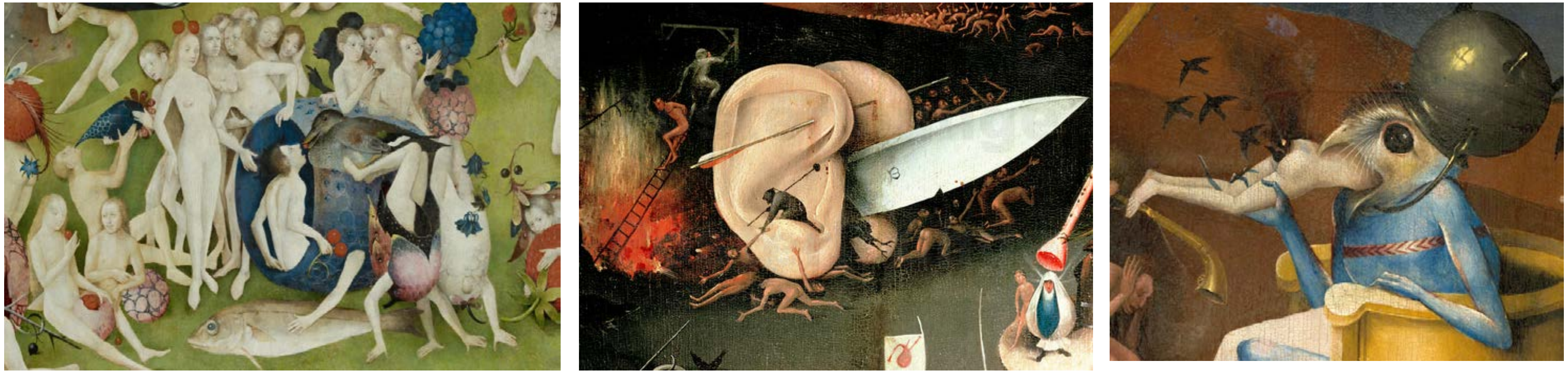
  CHNS ≡ Cahn-Hilliard Navier-Stokes
- Single Eddy Problem
- CHNS Turbulence
- Transport and Beyond
- Lessons

What is Turbulence?

Turbulence (after Kadomtsev)

"The Garden of Earthly Delights", Hieronymous Bosch





Model

- Navier-Stokes Equation:

$$\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\nabla\vec{v}-\nu\nabla^2\vec{v}\right)=-\nabla P+\tilde{f}$$

$$\nabla\cdot\vec{v}=0$$

Random forcing
(usually large scale)

  - Finite domain, closed, periodic
  - $Re = v\cdot\nabla v\,/\nu\nabla^2 v \sim VL/\nu \quad ; \quad Re \gg 1$

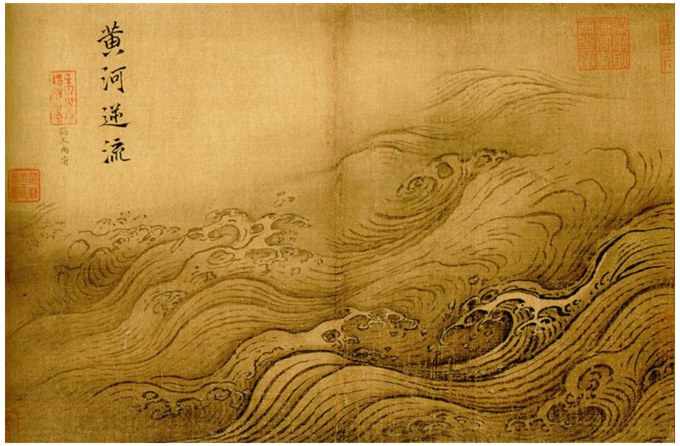
- Variants:
  - 2D, QG
  - Compressible flow
  - Pipe flow – inhomogeneity
  - MHD, etc.

What is turbulence? (3D)

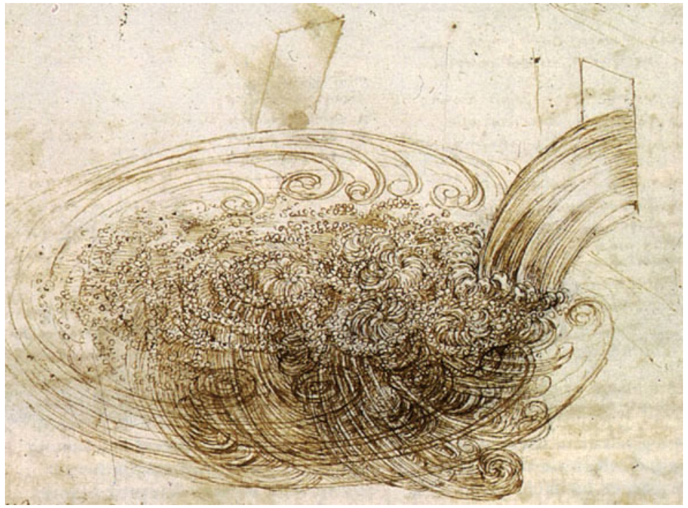
- Spatio-temporal “disorder”
- Broad range of space-time scales
- Power transfer / flux thru broad range of scales *
- Energy dissipation and irreversibility as $Re \to \infty$ *

And:

- Decay of large scales
- Irreversible mixing
- Intermittency / burstiness



Ma Yuan

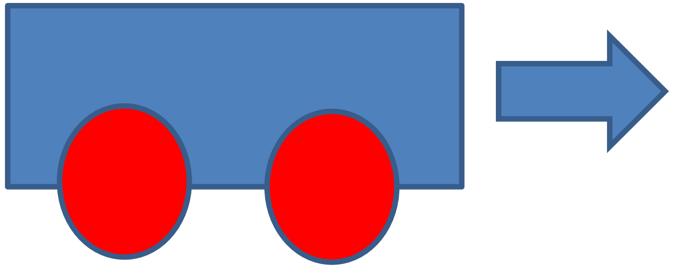


Leonardo

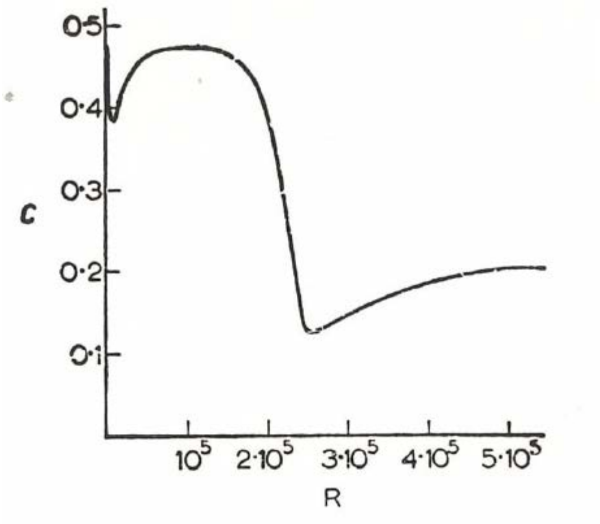
Why broad range scales?
What motivates cascade concept?

A) Planes, trains, automobiles…

<u>DRAG</u>



- Recall: $F_d \sim c_D \rho A V^2$
- $C_D = C_D(Re)$ → drag coefficient



$C_D \sim Re^{(0)}$ as $Re \to \infty$

- The Point:
  - Energy dissipation is finite, and due to viscosity, yet does not depend explicitly on viscosity ➔ $\boxed{\text{ANOMALY}}$
  - ‘Irreversibility persists as symmetry breaking factors vanish’

i.e. $\frac{dE}{dt} \sim F_d\, V \sim C_D \rho A V^3$

$\frac{dE}{dt} \sim \frac{V^3}{l_0} \equiv \epsilon$ → dissipation rate l_0→ macro length scale

- Where does the energy go?

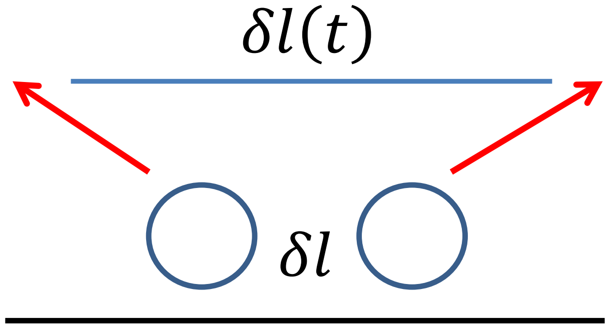
Steady state $\nu\langle(\nabla\vec{v})^2\rangle = \langle\vec{f}\cdot\vec{v}\rangle = \epsilon$

- So $\epsilon = \nu\langle(\nabla \mathrm{v})^2\rangle$ ← independent of ν
- $(\nabla v)_{rms} \sim \frac{1}{\nu^{1/2}}$ ➔ <u>suggests</u> → singular velocity gradients (small scale)

$\therefore$

- Flat C_D in Re → turbulence must access small scales as $Re \rightarrow \infty$
- Obviously consistent with broad spectrum, via nonlinear coupling

B) … and balloons



- Study of 'test particles' in turbulence:
- Anecdotal:

  Titus Lucretius Caro: 99-55 BC

  "De rerum Nature" cf. section V, line 500

- Systematic:

  L.F. Richardson: - probed atmospheric turbulence by study of balloon separation

  Noted: $\langle \delta l^2 \rangle \sim t^3$ → <u>super-diffusive</u>

  - not ~ t, ala' diffusion, noise
  - not exponential, ala' smooth chaotic flow

Upshot:

$\delta V(l) = \left(\left(\vec{v}(\vec{r}+\vec{l}) - \vec{v}(\vec{r})\right) \cdot \frac{\vec{l}}{|\vec{l}|}\right)$ → structure function → velocity differential across scale

Then: $\delta V \sim l^{\alpha}$

so, $\frac{dl}{dt} \sim l^{\alpha}$ → growth of separation

$\rightarrow \langle l^2 \rangle \sim t^{\frac{2}{1-\alpha}} \sim t^3$

$$\boxed{\rightarrow \alpha = \frac{1}{3}}$$

so $\delta V(l) \sim l^{1/3}$, $\langle \delta l^2 \rangle \sim t^3$

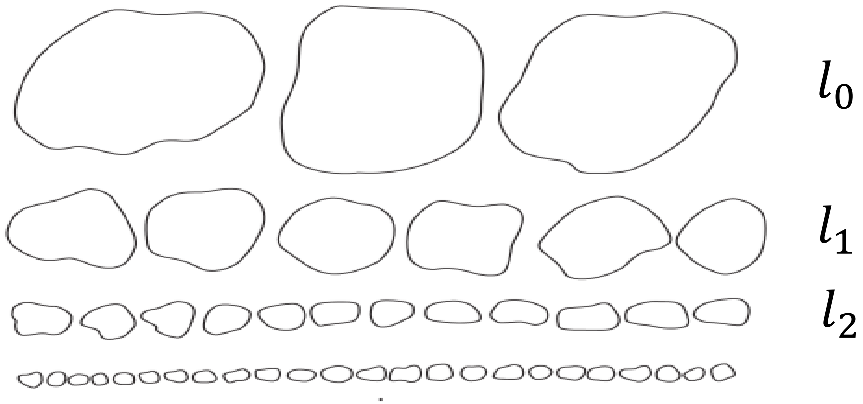
→ Points:

- large eddys have more energy, so rate of separation increases with scale
- Relative separation is excellent diagnostic of flow dynamics

cf: tetrads: Siggia and Shraiman

K41 Model (Phenomenological)

- Cascade → hierarchical fragmentation



- Broad range of scales, no gaps
- Described by structure function
- $\langle \delta V(l)^2 \rangle$, …. $\langle \delta V(l)^n \rangle$, …

Related to energy distribution
←→ greatest interest

- $\langle \delta v(l)^2 \rangle \leftrightarrow$ energy,
of great interest

- higher moments
more challenging

- Input:
- 2/3 law (empirical)

  $S_2(l) \sim l^{2/3}$

- 4/5 law (Rigorous) - TBD

  $\langle \delta V(l)^3 \rangle = -\frac{4}{5}\epsilon l$

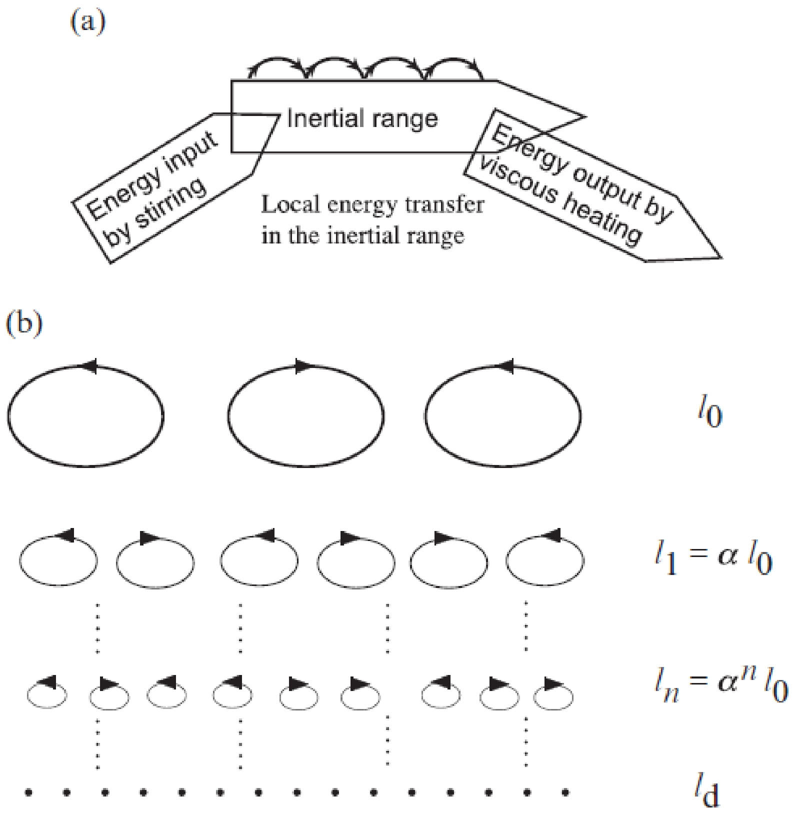
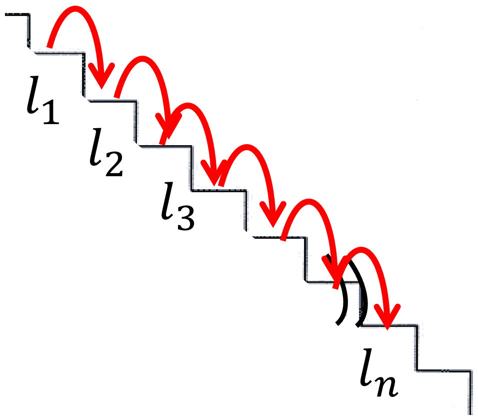


Fig. 2.12. Basic cartoon explanation of the Richardson–Kolmogorov cascade. Energy transfer in Fourier–space (a), and real scale (b)

→ Ideas:

- Flux of energy in scale space from l_0 (input/integral scale) to l_d (dissipation) scale – set by ν
- Energy flux is same at all scales between l_0, l_d <-> self-similarity

→ So



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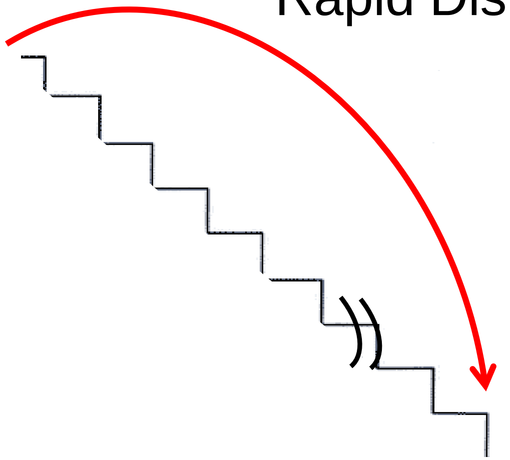


exception:
Rapid Distortion Theory

→ $\epsilon \sim V(l)^2 / \tau(l) \sim V(l)^3 / l$ → $V(l) \sim (\epsilon l)^{1/3}$; $1/ \tau(l) \sim (\epsilon / l^2)^{1/3}$

→ $V(l)^2 \sim V_0^2 \; (l \,/\, l_0)^{2/3}$ (transfer rate increases as scale decreases)

And

→ $E(k) \sim \epsilon^{2/3} \, k^{-5/3}$ $\qquad E = \int dk E(k)$

→ Where does it end?

- Dissipation scale
  - cut-off at $1/\tau(l) \sim \nu/l^2$ i.e. $Re(l) \to 1$
  - $l_d \sim \nu^{3/4}\,/\epsilon^{1/4}$
- Degrees of freedom

$$\#DOFs \sim \left(\frac{l_0}{l_d}\right)^3 \sim Re^{9/4}$$

For $l_o \sim 1km$, $l_d \sim 1mm$ (PBL)

→ $N \sim 10^{18}$

The Theoretical Problem

- “We don’t want to *think* anything, man. We want to *know*.”

  – “Pulp Fiction” (Quentin Tarantino)

- What do we know?
  - 4/5 Law (and not much else...)

$\langle V(l)^3 \rangle = -\frac{4}{5}\epsilon l$ → asymptotic for finite $l, \nu \to 0$

$$S_2 = \langle \delta V(l)^2 \rangle$$
$$S_3 = \langle \delta V(l)^3 \rangle$$

from: $\frac{\partial S_2}{\partial t} = -\frac{1}{3l^4}\frac{\partial}{\partial l}(l^4 S_3) - \frac{4}{3}\epsilon + \frac{2\nu}{l^4}\frac{\partial}{\partial l}\left(l^4 \frac{\partial S_2}{\partial l}\right)$

(Karman-Howarth) flux in scale dissipation

- Stationarity, $\nu \to 0$

4/5 Law

- Asymptotically exact $\nu \to 0$, l finite
- Unique, rigorous result

- $\boxed{S_3(l) = -\frac{4}{5}\epsilon l}$
- Energy thru-put balance $\langle \delta V(l)^3 \rangle / l \;\leftrightarrow\; \epsilon$
- Notable:
  - Euler: $\partial_t v + v \cdot \nabla v + \nabla P/\rho = 0$; reversible; $t \to -t, v \to -v$
  - N-S: $\partial_t v + v \cdot \nabla v + \nabla P/\rho = \nu \nabla^2 v$; time reversal broken by viscosity
  - $S_3(l)$: $S_3(l) = -\frac{4}{5}\epsilon l$; reversibility breaking maintained as $\nu \to 0$

Anomaly

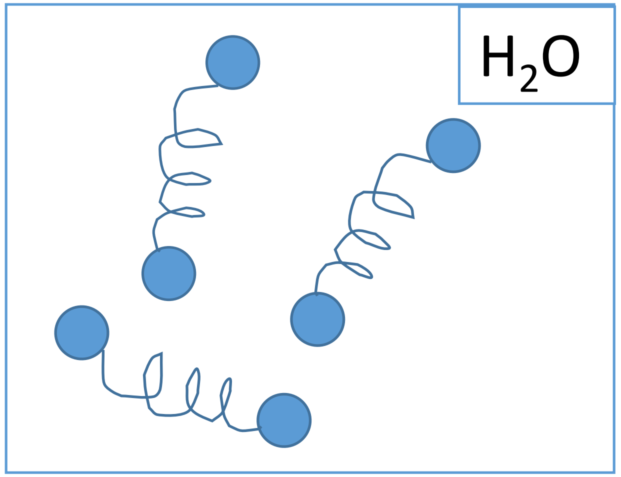
- N.B.: A little history; philosophy:
  - ‘Anomaly’ in turbulence → Kolmogorov, 1941
  - Anomaly in QFT → J. Schwinger, 1951 (regularization for vacuum polarization)
- Speaking of QFT, what of renormalized perturbation theory?
  - Renormalization gives some success to low order moments, identifies relevant scales
  - Useful in complex problems (i.e. plasmas) and problems where τ_{int} is not obvious
  - Rather few fundamental insights have emerged from R.P.T

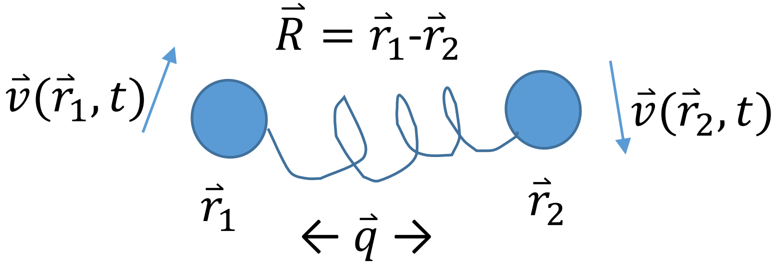
    Caveat Emptor

What and Why of Elastic Fluids?

Elastic Fluid -> Oldroyd-B Family Models

→ Solution of Dumbells





Internal DoF
i.e. polymers

- $\gamma\left(\frac{d\vec{r}_{1,2}}{dt} - \vec{v}(\vec{r}_{1,2}, t)\right) = -\frac{\partial U}{\partial \vec{r}_{1,2}} + \vec{\xi}$, where $U = \frac{k}{2}(\vec{r}_1 - \vec{r}_2)^2 + \cdots$

  stokes drag

  entropic spring

  noise

- so $\frac{d\vec{R}}{dt} = \vec{v}(\vec{R}, t) + \vec{\xi}/\gamma$, and $\frac{d\vec{q}}{dt} = \vec{q} \cdot \nabla\, \vec{v}(\vec{R}, t) - \frac{2}{\gamma}\frac{\partial U}{\partial \vec{q}}$ + noise

Seek $f(\vec{q},\vec{R},t|\vec{v},\dots) \rightarrow$ distribution

- $$\partial_t f + \partial_{\vec{R}} \cdot \left[\vec{v}(\vec{R},t) f\right] + \partial_{\vec{q}} \cdot \left[\vec{q} \cdot \nabla \vec{v}(\vec{R},t) f - \frac{2}{\gamma}\frac{\partial U}{\partial \vec{q}} f\right]$$
  $$= \partial_{\vec{R}} \cdot \mathbf{D}_0 \cdot \frac{\partial f}{\partial \vec{R}} + \partial_{\vec{q}} \cdot \mathbf{D}_\mathrm{q} \cdot \frac{\partial f}{\partial \vec{q}}$$

Is F.P. valid?!

- and moments:

  $$Q_{ij}(\vec{R},t) = \int d^3q \; q_i q_j f(\vec{q},\vec{R},t)$$ → electric energy field (tensor)

- so:

  strain

  $$\partial_t Q_{ij} + \vec{v} \cdot \nabla Q_{ij} = Q_{i\gamma}\partial_\gamma v_j + Q_{j\gamma}\partial_\gamma v_i$$
  $$-\omega_z Q_{ij} + D_0 \nabla^2 Q_{ij} + 4\frac{k_B T}{\gamma}\delta_{ij}$$

  relaxation

and concentration equation

- Defines Deborah number: $\nabla\vec{v}/\omega_z$

Reaction on Dynamics

- $\rho[\partial_t v_i + \vec{v} \cdot \nabla v_i] = -\nabla_i P + \nabla_i \cdot [\underset{\underset{\text{elastic stress}}{\uparrow}}{c_p} k Q_{ij}] + \eta \nabla^2 v_i + f_i$
- Classic systems; Oldroyd-B (1950).
- Extend to nonlinear springs (FENE), rods, rods + springs, networks, director fields, etc…
- Supports elastic *waves* and fluid dynamics, depending on Deborah number.
- Oldroyd-B ↔ *active tensor* field

Constitutive Relations

- J. C. Maxwell:

$$(\text{stress}) + \overset{\text{relaxation}}{\overset{\downarrow}{\tau_R}} \frac{d(\text{stress})}{dt} = \overset{\text{viscosity}}{\overset{\downarrow}{\eta}} \frac{d}{dt}(\text{strain})$$

- If $\tau_R/T = D \ll 1$, stress = $\eta \frac{d}{dt}(\text{strain})$

$$\sigma = -\eta \nabla \vec{v}$$

- If $\tau_R/T = D \gg 1$, stress $\cong \frac{\eta}{\tau_R}(\text{strain})$

$$\sim E\ (\text{strain})$$

- Limit of “freezing-in”: D>1 is criterion.

$T \equiv$ dynamic time scale

- D ~ Deborah Number ~ $|\nabla V|/\omega_Z$ ~ τ_{relax}/τ_{dyn}
- Limit for elasticity: $D \gg 1$ → limit for elasticity
- Why “Deborah”? →

Hebrew Prophetess Deborah:

“The moutains flowed before the Lord.” (Judges)

$\therefore$

- Revisit Heraclitus (1500 years later):

→ “All things flow” – if you can wait long enough

Relation to MHD?!

$\mathbf{T} \equiv$ stress

- Re-writing Oldroyd-B:

$$\frac{\partial}{\partial_t}\mathbf{T} + \vec{v} \cdot \nabla \mathbf{T} - \mathbf{T} \cdot \nabla \vec{v} - (\nabla \vec{v})^T \cdot \mathbf{T} = \frac{1}{\tau}\left(\mathbf{T} - \frac{\mu}{\tau}\mathbf{I}\right)$$

- MHD: $\mathbf{T}_m = \frac{\vec{B}\vec{B}}{4\pi}$

$$\partial_t \vec{B} + \vec{v} \cdot \nabla \vec{B} = \vec{B} \cdot \nabla \vec{v} + \eta \nabla^2 \vec{B}$$

- So

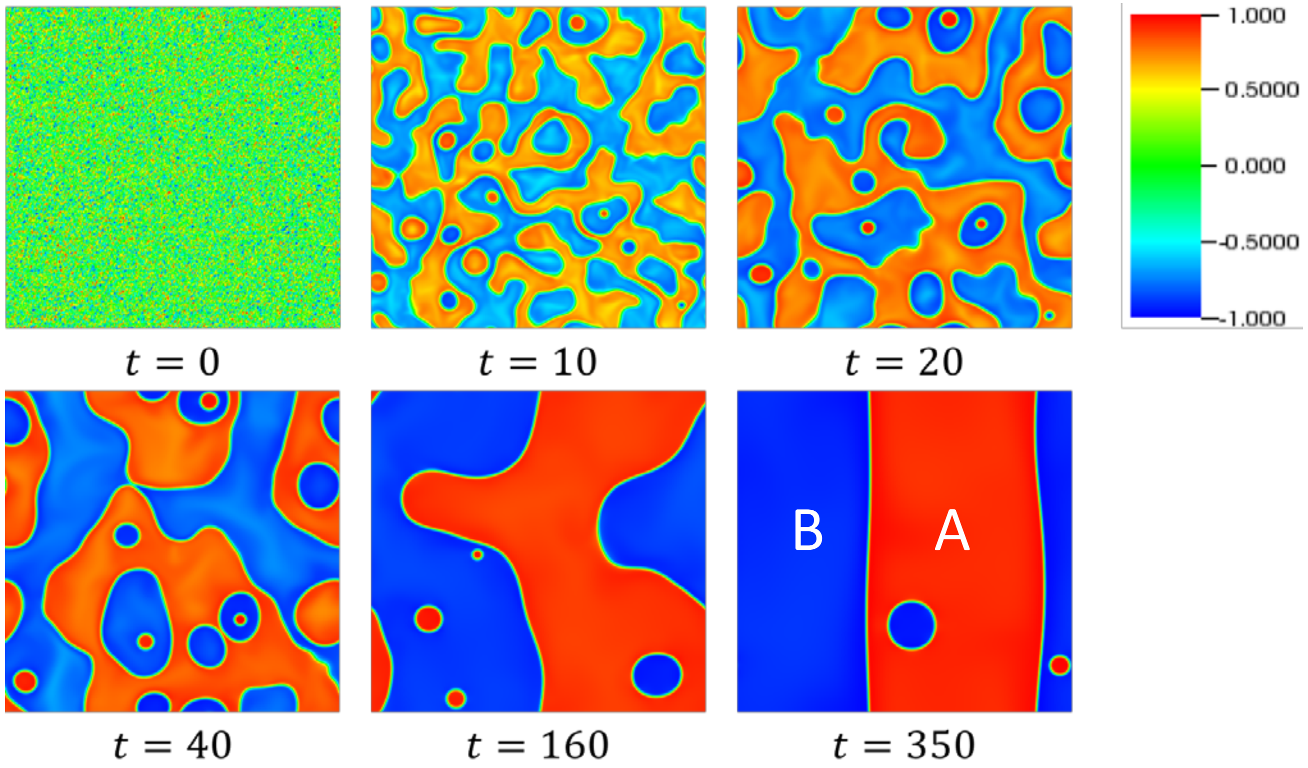
$$\frac{\partial}{\partial_t}\mathbf{T}_m + \vec{v} \cdot \nabla \mathbf{T}_m - \mathbf{T}_m \cdot \nabla \vec{v} - (\nabla \vec{v})^T \cdot \mathbf{T}_m = \eta[\vec{B}\nabla^2\vec{B} + (\nabla^2\vec{B})\vec{B}]$$

- $\boxed{\lim_{D\to\infty} \text{(Oldroyd-B)} \Leftrightarrow \lim_{R_m\to\infty} \text{(MHD)}}$

c.f. Ogilvie and Proctor

Elastic Media -- What Is the CHNS System?

- ➢Elastic media – Fluid with internal DoFs → "springiness"
- ➢The Cahn-Hilliard Navier-Stokes (CHNS) system describes ***phase separation*** for binary fluid (i.e. ***Spinodal Decomposition***)



$t = 0$ $t = 10$ $t = 20$

$t = 40$ $t = 160$ $t = 350$

[Fan *et.al*. Phys. Rev. Fluids 2016]

Miscible phase
→ Immiscible phase

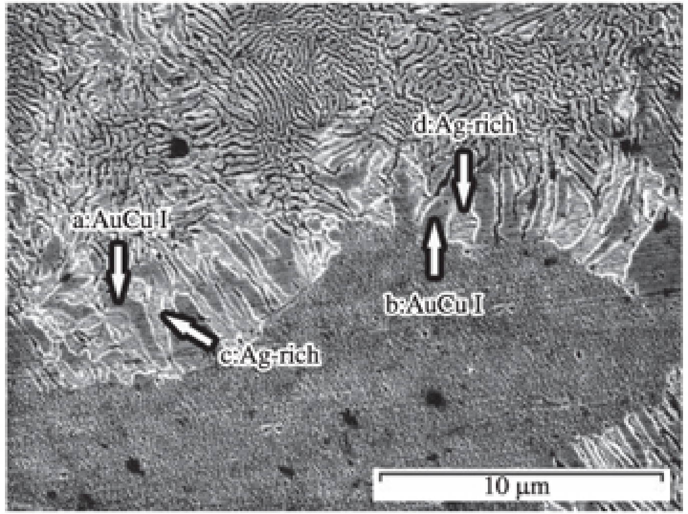


Figure 5. FE-SEM micrograph of specimen aged at 400 °C for 5000 minutes.

[Kim *et.al.* 2012]

Elastic Media? -- What Is the CHNS System?

- How to describe the system: the concentration field
- $\psi(\vec{r},t) \overset{\text{def}}{=} [\rho_A(\vec{r},t) - \rho_B(\vec{r},t)]/\rho$: scalar field → density contrast
- $\psi \in [-1,1]$
- CHNS equations (2D):

$$\boxed{\begin{gathered}\partial_t\psi + \vec{v}\cdot\nabla\psi = D\nabla^2(-\psi + \psi^3 - \xi^2\nabla^2\psi) \\ \partial_t\omega + \vec{v}\cdot\nabla\omega = \frac{\xi^2}{\rho}\vec{B}_\psi\cdot\nabla\nabla^2\psi + \nu\nabla^2\omega\end{gathered}}$$

Why Should a Plasma Physicist Care?

- Useful to examine familiar themes in plasma turbulence from new vantage point
- Some key issues in plasma turbulence:

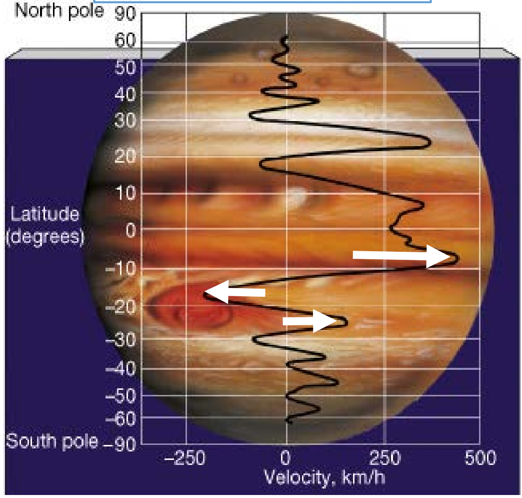
1. Electromagnetic Turbulence
    - CHNS vs 2D MHD: analogous, with interesting differences.
    - Both CHNS and 2D MHD are ***elastic*** systems
    - Most systems = 2D/Reduced MHD + many linear effects
        - Physics of dual cascades and constrained relaxation → relative importance, selective decay…
        - Physics of wave-eddy interaction effects on nonlinear transfer (i.e. Alfven effect ←→ Kraichnan)

MHD ←→ CHNS

Why Care?

2. Zonal flow formation → negative viscosity phenomena
    - ZF can be viewed as a “spinodal decomposition” of momentum.
    - What determines scale?

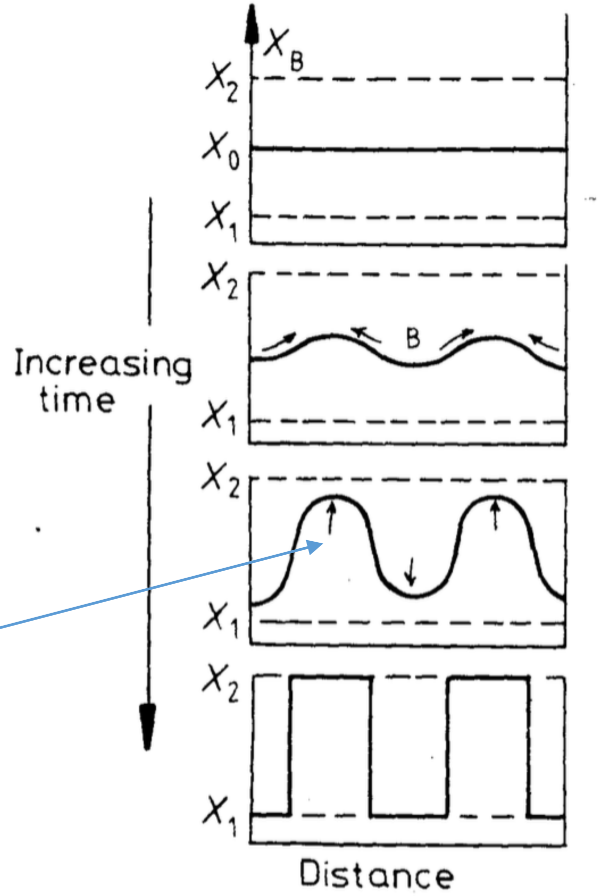
Zonal Flow



http://astronomy.nju.edu.cn/~lixd/GA/AT4/AT411/HTML/AT41102.htm

Arrows:
ψ for CHNS;
flow for ZF.

Spinodal Decomposition



[Porter 1981]

Why Care?

3. “Blobby Turbulence”
   - CHNS is a naturally blobby system of turbulence.
   - What is the role of structure in interaction?
   - How to understand blob coalescence and relation to cascades?
   - How to understand multiple cascades of blobs and energy?

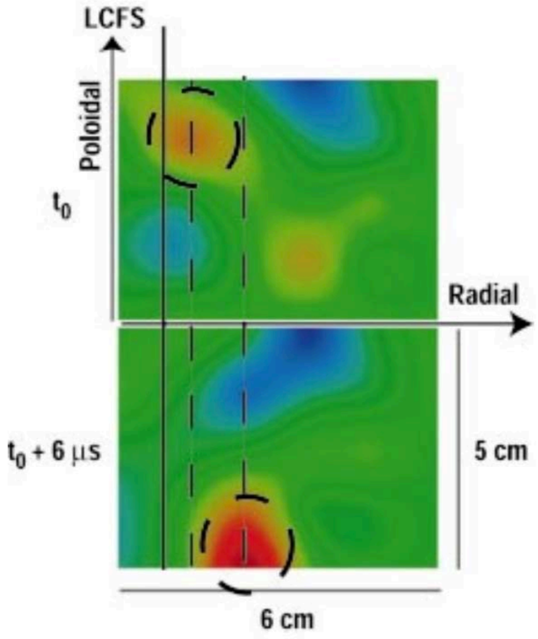


FIG. 4. (Color) Two frames from BES showing 2-D density plots. There is a time difference of 6 μs between frames. Red indicates high density and blue low density. A structure, marked with a dashed circle and shown in both frames, features poloidal and radial motion.

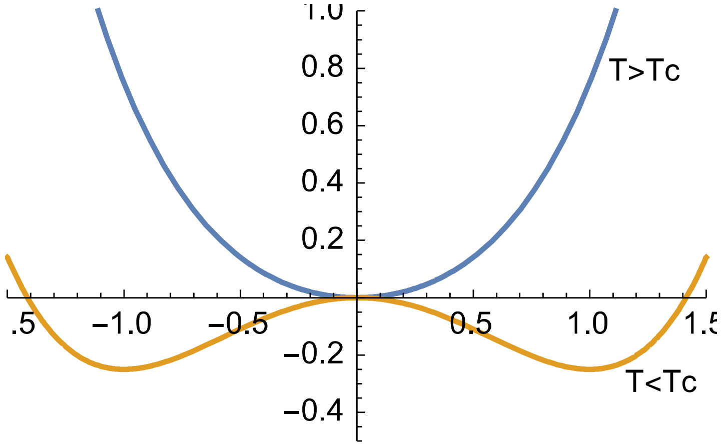
[J. A. Boedo et.al. 2003]

- CHNS exhibits all of the above, with many new twists

A Brief Derivation of the CHNS Model

- Second order phase transition → Landau Theory.
- Order parameter: $\psi(\vec{r},t) \overset{\text{def}}{=} [\rho_A(\vec{r},t) - \rho_B(\vec{r},t)]/\rho$
- Free energy:

$$\mathrm{F}(\psi) = \int d\vec{r}\Big(\underbrace{\frac{1}{2}C_1\psi^2 + \frac{1}{4}C_2\psi^4}_{\text{Phase Transition}} + \underbrace{\frac{\xi^2}{2}|\nabla\psi|^2}_{\text{Gradient Penalty}}\Big)$$



- $C_1(T), C_2(T)$.
- Isothermal $T < T_C$. Set $C_2 = -C_1 = 1$:

$$\mathrm{F}(\psi) = \int d\vec{r}\Big(-\frac{1}{2}\psi^2 + \frac{1}{4}\psi^4 + \frac{\xi^2}{2}|\nabla\psi|^2\Big)$$

A Brief Derivation of the CHNS Model

- Continuity equation: $\frac{d\psi}{dt} + \nabla \cdot \vec{J} = 0$. Fick's Law: $\vec{J} = -D\nabla\mu$.
- Chemical potential: $\mu = \frac{\delta F(\psi)}{\delta\psi} = -\psi + \psi^3 - \xi^2\nabla^2\psi$.
- Combining above → Cahn Hilliard equation:

$$\boxed{\frac{d\psi}{dt} = D\nabla^2\mu = D\nabla^2(-\psi + \psi^3 - \xi^2\nabla^2\psi)}$$

- $d_t = \partial_t + \vec{v} \cdot \nabla$. Surface tension: force in Navier-Stokes equation:

$$\boxed{\partial_t\vec{v} + \vec{v} \cdot \nabla\vec{v} = -\frac{\nabla p}{\rho} - \psi\nabla\mu + \nu\nabla^2\vec{v}}$$

- For incompressible fluid, $\nabla \cdot \vec{v} = 0$.

2D CHNS and 2D MHD

- 2D CHNS Equations:

$$\partial_t\psi + \vec{v}\cdot\nabla\psi = D\nabla^2(-\psi + \psi^3 - \xi^2\nabla^2\psi)$$

$$\partial_t\omega + \vec{v}\cdot\nabla\omega = \frac{\xi^2}{\rho}\vec{B}_\psi\cdot\nabla\nabla^2\psi + \nu\nabla^2\omega$$

$-\psi$: Negative diffusion term

ψ^3: Self nonlinear term

$-\xi^2\nabla^2\psi$: Hyper-diffusion term

With $\vec{v}$=$\hat{\hat{z}}\times\nabla\phi$, $\omega = \nabla^2\phi$, $\vec{B}_\psi = \hat{\hat{z}}\times\nabla\psi$, $j_\psi = \xi^2\nabla^2\psi$.

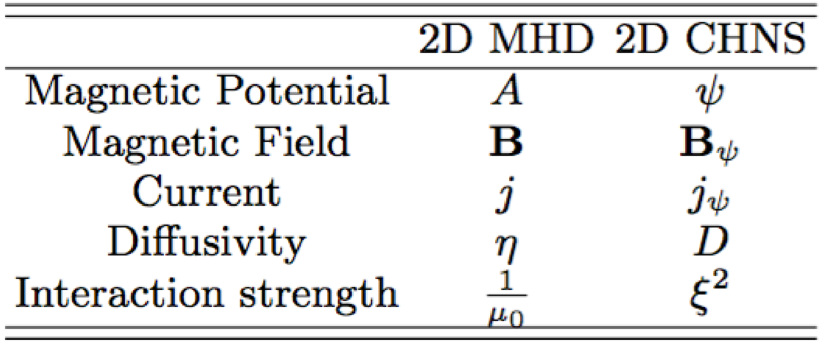
- 2D MHD Equations:

$$\partial_t A + \vec{v}\cdot\nabla A = \eta\nabla^2 A$$

$$\partial_t\omega + \vec{v}\cdot\nabla\omega = \frac{1}{\mu_0\rho}\vec{B}\cdot\nabla\nabla^2 A + \nu\nabla^2\omega$$

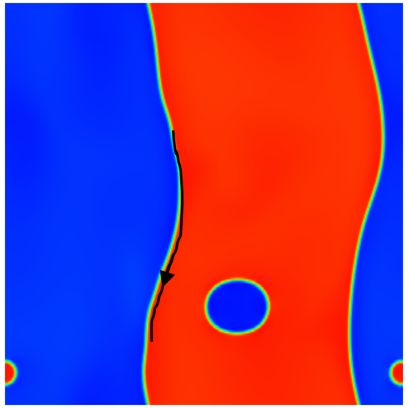
A: Simple diffusion term

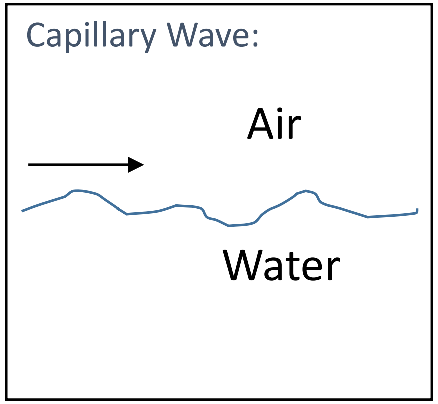
With $\vec{v}$=$\hat{\hat{z}}\times\nabla\phi$, $\omega = \nabla^2\phi$, $\vec{B} = \hat{\hat{z}}\times\nabla A$, $j = \frac{1}{\mu_0}\nabla^2 A$.



	2D MHD	2D CHNS
Magnetic Potential	A	ψ
Magnetic Field	$\mathbf{B}$	$\mathbf{B}_\psi$
Current	j	j_ψ
Diffusivity	η	D
Interaction strength	$\frac{1}{\mu_0}$	ξ^2

Linear Wave





- CHNS supports linear “elastic” wave:

$$\omega(k) = \pm\sqrt{\frac{\xi^2}{\rho}}\left|\vec{k}\times\vec{B}_{\psi 0}\right| - \frac{1}{2}i(CD+\nu)k^2$$

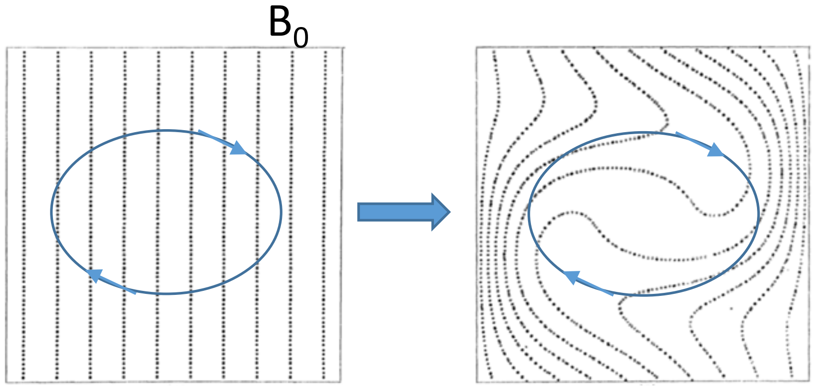
Where $C \equiv [-1 - 6\psi_0\nabla^2\psi_0/k^2 - 6(\nabla\psi_0)^2/k^2 - 6\psi_0\nabla\psi_0\cdot i\mathbf{k}/k^2 + 3\psi_0^2 + \xi^2k^2]$

- Akin to capillary wave at phase interface. Propagates ***only*** along the interface of the two fluids, where $|\vec{B}_\psi| = |\nabla\psi| \neq 0$.
- Analogue of Alfven wave.
- Important differences:
    - $\vec{B}_\psi$ in CHNS is large only in the interfacial regions.
    - Elastic wave activity does not fill space.

What of a Single Eddy?
(Homogenization)

Flux Expulsion

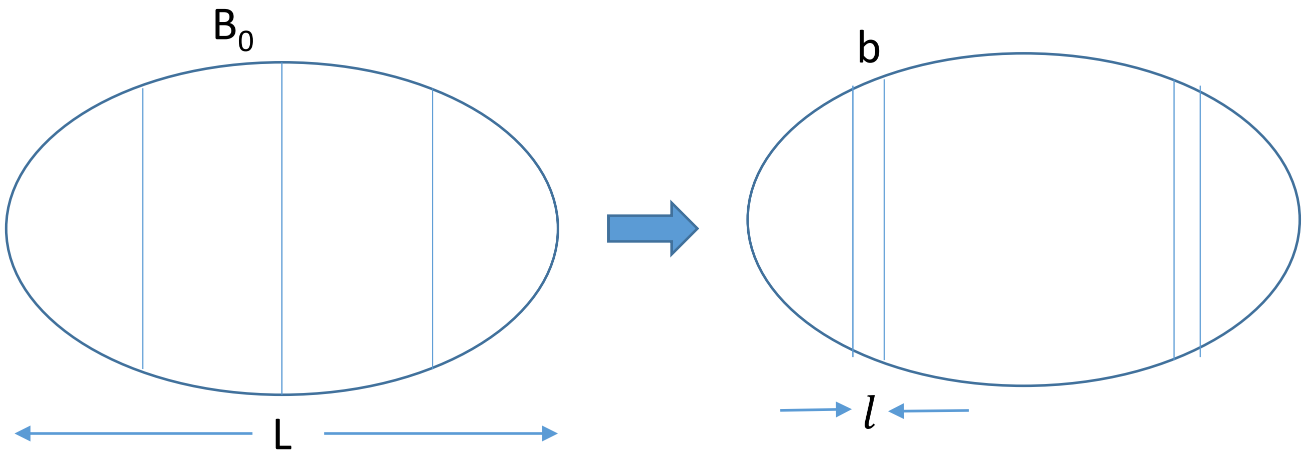
- Simplest dynamical problem in MHD (Weiss ‘66, et. seq.)
- Closely related to “PV Homogenization”



$Rm \sim vL/\eta \gg 1$

- Field wound-up, “expelled” from eddy
- For large Rm, field concentrated in boundary layer of eddy
- Ultimately, back-reaction asserts itself for sufficient B_0

How to Describe?



after n turns:
nl=L

- Flux conservation: B_0L~bl   Wind up: $b=nB_0$ (field stretched)
- Rate balance: wind-up ~ dissipation

$$\frac{v}{L}B_0 \sim \frac{\eta}{l^2}b\ .\quad \tau_{expulsion} \sim \left(\frac{L}{v_0}\right)Rm^{1/3}.$$

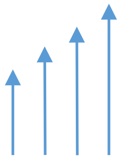
$$l \sim \delta_{BL} \sim L/Rm^{1/3}\ .\quad b \sim Rm^{1/3}B_0\ .$$

N.B. differs from Sweet-Parker!

What’s the Physics?

- Shear dispersion! (Moffatt, Kamkar ‘82)

$$\partial_t A + \vec{v} \cdot \nabla A = \eta \nabla^2 A \quad \text{(Shearing coordinates)}$$



$$v_y = v_y(x) = v_{y0} + x v_y' + \cdots$$

$$\frac{dk_x}{dt} = -k_y v_y' \, , \frac{dk_y}{dt} = 0$$

$$\partial_t A + x v_y' \partial_y A - \eta\left(\partial_x^2 + \partial_y^2\right) A = 0$$

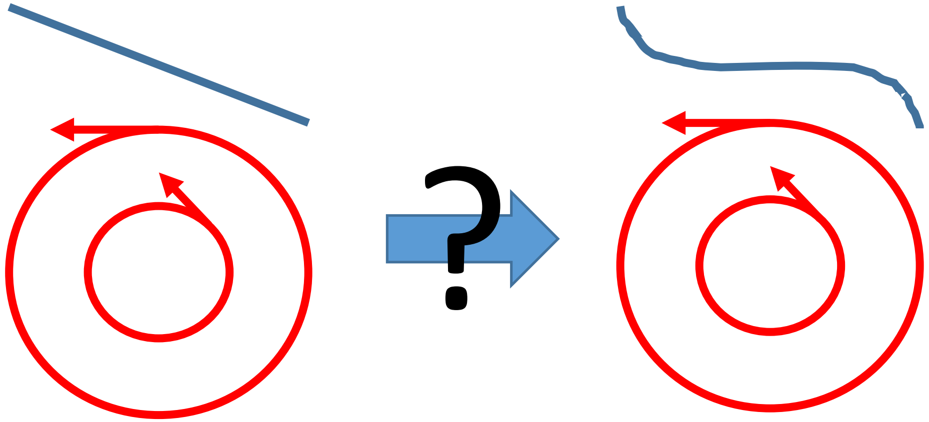
$$A = A(t) \exp i(\vec{k}(t) \cdot \vec{x})$$

(Shear enhanced dissipation annihilates interior field)

- So $\tau_{mix} \cong \tau_{shear} Rm^{1/3} = ({v_y'}^{-1}) Rm^{1/3}$

Single Eddy Mixing -- Cahn-Hilliard

- Structures are the key → need understand how a ***single eddy*** interacts with ψ field
- Mixing of $\nabla\psi$ by a single eddy → characteristic time scales?
- Evolution of structure?
- Analogous to flux expulsion in MHD (Weiss, '66)



$\nabla\psi \leftrightarrow \vec{B}$

Transport / Relaxation

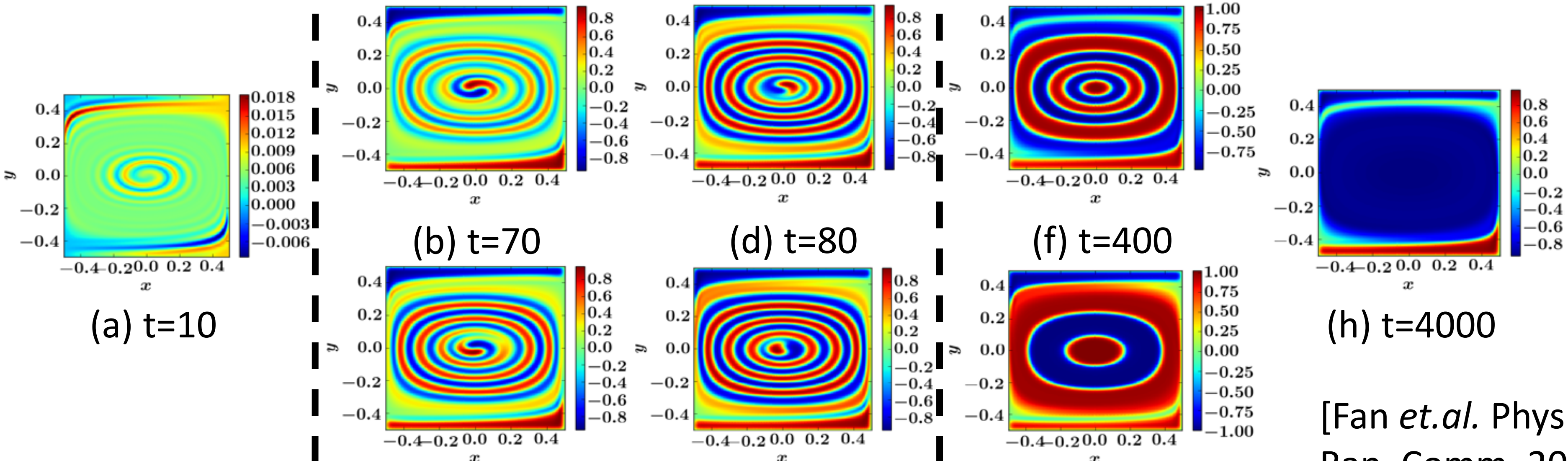
Single Eddy Mixing -- Cahn-Hilliard

- 3 stages: (A) the *"jelly roll"* stage, (B) the *topological evolution* stage, and (C) the *target pattern* stage.
- ψ ultimately homogenized in slow time scale, but metastable target patterns formed and merge.

A: Jelly roll

B: reconnection

C: Target



(a) t=10

(b) t=70

(c) t=75

(d) t=80

(e) t=85

(f) t=400

(g) t=1500

(h) t=4000

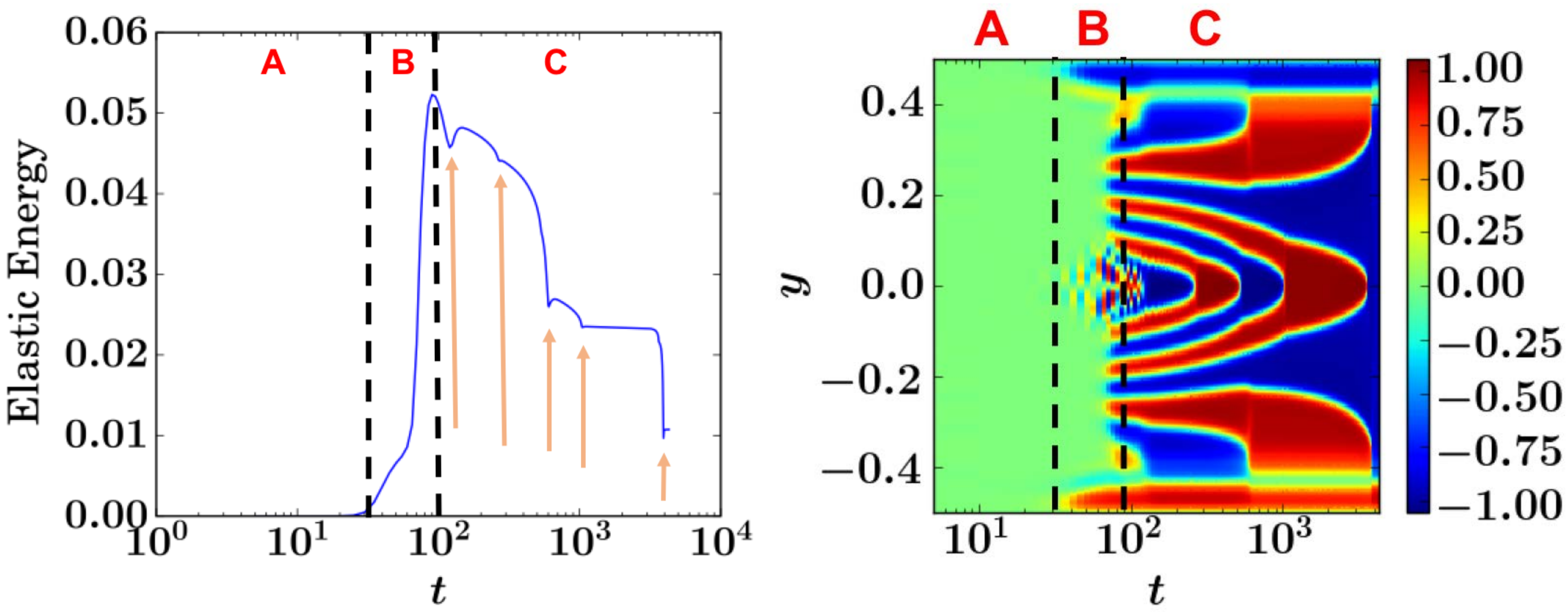
[Fan *et.al.* Phys. Rev. E Rap. Comm. 2017]

- Additional mixing time emerges.

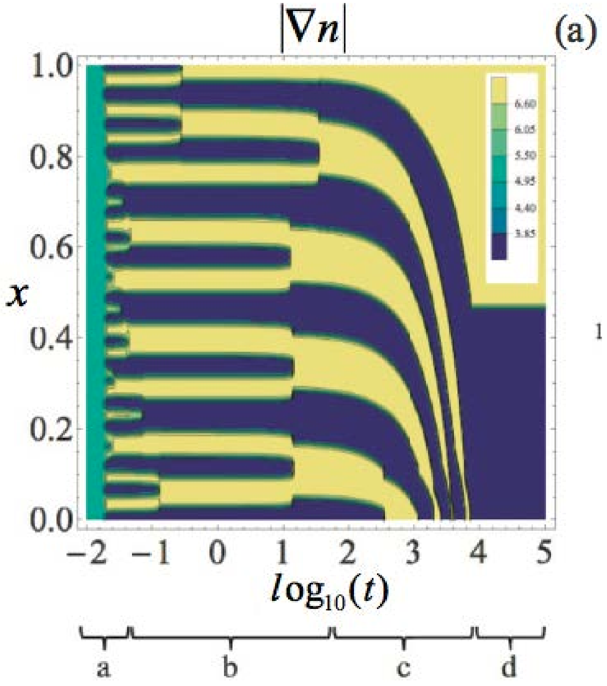
Note coarsening!

Single Eddy Mixing

- ➢The bands merge on a time scale long relative to eddy turnover time.
- ➢The 3 stages are reflected in the elastic energy plot.
- ➢The target bands mergers are related to the dips in the target pattern stage.
- ➢The band merger process is similar to the step merger in drift-ZF staircases.



Episodic relaxation-coarsening Cahn-Hilliard dynamics



[Ashourvan *et.al.* 2016]

Back Reaction – Vortex Disruption

- (MHD only) (A. Gilbert et.al. ‘16; J. Mak et.al. ‘17)
- Demise of kinematic expulsion?
  - Magnetic *tension* grows to react on vorticity evolution!
- Recall: $b \sim B_0(Rm^{1/3})$
  - B.L. field stretched!
- and $\vec{B} \cdot \nabla \vec{B} = -\frac{|B|^2}{r_c}\hat{n} + \frac{d}{ds}(\frac{|B|^2}{2})\hat{t}$
- $\boxed{|\vec{B} \cdot \nabla \vec{B}| \cong b^2/L_0}$

$$\left.\begin{array}{l} r_c \sim L_0 \\ \frac{d}{ds} \sim L_0^{-1} \end{array}\right\} \text{vortex scale}$$

Back Reaction – Vortex Disruption

- So $\rho \frac{d\omega}{dt} = \hat{z} \cdot [\nabla \times (\vec{B} \cdot \nabla \vec{B})]$

  $\rightarrow \rho u \cdot \nabla\omega \sim b^2 / \underset{\uparrow}{l} L_0$

  small BL scale enters

$$v_{A0}^2 = B_0^2 / 4\pi\rho$$

- Feedback → 1 for: $\boxed{Rm \left(\frac{v_{A0}}{u}\right)^2 \sim 1}$

__Remember this!__

- Critical value to disrupt vortex, end kinematics
- Related Alfven wave emission.
- Note for $Rm \gg 1 \rightarrow$ strong field *__not__* required
- Will re-appear…

Some Aspects of CHNS Turbulence

MHD Turbulence – Quick Primer

- (Weak magnetization / 2D)
- Enstrophy conservation broken
- Alfvenic in B_{rms} field – "magneto-elastic" (E. Fermi '49)

$$\epsilon = \frac{\langle \tilde{v}^2 \rangle^2}{l^2} \frac{l}{B_{rms}} \implies E(k) = (\epsilon B_{rms})^{1/2} k^{-3/2}$$

- Dual cascade: Forward in energy
Inverse in $\langle A^2 \rangle \sim k^{-7/3}$

reduced transfer rate:
Kraichnan

- What is dominant (A. Pouquet)?
  - conventional wisdom focuses on energy
  - yet $\langle A^2 \rangle$ conservation – freezing-in law!?

→ Is the inverse cascade of $\langle A^2 \rangle$ the 'real' process, with energy dragged to small scale by fluid?

Ideal Quadratic Conserved Quantities

- **2D MHD**

1. Energy

$$E = E^K + E^B = \int (\frac{v^2}{2} + \frac{B^2}{2\mu_0}) d^2x$$

2. Mean Square Magnetic Potential

$$H^A = \int A^2 \, d^2x$$

3. Cross Helicity

$$H^C = \int \vec{v} \cdot \vec{B} d^2x$$

- **2D CHNS**

1. Energy

$$E = E^K + E^B = \int (\frac{v^2}{2} + \frac{\xi^2 B_\psi^2}{2}) d^2x$$

2. Mean Square Concentration

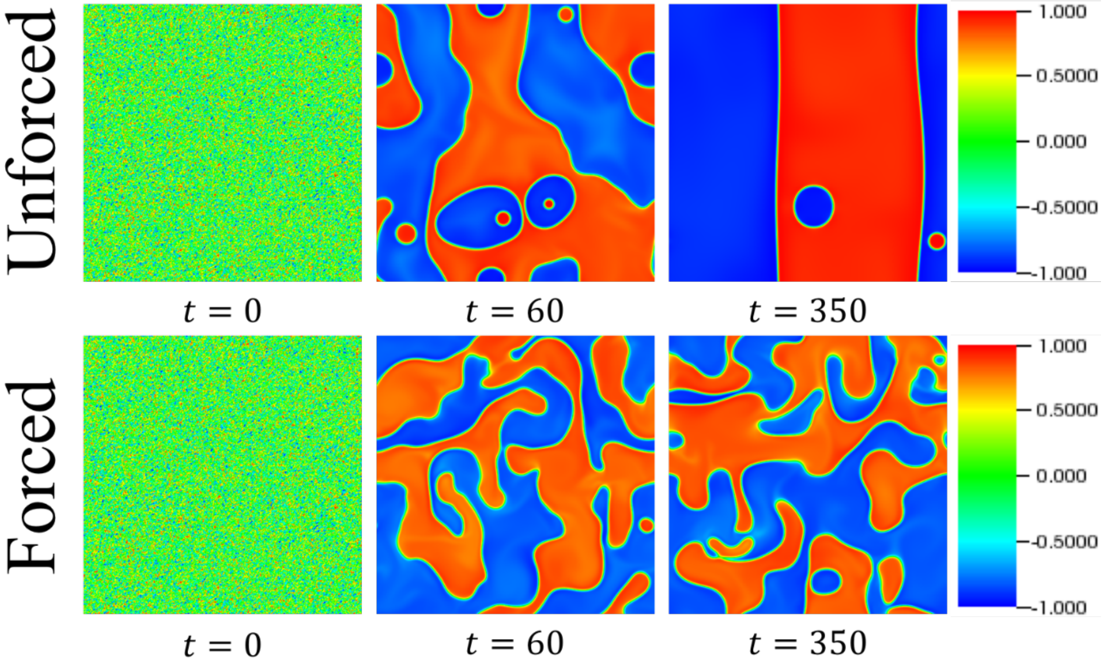
$$H^\psi = \int \psi^2 \, d^2x$$

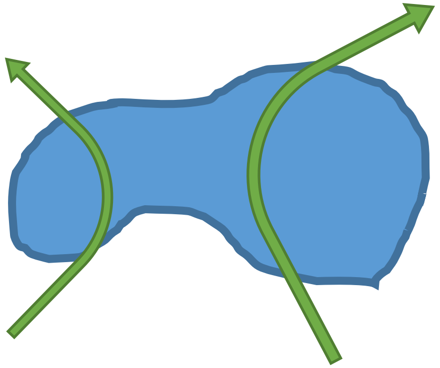
3. Cross Helicity

$$H^C = \int \vec{v} \cdot \vec{B}_\psi \, d^2x$$

Dual cascade expected!

Scales, Ranges, Trends





How big is a raindrop?

- Turbulent straining vs capillarity.
- ρv^2 vs σ / l.

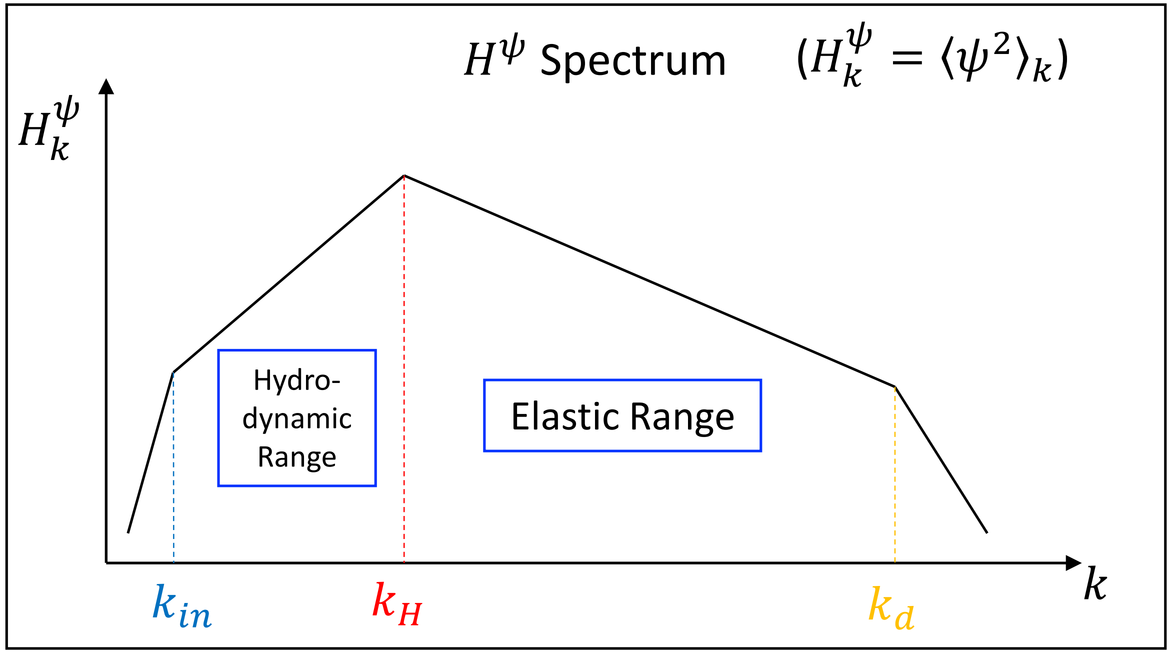
[Hinze 1955]

- ➢Fluid forcing → Fluid straining vs Blob coalescence
- ➢Straining vs coalescence is fundamental struggle of CHNS turbulence
- ➢Scale where turbulent straining ~ elastic restoring force (due surface tension): *Hinze Scale*

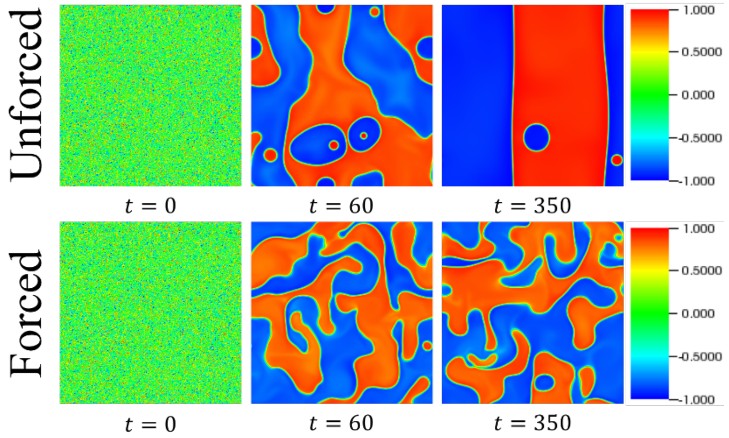
$$L_H \sim (\frac{\rho}{\xi})^{-1/3} \epsilon_\Omega^{-2/9}$$

Scales, Ranges, Trends

- Elastic range: $L_H > l > L_d$: where elastic effects matter.
- $L_H/L_d \sim (\frac{\rho}{\xi})^{-1/3} \nu^{-1/2} \epsilon_\Omega^{-1/18}$ → Extent of the elastic range
- $L_H \gg L_d$ required for large elastic range → case of interest

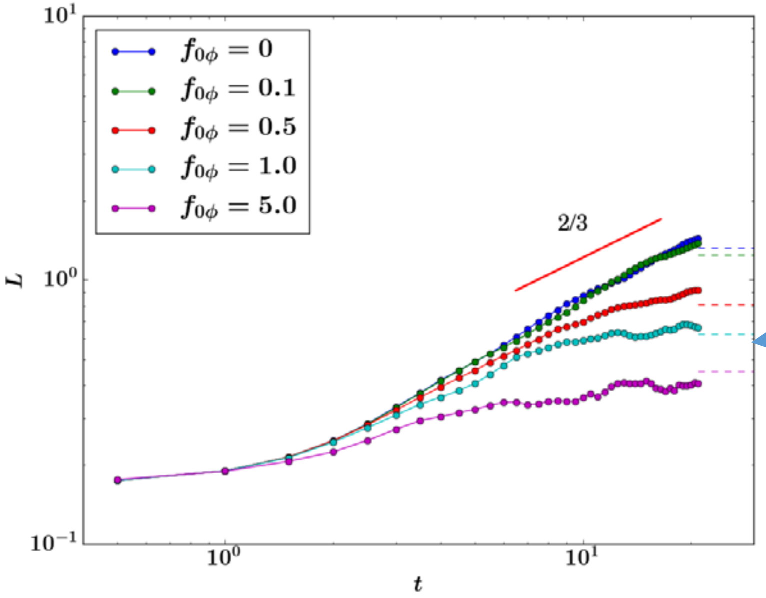


Scales, Ranges, Trends



- Key elastic range physics: ***Blob coalescence***
- Unforced case: $L(t) \sim t^{2/3}$.

  (Derivation: $\vec{v} \cdot \nabla \vec{v} \sim \frac{\xi^2}{\rho} \nabla^2 \psi \nabla \psi \Rightarrow \frac{\dot{L}^2}{L} \sim \frac{\sigma}{\rho} \frac{1}{L^2}$)
- Forced case: blob coalescence arrested at Hinze scale L_H.

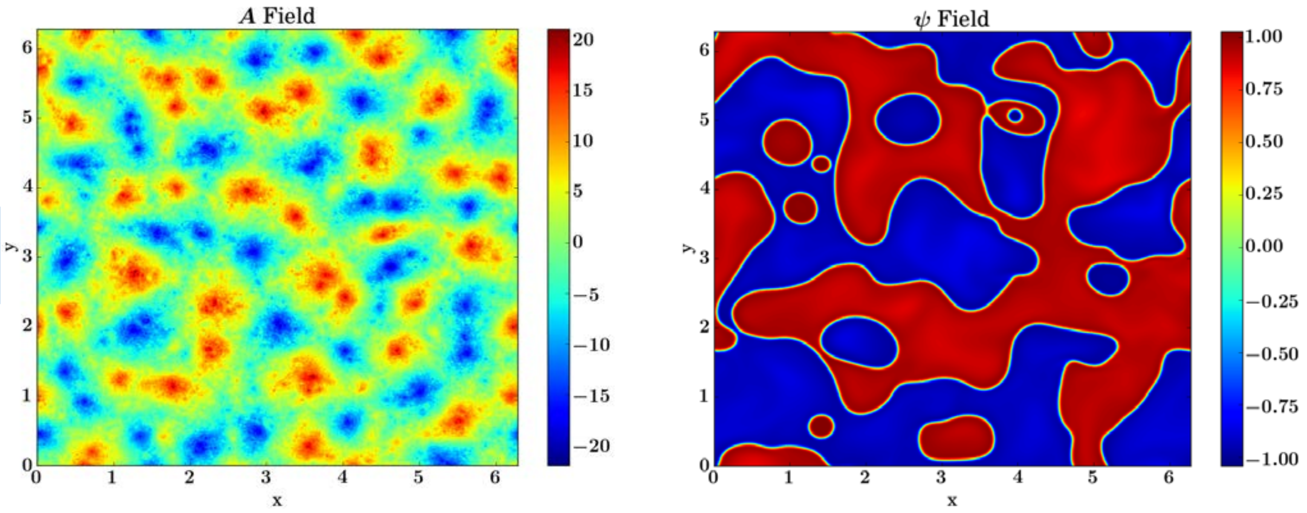


- $L(t) \sim t^{2/3}$ recovered
- Blob growth arrest observed
- Blob growth saturation scale tracks Hinze scale (dashed lines)

- Blob coalescence suggests <u>inverse cascade</u> <u>is</u> <u>fundamental here</u>.

Cascades: Comparing the Systems

MHD



CHNS

- ➢Blob coalescence in the elastic range of CHNS is analogous to flux coalescence in 2D MHD.
- ➢Suggests ***<u>inverse cascade</u>*** of $\langle \psi^2 \rangle$ in CHNS.
- ➢Supported by statistical mechanics studies (absolute equilibrium distributions).
- ➢Arrested by straining.

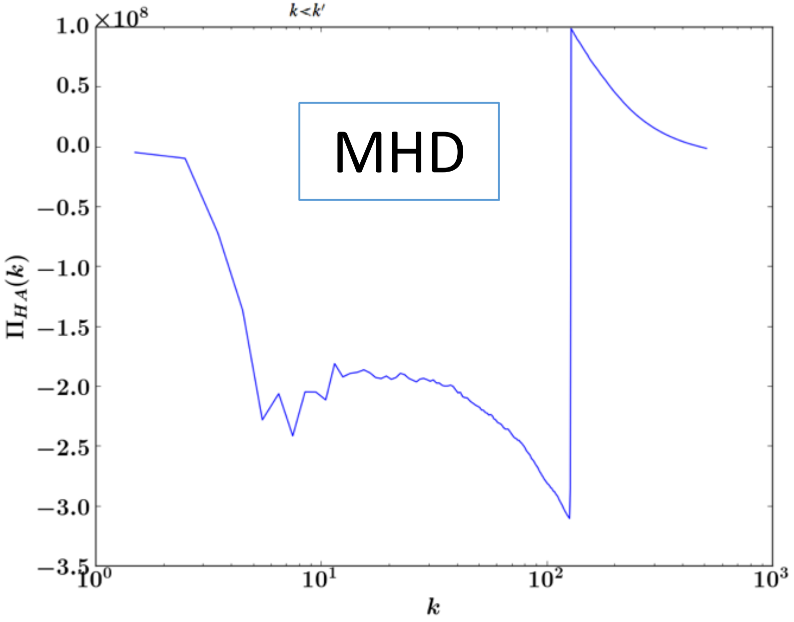
Cascades - the Story

- So, *dual cascade*:
    - ***Inverse*** cascade of $\langle \psi^2 \rangle$
    - ***Forward*** cascade of E
- Inverse cascade of $\langle \psi^2 \rangle$ is formal expression of blob coalescence process → generate larger scale structures till limited by straining
- Forward cascade of E as usual, as elastic force breaks enstrophy conservation
- Forward cascade of energy is analogous to counterpart in 2D MHD

Cascades

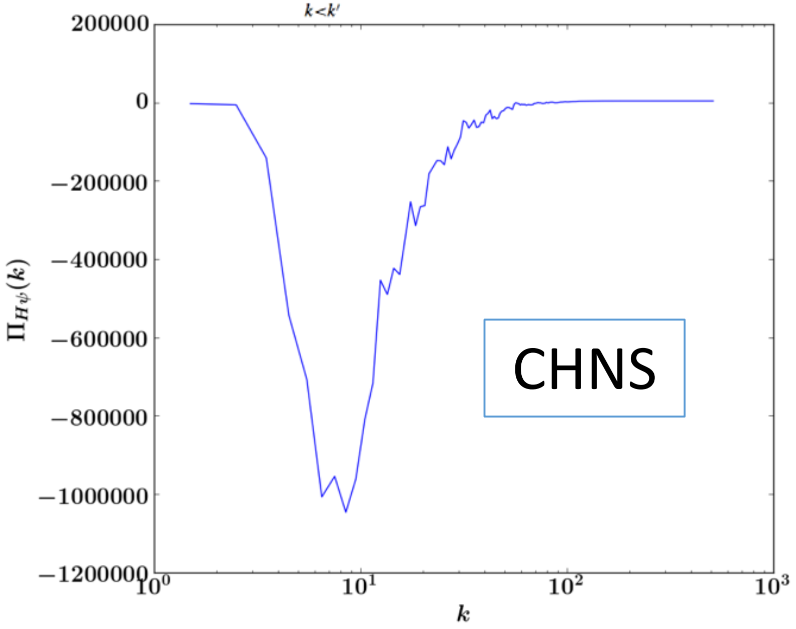
- Spectral flux of $\langle A^2 \rangle$:

$$\Pi_{HA}(k) = \sum_{k<k'} T_{HA}(k'), \quad \text{where } T_{HA}(k) = \langle A_k^*(\mathbf{v}\cdot\nabla A)_k \rangle$$



Spectral flux of $\langle \psi^2 \rangle$:

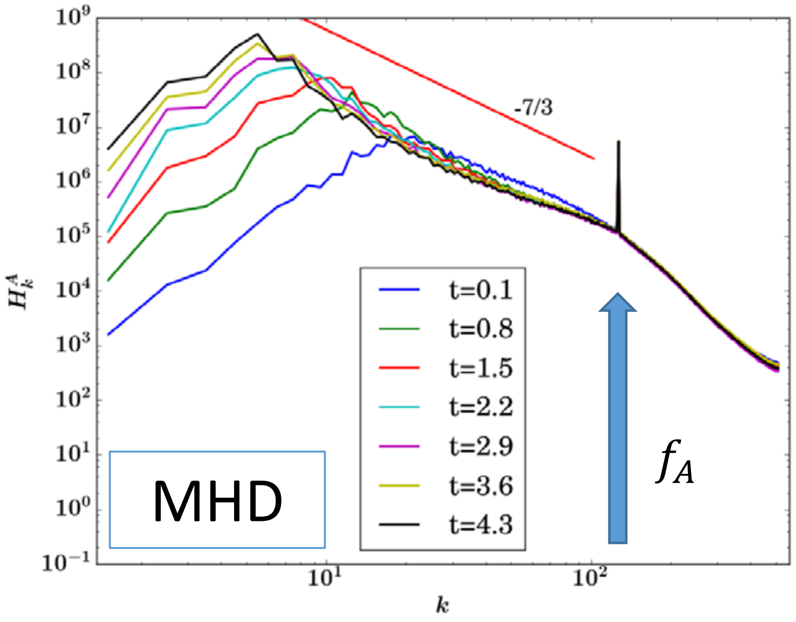
$$\Pi_{H\psi}(k) = \sum_{k<k'} T_{H\psi}(k'), \quad \text{where } T_{H\psi}(k) = \langle \psi_k^*(\mathbf{v}\cdot\nabla \psi)_k \rangle$$



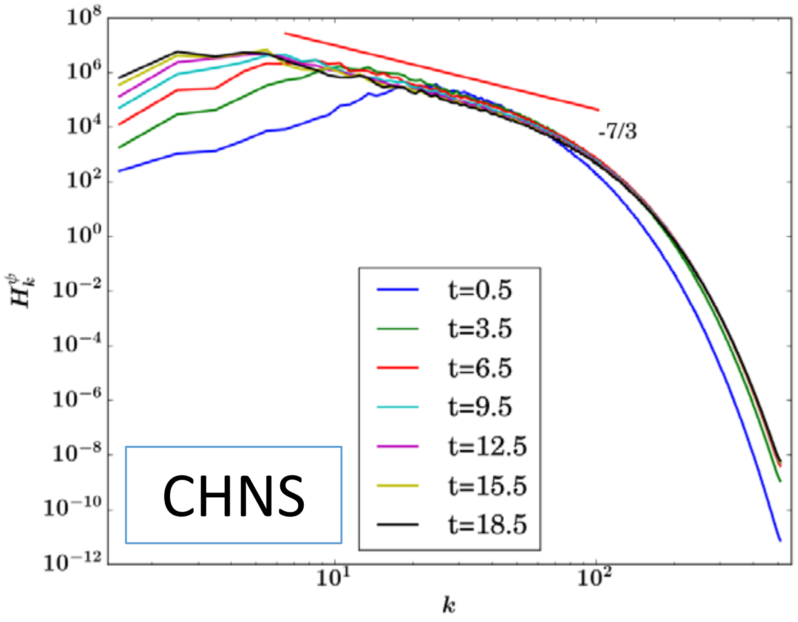
- MHD: weak small scale forcing on A drives inverse cascade
- CHNS: ψ is unforced → aggregates *naturally* ⟺ structure of free energy
- Both fluxes ***negative*** → ***inverse*** cascades

Power Laws

- $\langle A^2 \rangle$ spectrum:



$\langle \psi^2 \rangle$ spectrum:

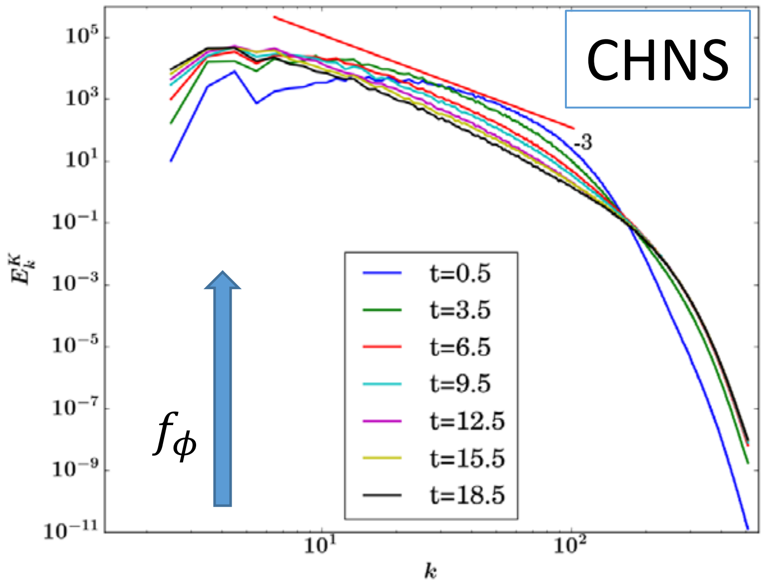


- Both systems exhibit $k^{-7/3}$ spectra.
- Inverse cascade of $\langle \psi^2 \rangle$ exhibits same power law scaling, so long as $L_H \gg L_d$, maintaining elastic range: Robust process.

Power Laws

- Derivation of -7/3 power law:
- For MHD, key assumptions:
  - Alfvenic equipartition ($\rho\langle v^2\rangle \sim \frac{1}{\mu_0}\langle B^2\rangle$)
  - Constant mean square magnetic potential dissipation rate ϵ_{HA}, so $\epsilon_{HA}\sim\frac{H^A}{\tau}\sim(H_k^A)^{\frac{3}{2}}k^{\frac{7}{2}}$.
- Similarly, assume the following for CHNS:
  - Elastic equipartition ($\rho\langle v^2\rangle \sim \xi^2\langle B_\psi^2\rangle$)
  - Constant mean square magnetic potential dissipation rate $\epsilon_{H\psi}$, so $\epsilon_{H\psi}\sim\frac{H^\psi}{\tau}\sim(H_k^\psi)^{\frac{3}{2}}k^{\frac{7}{2}}$.

More Power Laws



- Kinetic energy spectrum (Surprise!):
- 2D CHNS: $E_k^K \sim k^{-3}$;
- 2D MHD: $E_k^K \sim k^{-3/2}$.

!

- The -3 power law:
  - Closer to enstrophy cascade range scaling, in 2D Hydro turbulence.
  - Remarkable departure from expected -3/2 for MHD. ***Why?***
- Why does CHNS ←→ MHD correspondence hold well for $\langle \psi^2 \rangle_k \sim \langle A^2 \rangle_k \sim k^{-7/3}$, yet break down drastically for energy???
- ***What physics*** underpins this surprise??

Interface Packing Matters! – Pattern!

- Need to understand ***differences***, as well as similarities, between CHNS and MHD problems.

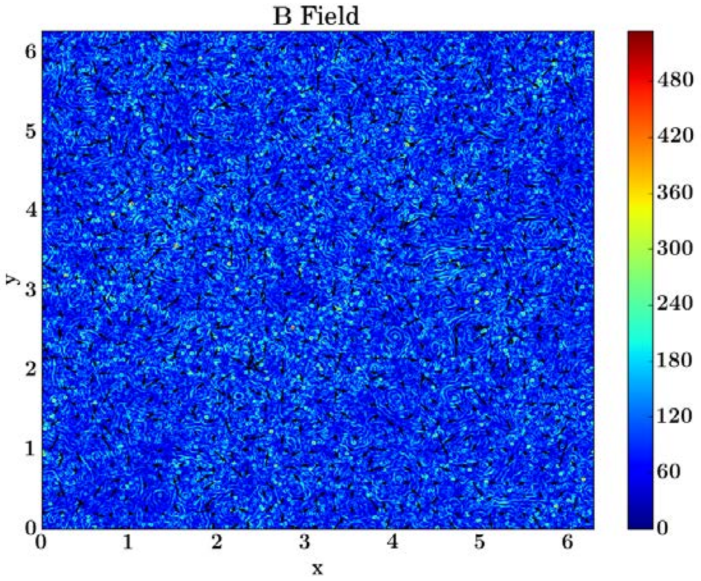
2D MHD:

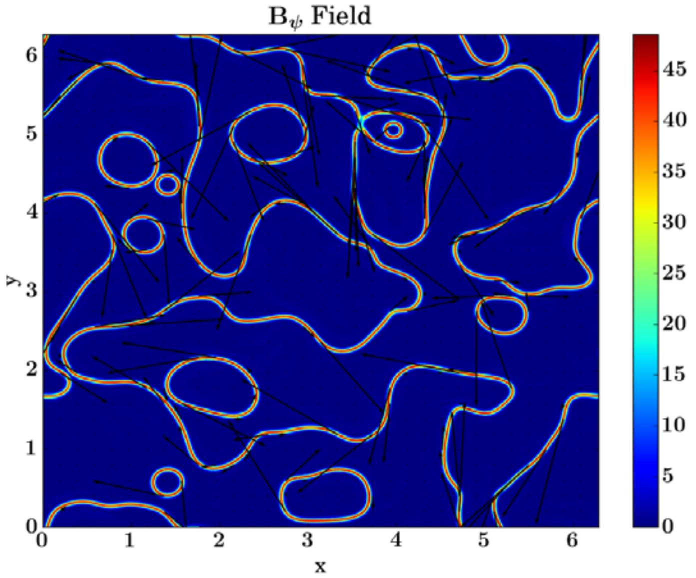
- Fields pervade system.

2D CHNS:

- Elastic back-reaction is limited to regions of density contrast i.e. $|\vec{B}_{\psi}| = |\nabla\psi| \neq 0$.
- As blobs coalesce, interfacial region diminished. 'Active region' of elasticity decays.

MHD





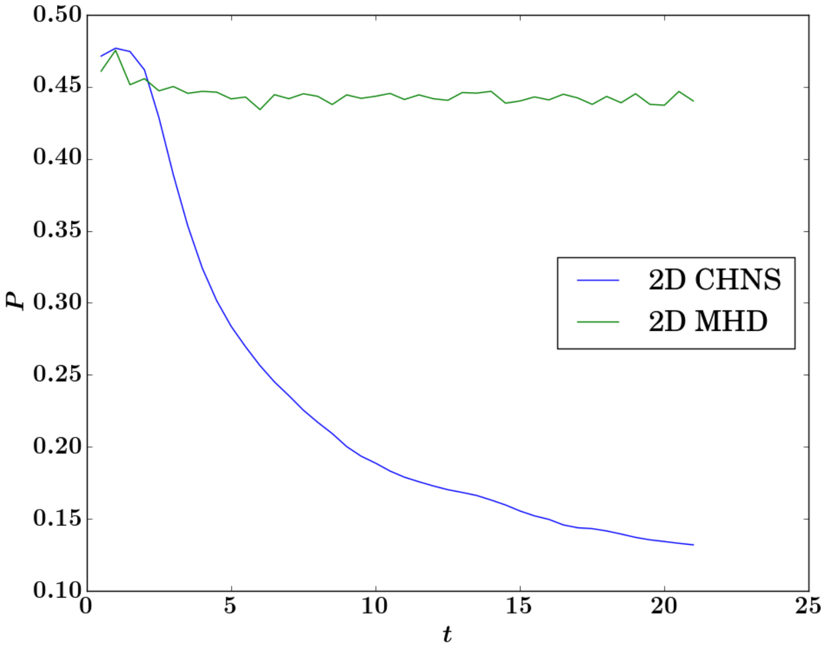
CHNS

Interface Packing Matters!

- Define the ***interface packing fraction*** P:

$$P = \frac{\text{\# of grid points where } |\vec{B}_\psi| > B_\psi^{rms}}{\text{\# of total grid points}}$$

  - P for CHNS decays;
  - P for MHD stationary!



- $\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega$: small P → local back reaction is weak.
- Weak back reaction → reduce to 2D hydro → k-spectra
- Blob coalescence coarsens interface network

What Are the Lessons?

- Avoid power law tunnel vision!
- ***<u>Real space</u>*** realization of the flow is necessary to understand key dynamics. Track interfaces and packing fraction P.
- One player in dual cascade (i.e. $\langle\psi^2\rangle$) can modify or constrain the dynamics of the other (i.e. E).
- Against conventional wisdom, $\langle\psi^2\rangle$ inverse cascade due to blob coalescence is the robust nonlinear transfer process in CHNS turbulence.
- Begs more attention to magnetic helicity in 3D MHD.

Transport and Beyond

- Active Scalar Transport
- Two Stage Evolution
- Revisiting Quenching

Physics: Active Scalar Transport

- Magnetic diffusion, ψ transport are cases of active scalar transport
- (Focus: 2D MHD) (Cattaneo, Vainshtein ’92, Gruzinov, P. D. ’94, ’95)

scalar mixing – the usual

$$\partial_t A + \nabla\phi \times \hat{z} \cdot \nabla A = \eta \nabla^2 A$$

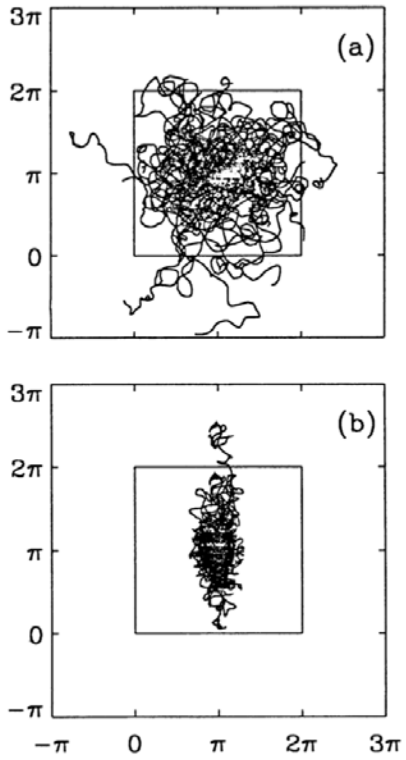
$$\partial_t \nabla^2 \phi + \nabla\phi \times \hat{z} \cdot \nabla\nabla^2 \phi = \nabla A \times \hat{z} \cdot \nabla\nabla^2 A + \nu \nabla^2 \nabla^2 \phi$$

turbulent resistivity

back-reaction

- Seek $\langle v_x A \rangle = -D_T \frac{\partial \langle A \rangle}{\partial x} - \eta \frac{\partial \langle A \rangle}{\partial x}$
- Point: $D_T \neq \sum_{\vec{k}} |v_{\vec{k}}|^2 \, \tau_{\vec{k}}^K$, often substantially less
- Why: Memory! ↔ Freezing-in
- Cross Phase

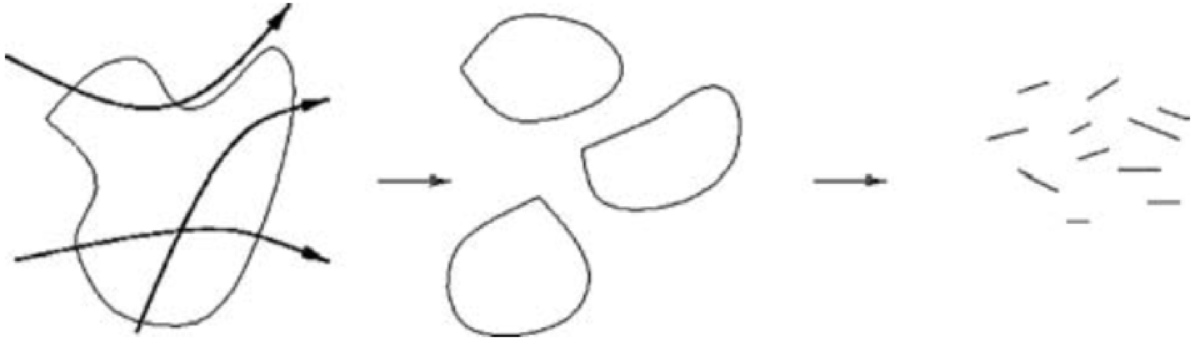
Conventional Wisdom



- [Cattaneo and Vainshtein 1991]: turbulent transport is suppressed even for a weak large scale magnetic field is present.
- Starting point: $\partial_t \langle A^2 \rangle = -2\eta \langle B^2 \rangle$
- Assumptions:
  - Energy equipartition: $\frac{1}{\mu_0 \rho}\langle B^2 \rangle \sim \langle v^2 \rangle$
  - Average B can be estimated by: $|\langle \mathbf{B} \rangle| \sim \sqrt{\langle A^2 \rangle}/L_0$
- Define Mach number as: $M^2 = \langle v_A \rangle^2 / \langle \tilde{v}^2 \rangle = \langle v^2 \rangle / v_A^2 = \langle v^2 \rangle / \frac{1}{\mu_0 \rho}\langle B^2 \rangle$
- Result for suppression stage: $\eta_T \sim \eta M^2$
- Fit together with kinematic stage result: $\boxed{\eta_T \sim \frac{ul}{1 + \mathrm{Rm}/M^2}}$
- Lack physics interpretation of η_T !

Origin of Memory?

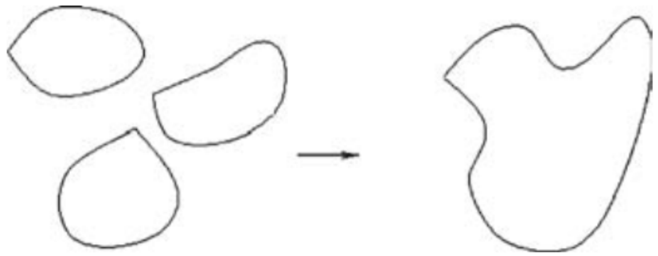
- (a) flux advection vs flux coalescence
  - intrinsic to 2D MHD (and CHNS)
  - rooted in inverse cascade of $\langle A^2 \rangle$ - dual cascades
- (b) tendency of (even weak) ***mean*** magnetic field to "Alfvenize" turbulence [cf: vortex disruption feedback threshold!]
- Re (a): Basic physics of 2D MHD



Forward transfer: fluid eddies chop up scalar A.

Memory Cont'd

- v.s.



Inverse transfer: current filaments and A-blobs attract and coagulate.

- Obvious analogy: straining vs coalescence; CHNS
- Upshot: closure calculation yields:

$$\Gamma_A = -\sum_{\vec{k}'}[\tau_c^{\phi}\langle v^2\rangle_{\vec{k}'} - \tau_c^{A}\langle B^2\rangle_{\vec{k}'}]\frac{\partial\langle A\rangle}{\partial x} + \cdots$$

flux of potential

competition

scalar advection vs. coalescence ("negative resistivity")

(+) (-)

N.B.:
Coalescence
→ Negative diffusion
→ Bifurcation

Conventional Wisdom, Cont'd

- Then calculate $\langle B^2\rangle$ in terms of $\langle v^2\rangle$. From:

$$\partial_t A + \mathbf{v}\cdot\nabla A = -v_x\frac{\partial\langle A\rangle}{\partial x} + \eta\nabla^2 A$$

- Multiplying by A and sum over all modes:

$$\frac{1}{2}[\cancel{\partial_t\langle A^2\rangle} + \cancel{\langle\nabla\cdot(\mathbf{v}A^2)\rangle}] = -\Gamma_A\frac{\partial\langle A\rangle}{\partial x} - \eta\langle B^2\rangle$$

Dropped stationary case

Dropped periodic boundary → introduce nonlocality?!

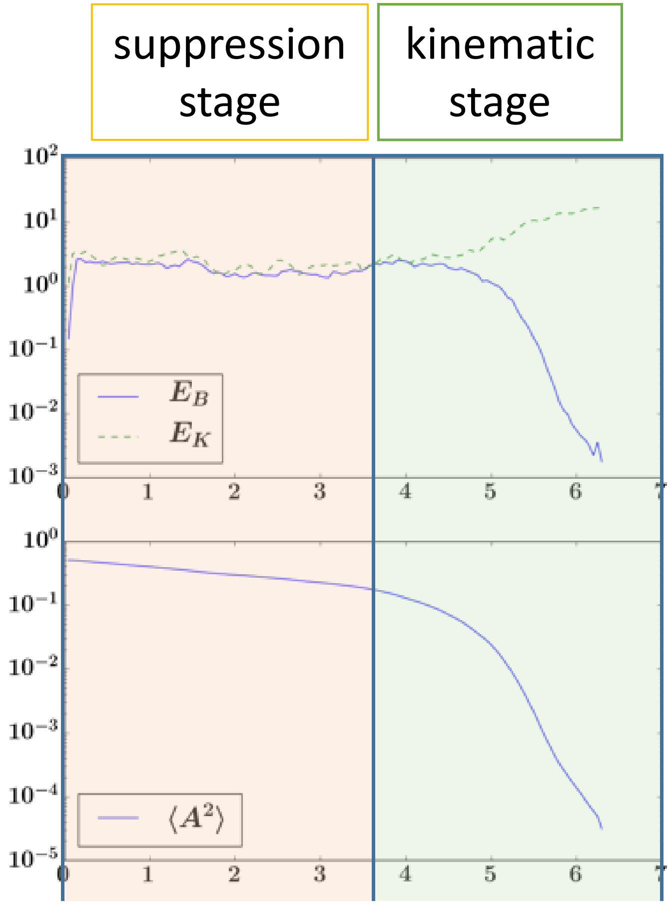
- Therefore: $$\langle B^2\rangle = -\frac{\Gamma_A}{\eta}\frac{\partial\langle A\rangle}{\partial x} = \frac{\eta_T}{\eta}B_0^2$$
- Define Mach number as: $$M^2 \equiv \langle v^2\rangle/v_{A0}^2 = \langle v^2\rangle/(\frac{1}{\mu_0\rho}B_0^2)$$
- Result: $$\boxed{\eta_T = \frac{\sum_{\mathbf{k}}\tau_c\langle v^2\rangle_{\mathbf{k}}}{1+\mathrm{Rm}/M^2} = \frac{ul}{1+\mathrm{Rm}/M^2}}$$
- This theory is not able to describe $B_0 \rightarrow 0$, though may be extended (?!)

Is this story “the truth, the whole truth and nothing but the truth’?

➔ A Closer Look

Two Stage Evolution:

- 1. The <u>suppression</u> <u>stage</u>: the (large scale) magnetic field is sufficiently strong so that the diffusion is suppressed.
- 2. The <u>kinematic decay stage</u>: the magnetic field is dissipated so the diffusion rate returns to the kinematic rate.
- Suppression is due to the memory induced by the magnetic field.

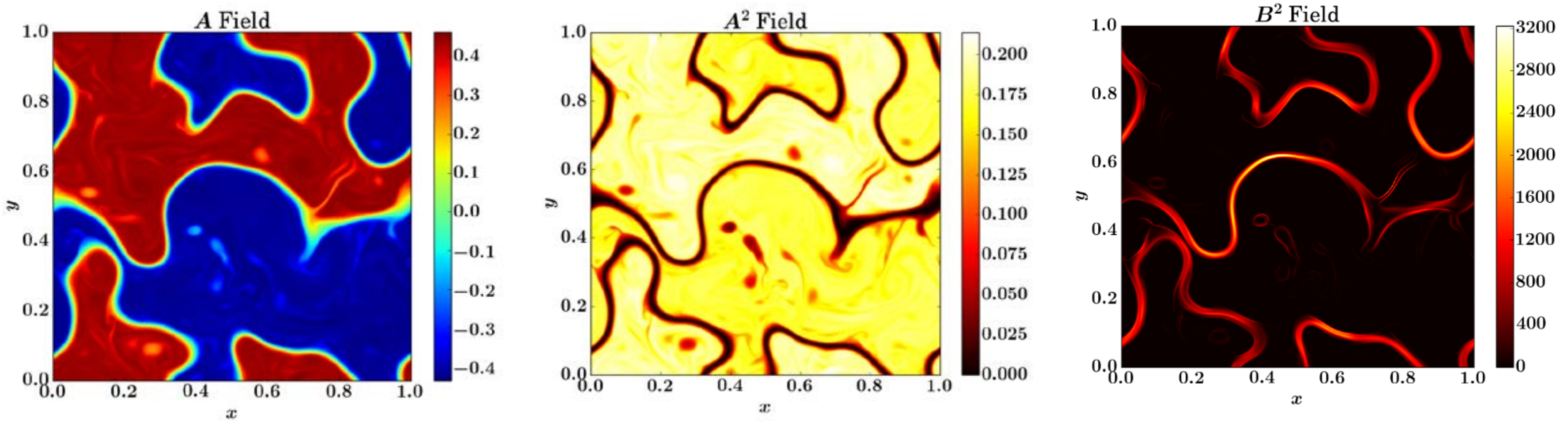


New Observations

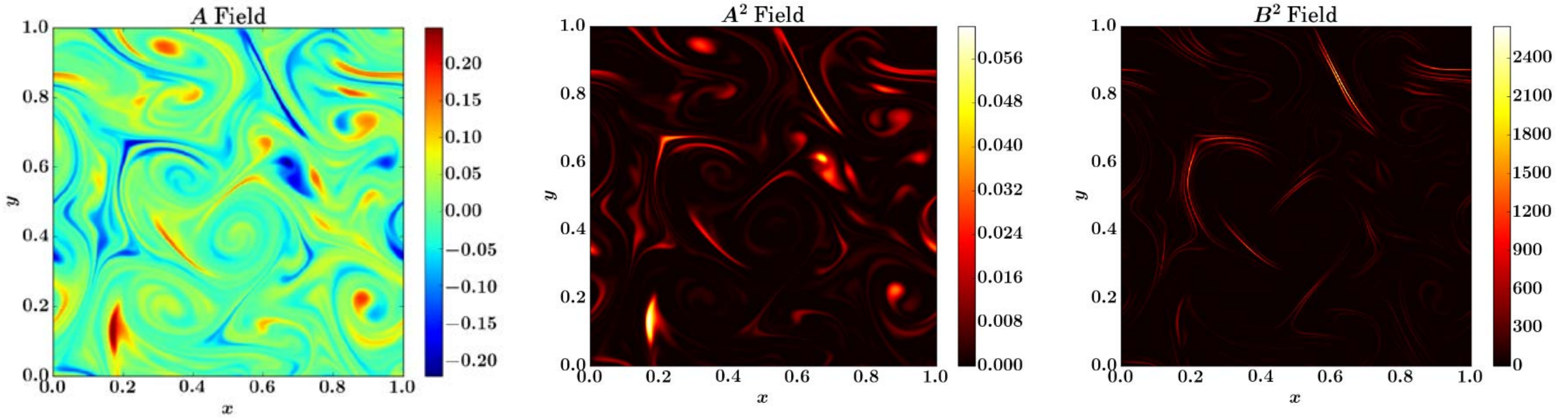
- With no imposed B_0, in suppression stage:

Field

Concentrated!



- v.s. same run, in kinematic stage (trivial):



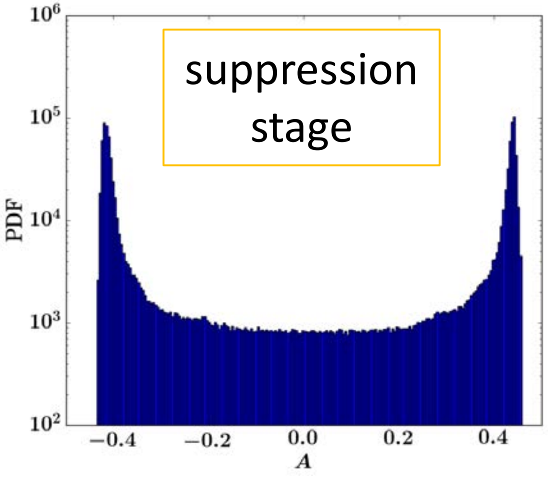
New Observations Cont'd

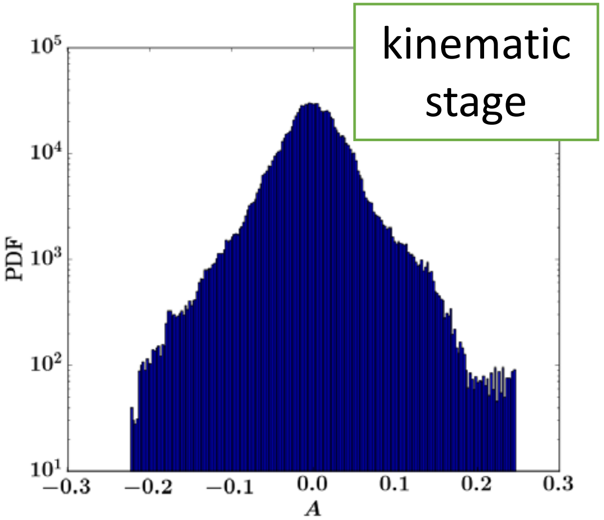
- Nontrivial structure formed in real space during the suppression stage.
- A field is evidently composed of "blobs".
- The low A^2 regions are 1-dimensional.
- The high B^2 regions are strongly correlated with low A^2 regions, and also are 1-dimensional.
- We call these 1-dimensional high B^2 regions ``barriers'', because these are the regions where mixing is reduced, relative to η_K.

➔ Story one of 'blobs and barriers'

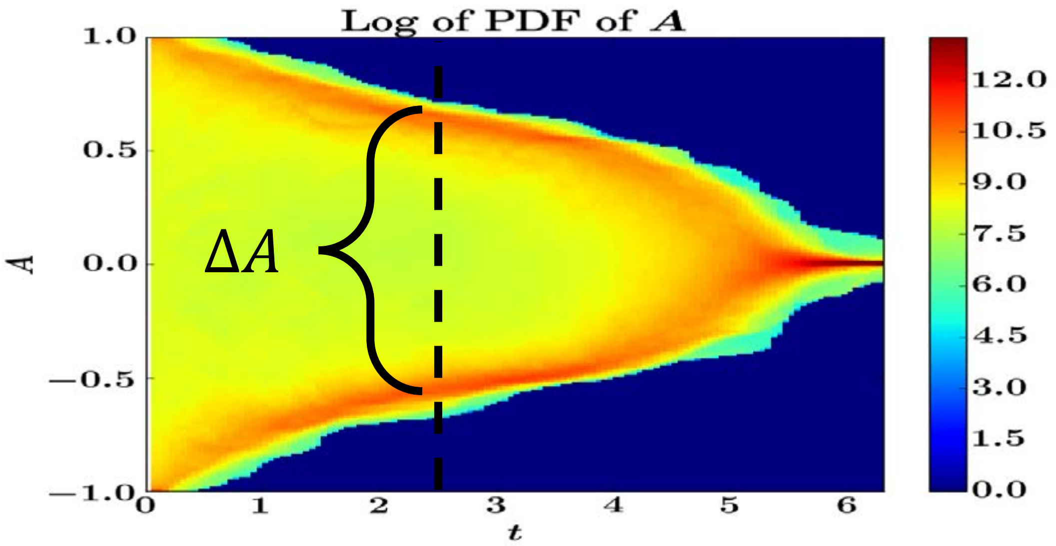
Evolution of PDF of A

- Probability Density Function (PDF) in two stage:



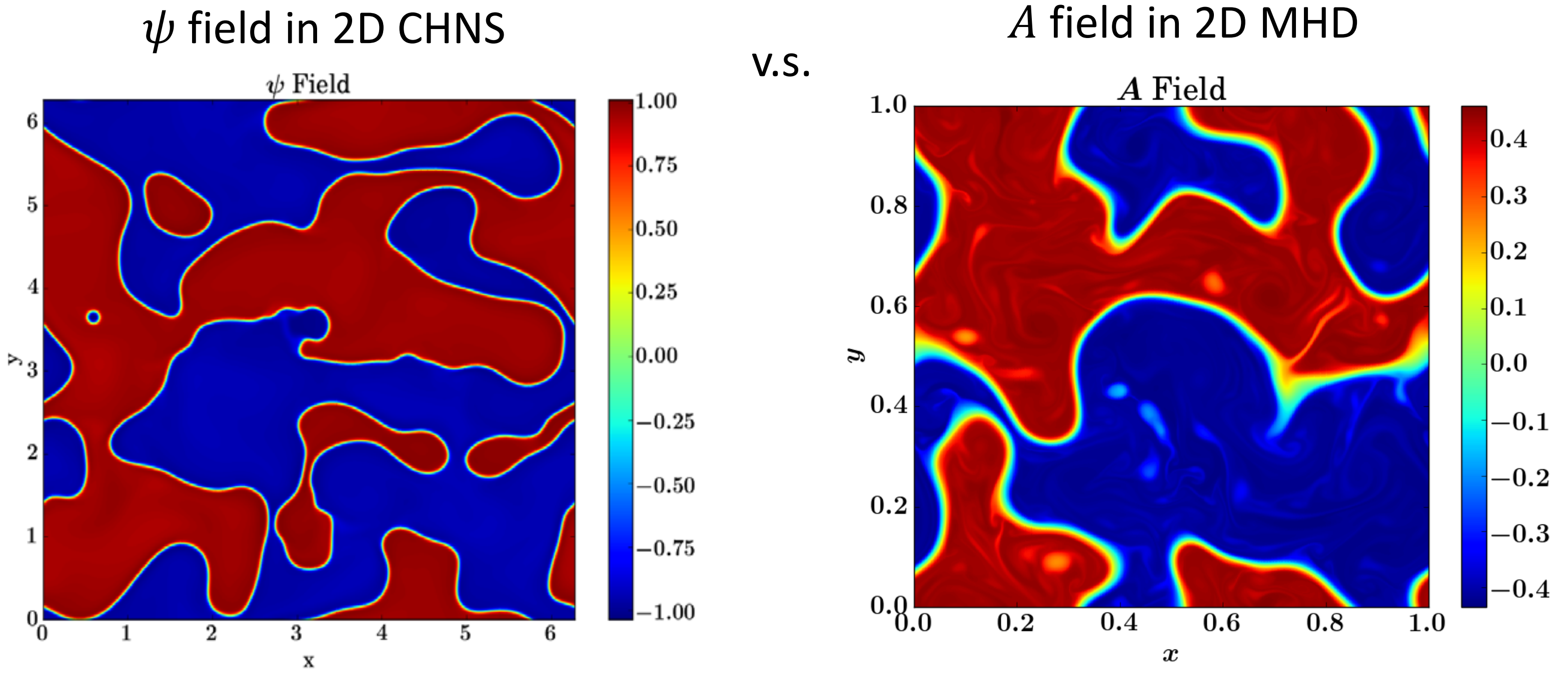


- Time evolution: horizontal “Y”.
  - The PDF changes from double peak to single peak as the system evolves from the suppression stage to the kinematic stage.



2D CHNS and 2D MHD

- The A field in 2D MHD in suppression stage is strikingly similar to the ψ field in 2D CHNS (Cahn-Hilliard Navier-Stokes) system:

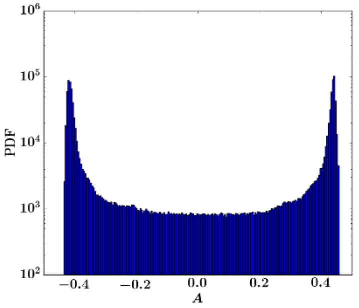

ψ field in 2D CHNS

v.s.

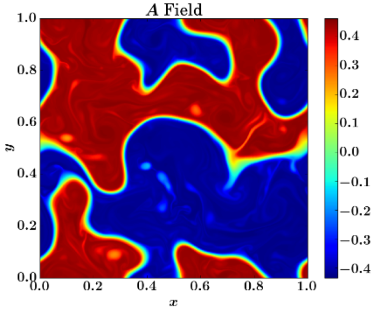
A field in 2D MHD

Unimodal Initial Condition

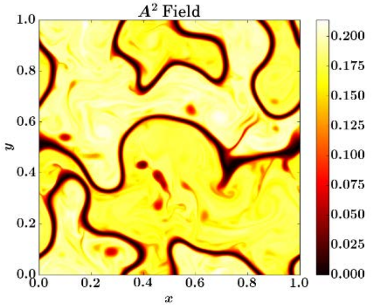
- One may question whether the bimodal PDF feature is purely due to the initial condition. The answer is No.
- Two non-zero peaks in PDF of A still arise, even if the initial condition is unimodal.



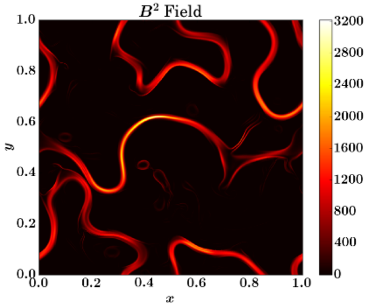
(a1)



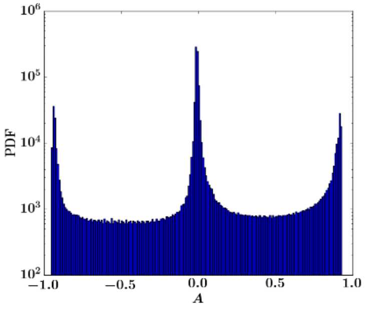
(a2)



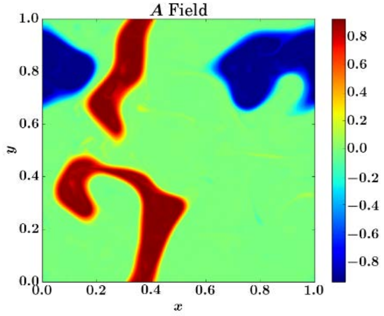
(a3)



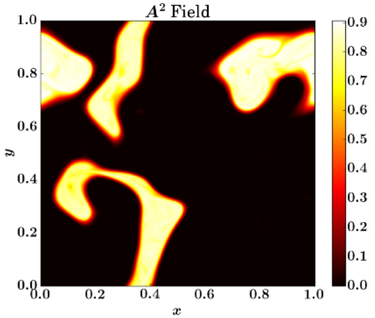
(a4)



(b1)



(b2)



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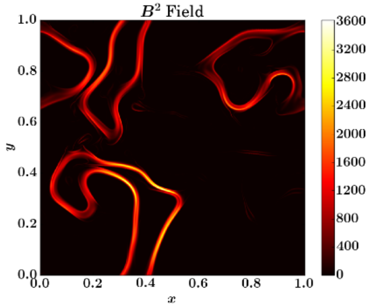
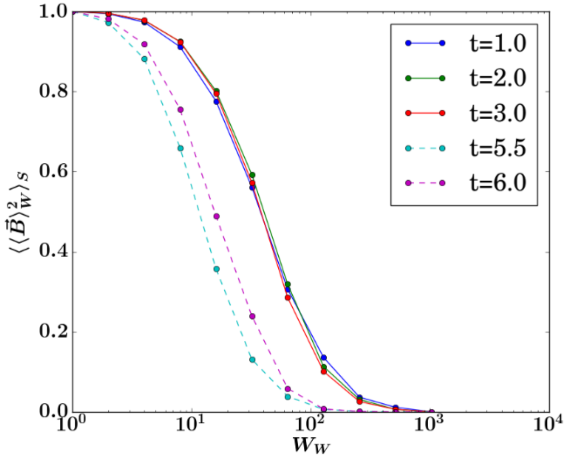
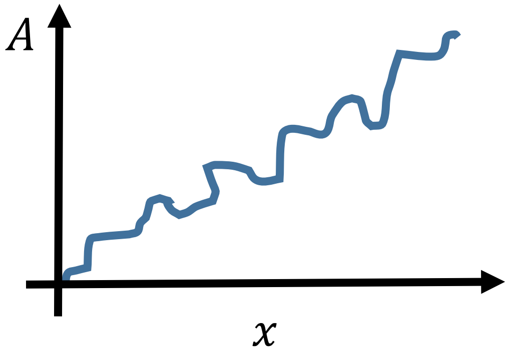


(b4)

The problem of the mean field $\langle \boldsymbol{B} \rangle$
→ What does mean mean?

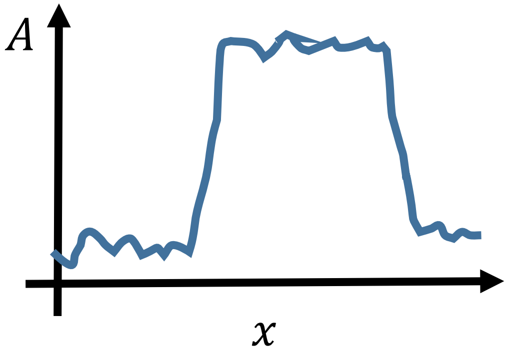
- $\langle B \rangle$ depends on the averaging window.
- With no imposed external field, B is highly intermittent, therefore the $\langle B \rangle$ is not well defined.





$|\langle B \rangle| \sim \sqrt{\langle A^2 \rangle}/L_0$ ✓

v.s.



$\langle B \rangle$ not well defined

Reality

Revisiting Quenching

New Understanding

- Summary of important length scales: $l < L_{stir} < L_{env} < L_0$
  - System size L_0
  - Envelope size L_{env} → emergent (blob)
  - Stirring length scale L_{stir}
  - Turbulence length scale l, here we use Taylor microscale λ
  - Barrier width W → emergent
- Quench is not uniform. Transport coefficients differ in different regions.
- In the regions where magnetic fields are strong, Rm/M^2 is dominant. They are regions of ***barriers***.
- In other regions, i.e. Inside blobs, Rm/M'^2 is what remains. $M'^2 \equiv \langle V^2 \rangle / \left(\frac{1}{\rho} \langle A^2 \rangle / L_{env}^2 \right)$

New Understanding, cont’d

- From $\partial_t\langle A^2\rangle = -\langle \mathbf{v}A\rangle\cdot\nabla\langle A\rangle - \nabla\cdot\langle \mathbf{v}A^2\rangle - \eta\langle B^2\rangle$
- Retain 2nd term on RHS. Average taken over an envelope/blob scale.
- Define diffusion (closure):

$$\langle \mathbf{v}A\rangle = -\eta_{T1}\nabla\langle A\rangle$$
$$\langle \mathbf{v}A^2\rangle = -\eta_{T2}\nabla\langle A^2\rangle$$

- Plugging in: $\partial_t\langle A^2\rangle = \eta_{T1}(\nabla\langle A\rangle)^2 + \nabla\eta_{T2}\cdot\nabla\langle A^2\rangle - \eta\langle B^2\rangle$
- For simplicity: $\langle B^2\rangle \sim \frac{\eta_T}{\eta}(\langle B\rangle^2 + \langle A^2\rangle/L_{env}^2)$
- where L_{env} is the envelope size. Scale of $\nabla^2\langle A^2\rangle$.
- Define new strength parameter: $M'^2 \equiv \langle v^2\rangle/(\frac{1}{\mu_0\rho}\langle A^2\rangle/L_{env}^2)$
- Result: $\boxed{\eta_T = \frac{ul}{1+\mathrm{Rm}/M^2+\mathrm{Rm}/M'^2} = \frac{ul}{1+\mathrm{Rm}\frac{1}{\mu_0\rho}\langle\mathbf{B}\rangle^2/\langle v^2\rangle + \mathrm{Rm}\frac{1}{\mu_0\rho}\langle A^2\rangle/L_{env}^2\langle v^2\rangle}}$

$$\eta_T \;= V\;l\;/\left[1+\frac{R_m}{M^2}+\frac{R_m}{M'^2}\right]$$

- Barriers:

Strong field

$$\eta_T \approx V\;l\;/\;\left[1+R_m\,\frac{\langle B\rangle^2}{\rho\langle\tilde{V}^2\rangle}\right]$$

- Blobs:

Weak effective field

$$\eta_T \approx V\;l\;/\left[1+R_m\,\frac{\langle A^2\rangle}{\rho L_{env}^2\;\langle\tilde{V}^2\rangle}\right]$$

- Quench stronger in barriers, ,non-uniform

Barrier Formation

Formation of Barriers

- How do the barriers form?

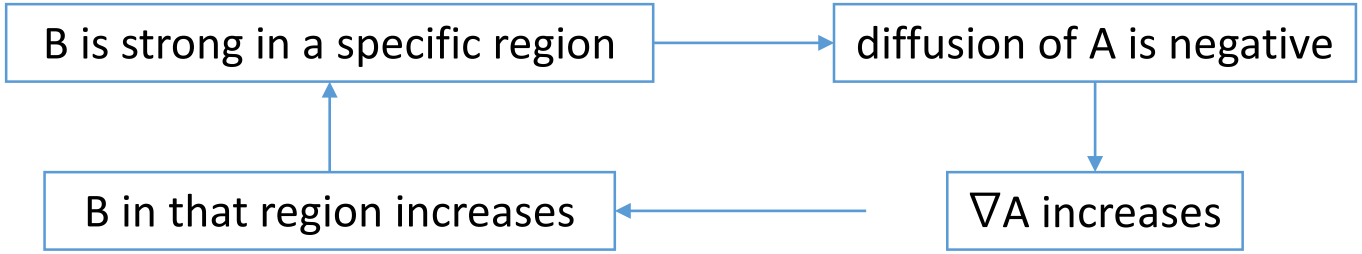
flux coalescence

$$\eta_T = \sum_{\mathbf{k}} \tau_c[\langle v^2\rangle_{\mathbf{k}} - \frac{1}{\mu_0\rho}\langle B^2\rangle_{\mathbf{k}}]$$

- From above, strong B regions can support negative incremental $\eta_T = \delta\Gamma_A/\delta(-\nabla A) < 0$, suggesting clustering

- $\langle\eta_T\rangle > 0$

- Positive feedback: a twist on a familiar theme



B is strong in a specific region

diffusion of A is negative

∇A increases

B in that region increases

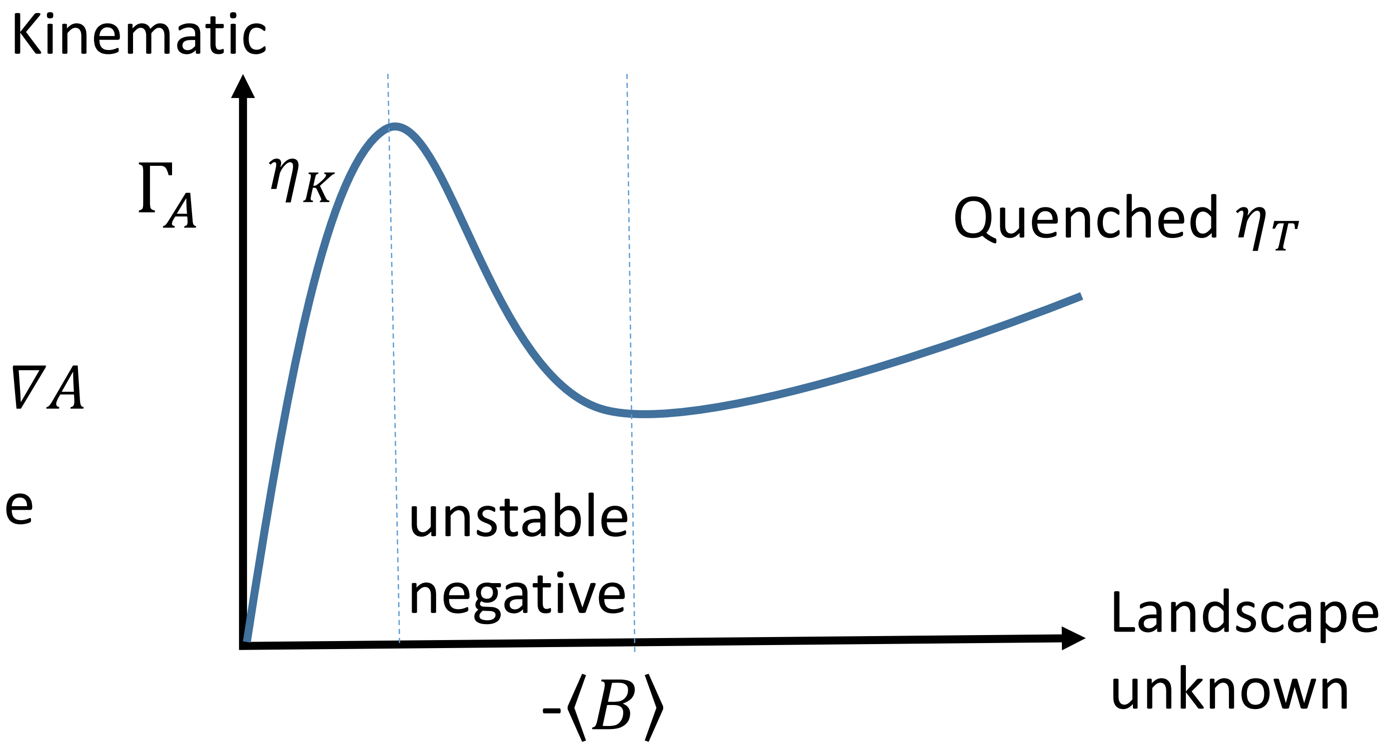
Formation of Barriers, Cont'd

- Negative resistivity leads to barrier formation.
- The S-curve reflects due to the dependence of Γ_A on B.
- When slope is negative → negative (incremental) resistivity.

Barriers ←→

Bistability of Γ_A vs ∇A

→ a familiar theme



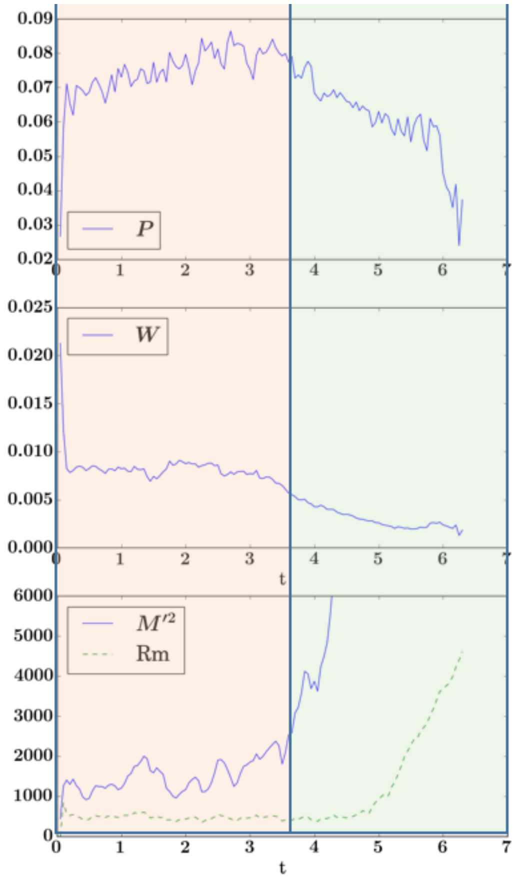
Describing the Barriers

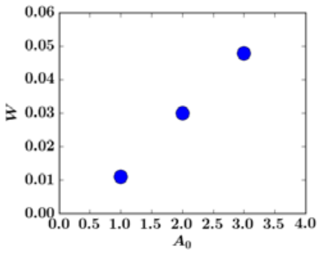
- How to measure the barrier width W.
- Starting point: $W \sim \Delta A / B_b$
- Use $\sqrt{\langle A^2 \rangle}$ to calculate ΔA
- Define the barrier regions as: $B(x,y) > \sqrt{\langle B^2 \rangle} * \overset{\text{arbitrary threshold}}{\overset{\downarrow}{2}}$
- Define barrier packing fracti $P \equiv \dfrac{\#\text{ of grid points for barrier regions}}{\#\text{ of total grid points}}$
- Use use the magnetic fields in the barrier regions to calculate the magnetic energy: $\boxed{\sum_{\text{barriers}} B_b^2 \sim \sum_{\text{system}} B^2}$
- Thus $\langle B_b^2 \rangle \sim \langle B^2 \rangle / P$
- So barrier width can be estimated by: $\boxed{W^2 \equiv \langle A^2 \rangle / (\langle B^2 \rangle / P)}$

N.B. All magnetic energy in the barriers

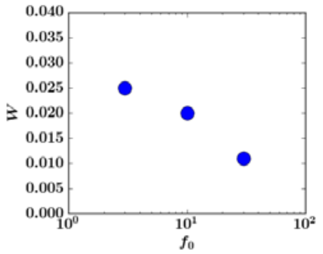
Describing the Barriers

- Time evolution of P and W:
  - P, W collapse in decay
  - M' rises
- Sensitivity of W:
  - A_0 or $1/\mu_0\rho$ greater → W greater;
  - f_0 greater, W smaller; (ala' Hinze)
  - W not sensitive to η or ν.





(a)



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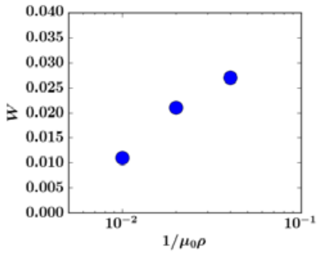

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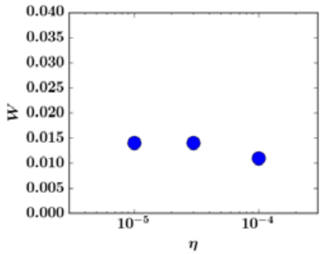

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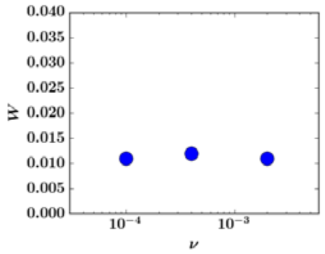
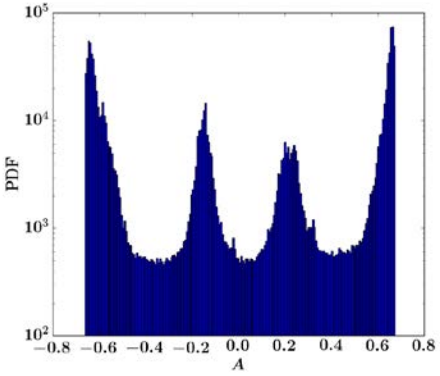


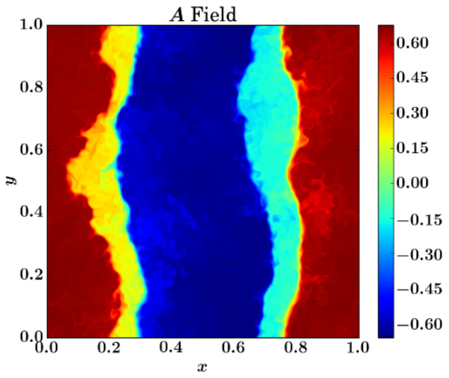
(e)

Staircase (inhomogeneous Mixing, Bistability)

- Staircases emerge spontaneously! - <u>Barriers</u>
- Initial condition is the usual cos function (bimodal)
- The only major sensitive parameter (from runs above) is the forcing scale is k=32 (for all runs above k=5).
- Resembles the staircase in MFE.



(1)



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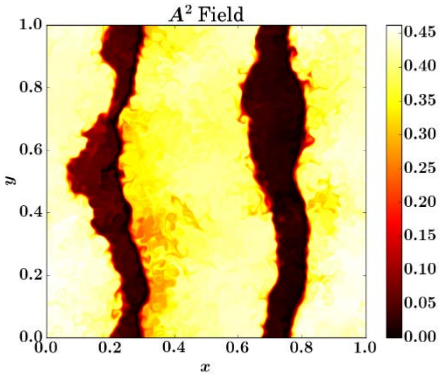
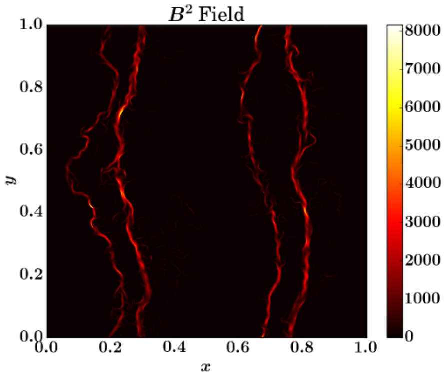


(3)



(4)

Conclusions / Summary

- Magnetic fields suppress turbulent diffusion in 2D MHD by: formation of intermittent ***<u>transport barriers</u>***.
- Magnetic structures:
  - Barriers – thin, 1D strong field regions
  - Blobs – 2D, weak field regions
- Quench not uniform:

$$\eta_T = \frac{ul}{1+\mathrm{Rm}\frac{1}{\mu_0\rho}\langle \mathbf{B}\rangle^2/\langle v^2\rangle + \mathrm{Rm}\frac{1}{\mu_0\rho}\langle A^2\rangle/L_{env}^2\langle v^2\rangle}$$

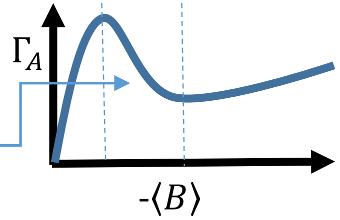
barriers, strong B

blobs, weak B, $\nabla^2\langle A^2\rangle$ remains

- Barriers form due to negative resistivity:

$$\eta_T = \sum_{\mathbf{k}} \tau_c[\langle v^2\rangle_{\mathbf{k}} - \frac{1}{\mu_0\rho}\langle B^2\rangle_{\mathbf{k}}]$$

flux coalescence



- Formation of “magnetic staircases” observed for some stirring scale

Future Works

- Extension of the transport study in MHD:
  - Numerical tests of the new η_T expression ?
  - What determines the barrier width and packing fraction ?
  - Why does layering appear when the forcing scale is small ?
  - What determines the step width, in the case of layering
  - The transport study may also be extended to 3D MHD ($\langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle$ important instead of $\langle A^2 \rangle$)
- Other similar systems can also be studied in this spirit. e.g. Oldroyd-B model for polymer solutions. (drag reduction)
- Reduced Model of Magnetic Staircase

General Conclusions

- Dual (or multiple) cascades can interact with each other, and one can modify another.
- We also show how a length scale, e.g. the Hinze scale in 2D CHNS, emerges from the balance of kinetic energy and elastic energy in blobby turbulence. → blob scale
- We see that negative incremental diffusion (flux/blob coalescence) can lead to novel real space structure in a simple system.
- Avoid fixation on k-spectra/power laws. Real space structure encodes info re: interactions.

Reading

Fan, P.D., Chacon:

- PRE Rap Comm 99, 041201 (2019)

  → Active Scalar Transport 2D MHD

- PoP 25, 055702 (2018)

  → Plasma/MHD Connection

- PRE Rap Comm 96, 041101 (2017)

  → Single Eddy

- Phys Rev Fluids 1, 054403 (2016)

  → Turbulence

Thank you!