Some Lessons From Simple Models

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Outlook

— Explore Fundamental Processes in Depth → develop intuition

— Exploit detailed comparative studies of simple systems

— Contrast to “the usual” romp through hideous complexity…

→ Skip the ballooning mode formalism…

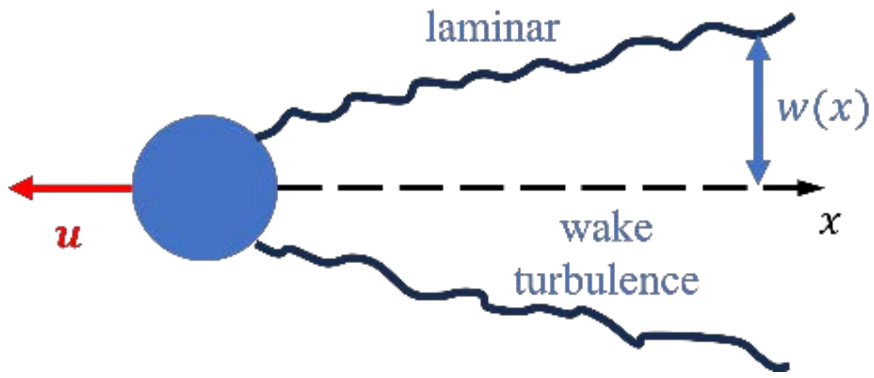
Topics

a.) Turbulence Spreading

b.) Staircase Formation and Evolution

Wake-Classic Example of Turbulence Spreading

⇨



Similarity Theory
Mixing Length Theory }

$W \sim (F_d/\rho U^2)^{1/3} X^{1/3}$,

$F_d \sim C_D \rho U^2 A_s$

C_D independent of viscosity at high Re

⇨ Physics: Entrainment of laminar region by expanding turbulent region. Key is turbulent mixing.

⇨ Townsend '49:

— Distinction between momentum transport — eddy viscosity—and fluctuation energy transport

— Jet Velocity: $V = \frac{<V_{perp} * V^2>}{<V^2>}$

C.f. Ting Long, this meeting

Spreading in MFE

⇨ Numerous gyrokinetic simulations

N.B. Basic studies absent ...

⇨ Diagnosis primarily by: - color VG

- tracking of "Front"

⇨ Theory ⇨ Nonlinear Intensity diffusion models

⇨ Reaction-Diffusion Equations

Recently:

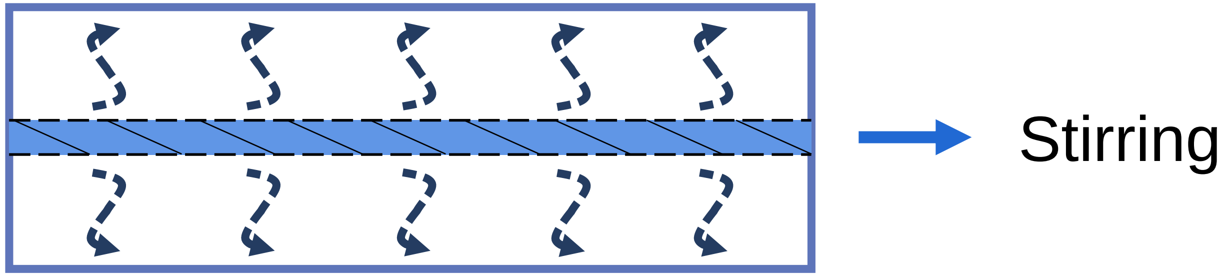
⇨ Renewed interest in context of λ_q broadening problem

⇨ Simulations measure correlation of spreading $\langle \tilde{V}_r \tilde{\rho} \tilde{\rho} \rangle$ with λ_q broadening

⇨ Intermittency effects

Spreading Studies

⇨ 2D Box, Localized Stirring Zone



⇨ Comparison of:

System	Features
2D Fluid	Selective Decay, Vortices How to Measure Spreading?
2D MHD with weak $\underline{B_0}$	Alfvenization, Vortex Bursting, Zeldovich number
Forced Hasegawa-Mima with Zonal Flow	Waves + Eddies + ZF Conversion, Dimits regime etc.

Sneak Preview of Results

System	Results
a.) 2D Fluid:	- Spreading as a selective decay process - Keeping Score: Enstrophy, Energy fluxes, jet velocity? - 'Ballistic' spreading $w_t \sim t$ $\Leftrightarrow$ dipole vortices
b.) 2D MHD + Weak *B_0*	- Saturation of spreading - Vortex bursting + Alfvenization - Zeldovich number $Z = R_m \frac{V_{A0}^2}{<V_{rms}^2>}$ as critical parameter
c.) Hasegawa-Mima + Zonal Flow (ongoing)	- Potential Enstrophy Flux sensitive to ZF damping - ZF blocking Dimits like regime

Numerics: 2D Dedalus simulation

Box Characteristics:

- Grid Size: 512×512
- Doubly Periodic boundary condition

Forcing Characteristics:

- Superposition of Sinusoidal Forcing
- Spectrum: Constant $E(k)$, ensuring uniform energy distribution across wave numbers.
- Correlation Length**:** Approximately $1/10$ of the box scale, some room for dual cascade.
- Localized through a Heaviside step function.
- Phase of forcing randomized every typical eddy turnover time

2D Fluid

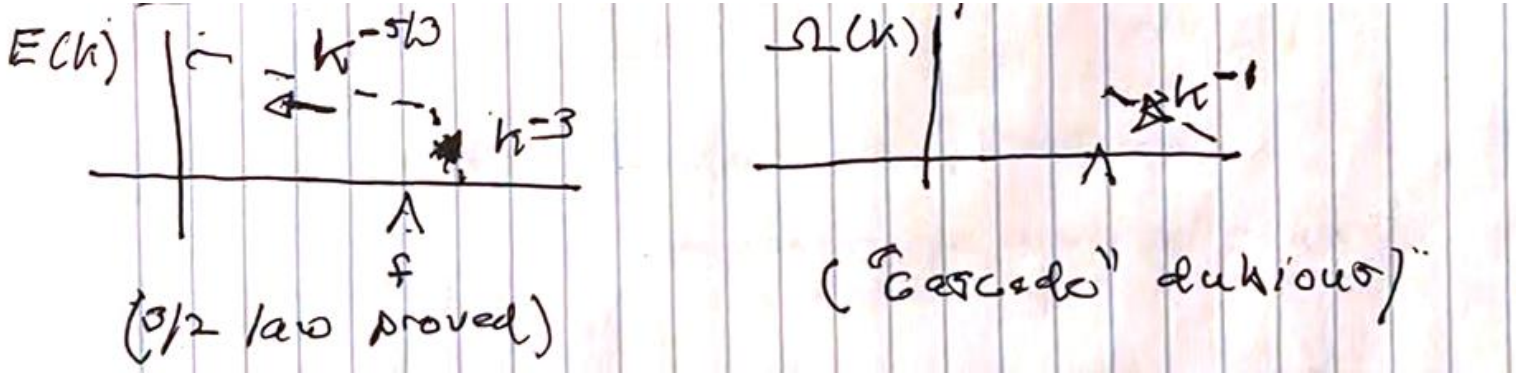
2D Fluid - the prototype

Vorticity Equation: $\frac{D\omega}{Dt} = \nu\nabla^2\omega - \alpha\omega$

Key Physics:

- Inviscid, unforced invariants → $\begin{cases} \text{Energy } E = \int d^2x(\nabla\varphi)^2/2 \\ \text{Enstrophy } \Omega = \int d^2x(\nabla^2\varphi)^2/2 \end{cases}$

⇨ Dual Cascade (Kraichnan)



2D Fluid, Cont'd

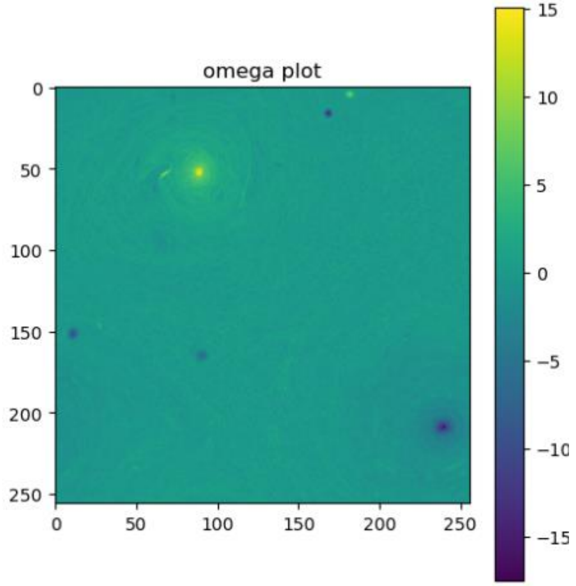
⇨ Selective Decay

Forward 'Cascade' enstrophy → Senses viscosity
Inverse 'Cascade' energy → Senses drag

For Final State of Decay:

$$\delta(\Omega + \lambda E) = 0$$ Bretherton + Haidvogel

⇨ Role Coherent Structures (Vortices)



- emergence isolated coherent vortices → survive decay

- $\frac{d}{dt}\nabla\omega = (s^2 - \omega^2)^{1/2}$

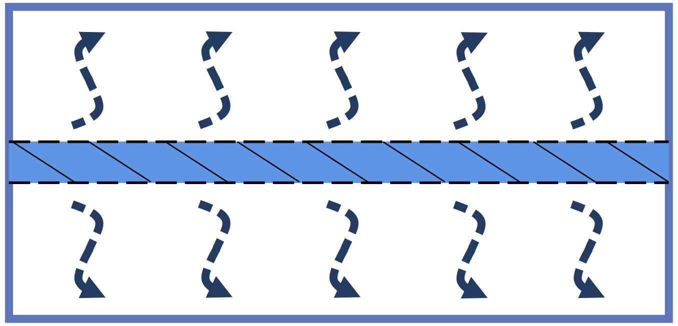
$\omega = \nabla^2\varphi$ → vorticity

$s = \partial^2_{xy}\varphi$ → shear

N.B. : Most of simulation domain is in decay state !

2D Fluid

⇨ Realize:



→ Forcing layer

- Most of system in state of Selective Decay !
- Need Consider / Compare :

$\langle V_y(\nabla^2\varphi)^2/2\rangle$ → Enstrophy Flux

⇩ ⇨ Physical Measures of Spreading
⇧

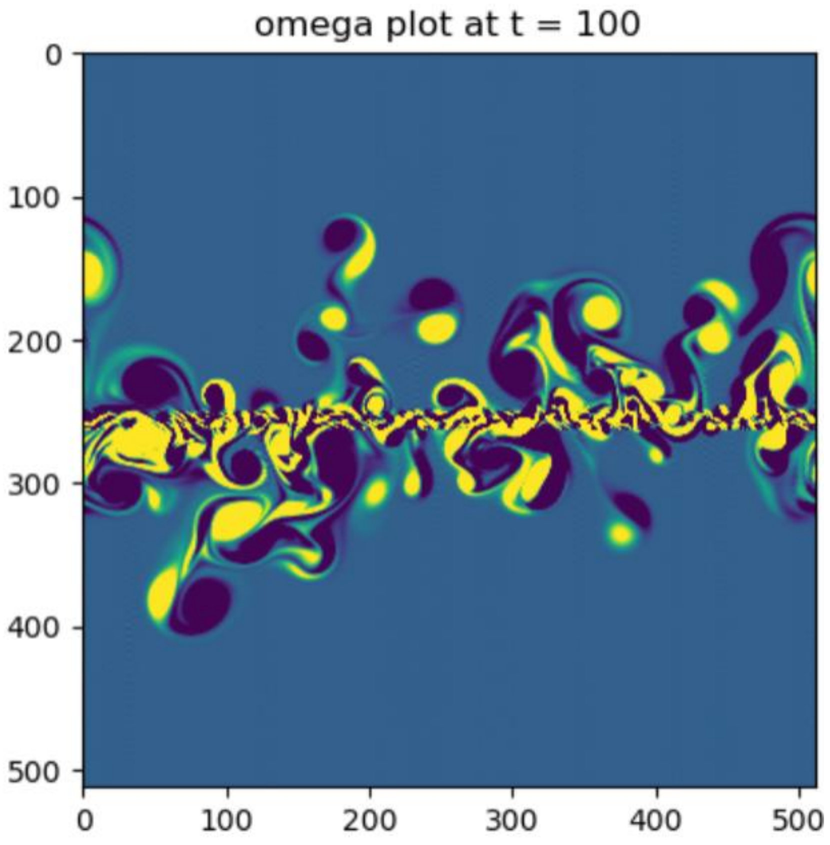
$\langle V_y(\nabla\varphi)^2/2\rangle$ → Energy Flux

as diagnostic of "intensity spreading".

What Happens ?

⇨ In Far Field, away from Forcing layer

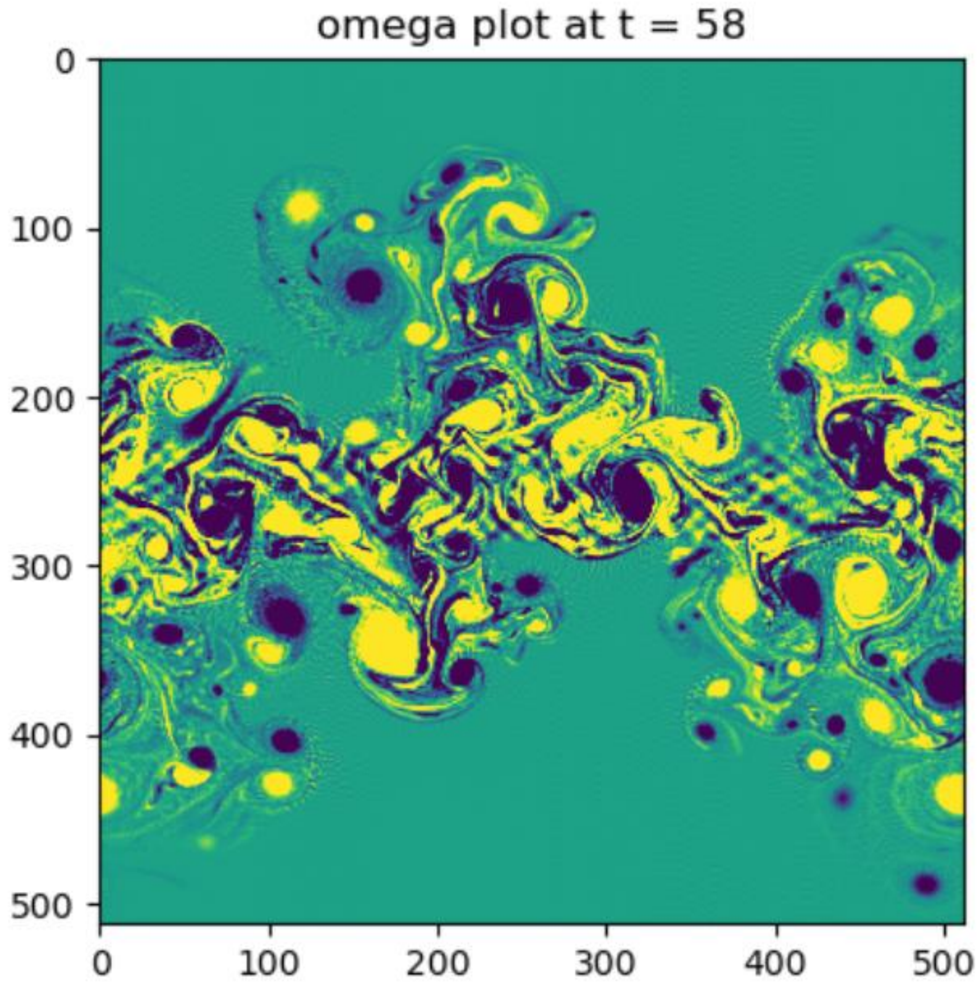
⇨ { Dipole Vortices emerge
  No apparent "Turbulence Front" } → Spreading is intermittent



Vorticity snapshot at Re~100

⇒ What Happens, cont'd

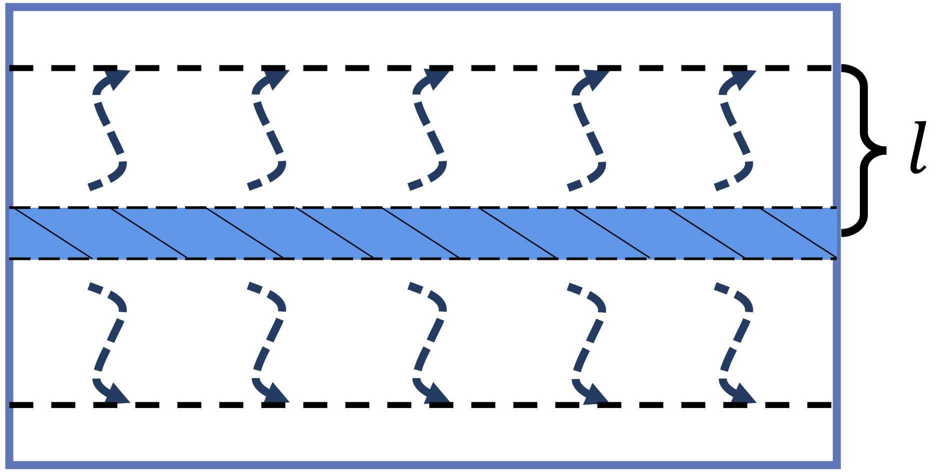
At Re ~ 2000 (marginal resolution):



- Dipoles, Filaments cluster
- Fractalized spreading front?!

On Keeping Score

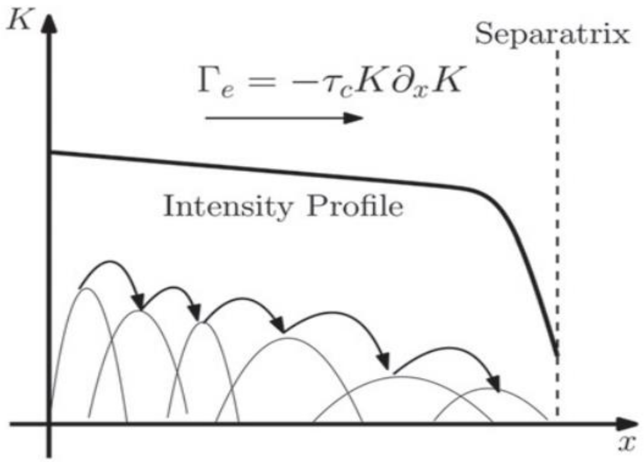
⇨ Loosely, interested in scaling of expansion of turbulent region



$$l \sim t^{\alpha}$$
$$\alpha\ ?$$

⇨ Many approaches to l…

MFE favorite :



Track footprint of $|\varphi|^2$
Plot vs time,
1D projection

Keeping Score, cont'd

⇨ Approaches 1

Table 1: Table describing various velocity and transport parameters.

Parameter	Symbol	Equation	Description
RMS Velocity	V_{rms}	$V_{rms} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} v_i^2}$	Root-mean-square velocity of turbulence, also known as turbulence intensity. This can either be measured near the forcing zone and averaged horizontally for a characteristic velocity as a basis of comparison, or measured globally to obtain global energy.
Quantity-Weighted RMS Distance	X_{W-rms}	$X_{W-rms} = \sqrt{\frac{\int \lvert\delta(x)\rvert^2 \lvert Q(x)\rvert\, dx}{\int \lvert Q(x)\rvert\, dx}}$	Quantity-weighted root-mean-square position represents the location of the quantity of interest, typically energy or enstrophy. One value is generated for each time. The quantity Q is usually energy or enstrophy.
Quantity-Weighted RMS Spreading Velocity	V_{W-rms}	V_{W-rms} is the slope of X_{W-rms} plotted against time	Quantity-Weighted RMS Spreading Velocity represents the bulk motion. This is more comprehensive than the front velocity.

N.B. :

— Quantity weighting can differ; depending on quantity

— RMS velocity sensitive to how computed

Keeping Score, cont'd

⇨ Approaches 1

— Front velocity is MFE favorite
sensitive to 1D projection, definition

— Transport Flux $\langle V_y E\rangle$, $\langle V_y \Omega\rangle$, most physical, clearest connection to dynamics of 2D Fluid

but: Sensitive to viscosity and selective decay

— Jet velocity very sensitive to viscosity, field chosen

Front Velocity	V_{front}	V_{front} is the slope obtained from tracking the outermost turbulent patch	This is usually comparable to V_{W-rms}, although a front doesn't exist for low Reynolds numbers.
Transport Flux Density of a certain quantity	Φ_Q	$\Phi_Q = < Q * V_{perpendicular} >$ 3	The amount of a certain quantity passing through a unit length per unit time; flux is the integral of flux density through the horizontal surface, which bounds half of the region and can be related to the rate of change of the quantity in that region.
Transport "jet" Velocity	V_Q	$V_Q = \frac{<Q*V_{perpendicular}>}{<Q>}$	Also known as normalized flux density. Average is usually taken horizontally. This velocity is separately obtained for each time.

Keeping Score, cont'd

Observation:

— Lower Re → Significant speed, 'front' fluctuations due to variability in dipole population

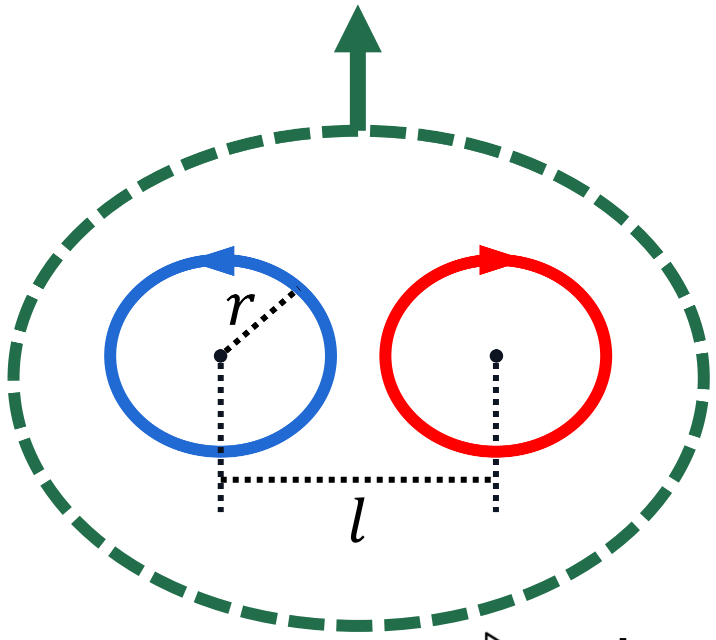
— Transport velocities quite sensitive to viscosity and selective decay

i.e. $\langle V_y \Omega \rangle$ drops
$\langle V_y / \Omega \rangle / \langle \Omega \rangle$ rises
} For higher viscosity

— Formation of dipoles follows decay of enstrophy

— Dipoles ultimately determine spreading

⇒ N.B. Dipole Vortex



— Uniform speed due to mutual induction

— $C = \frac{\Gamma}{l} = \frac{vr}{l}$

⇨ Dipole Vortices propagate at constant speed

⇨ Physical origin of "ballistic spreading" ? !

i.e. ensemble dipoles expands linearly in time

Results

Re ~ 5000

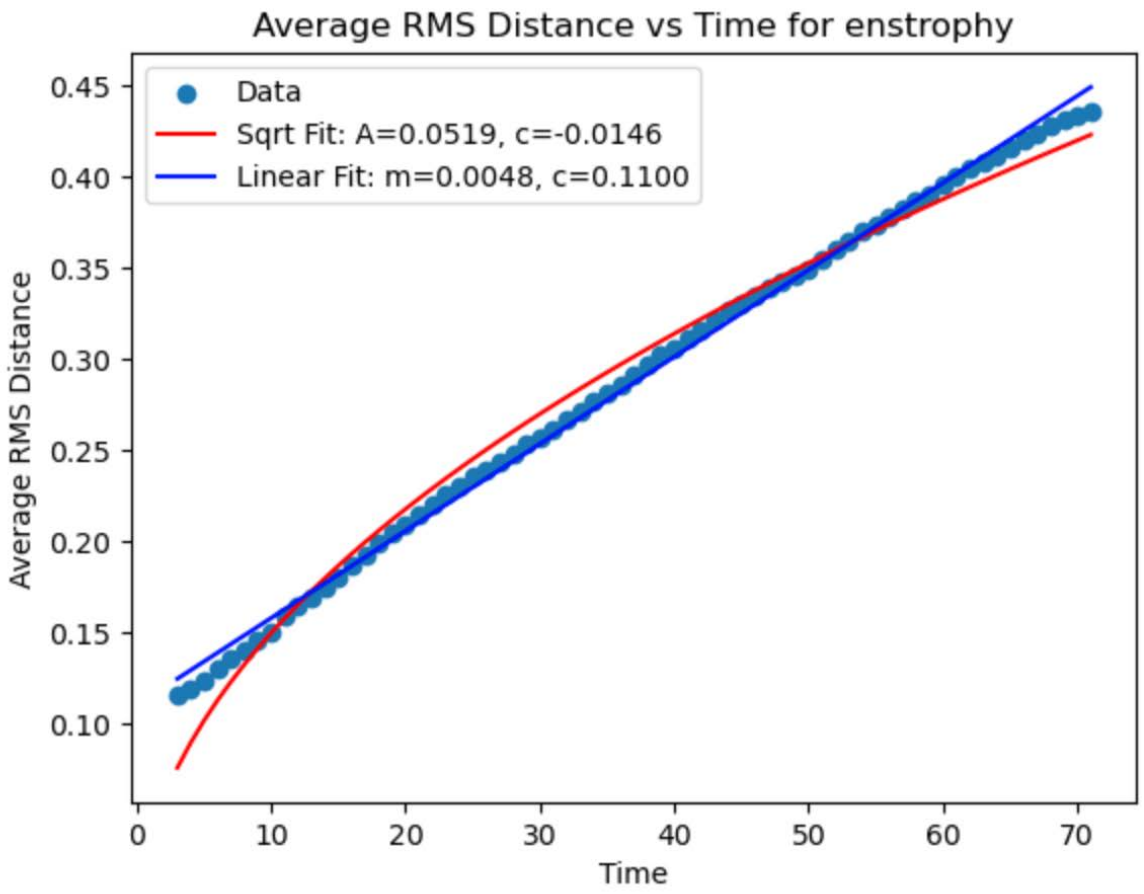
Ω–weighted
rms distance

— Constant spreading speed for enstrophy, i.e., $l \sim ct$

$\underline{\alpha = 1}$

— $c/V_{rms} \sim 0.1$

— Consistent with picture of dipole vortices carrying spreading flux



Results, cont'd

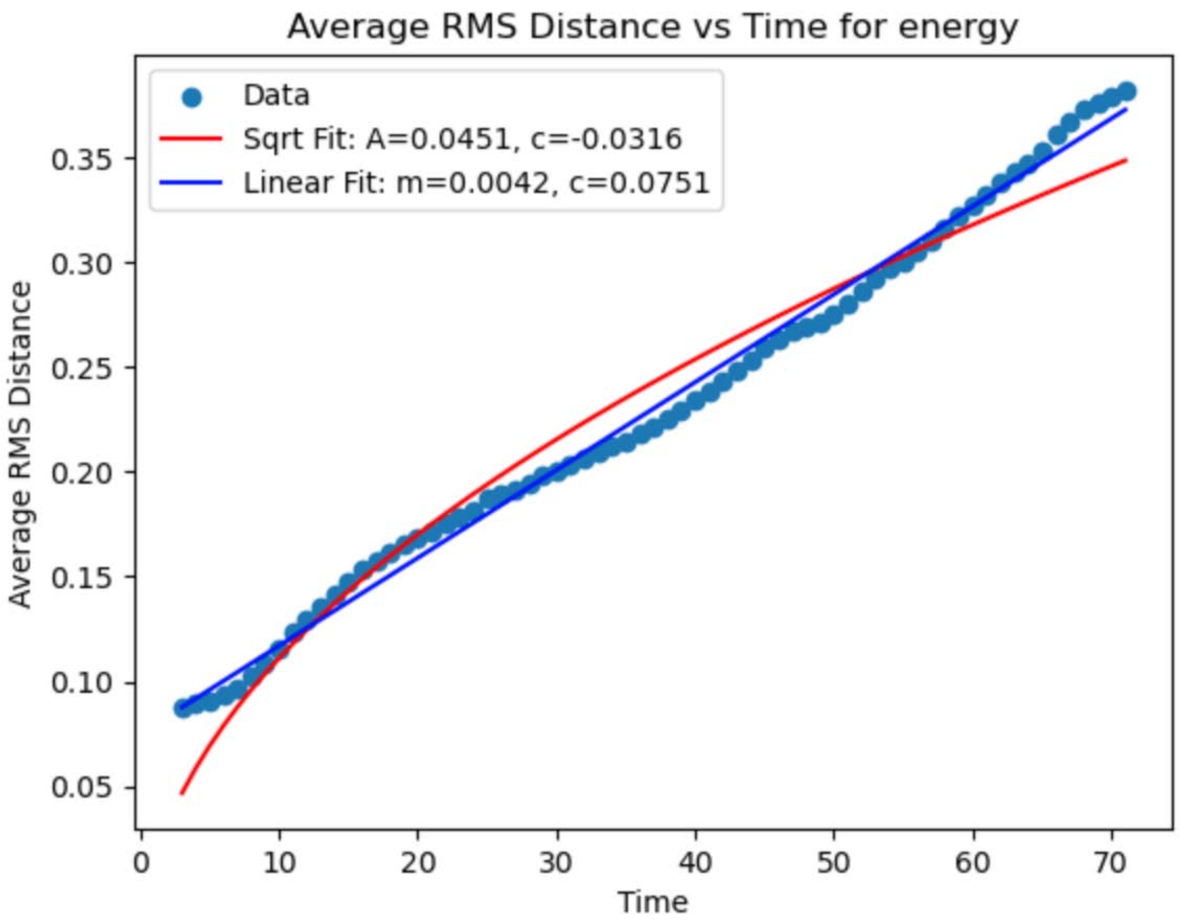
Re ~ 5000

E–weighted
rms distance

— Constant spreading speed for energy, i.e., $\alpha \simeq 1$

— $c/V_{rms} \sim 0.1$

— Lager dipoles ↔ more energy → increases fluctuations relative to enstrophy case

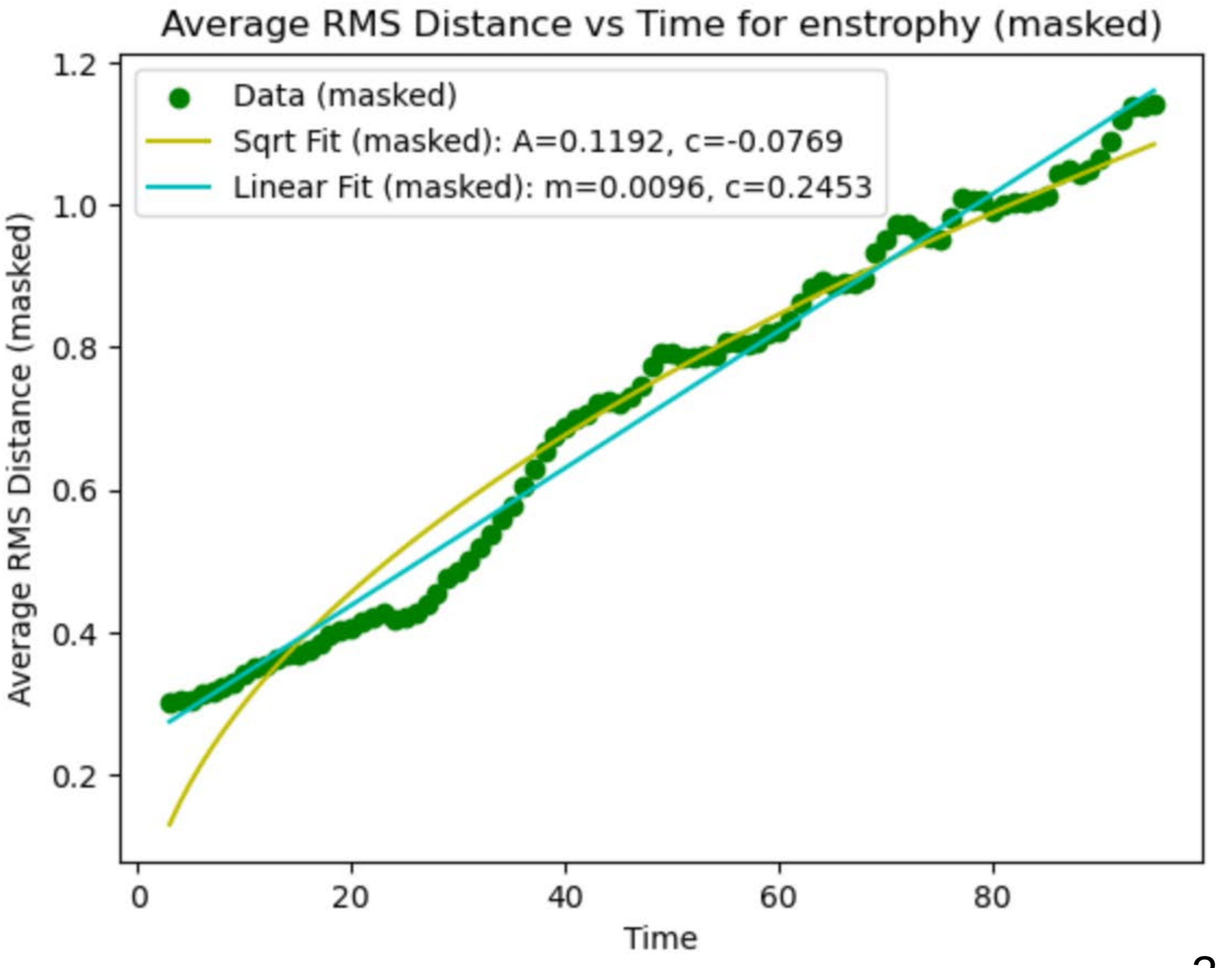


<u>Results</u>, cont'd

Re ~ 200

— Low Re → increased scatter in L vs t

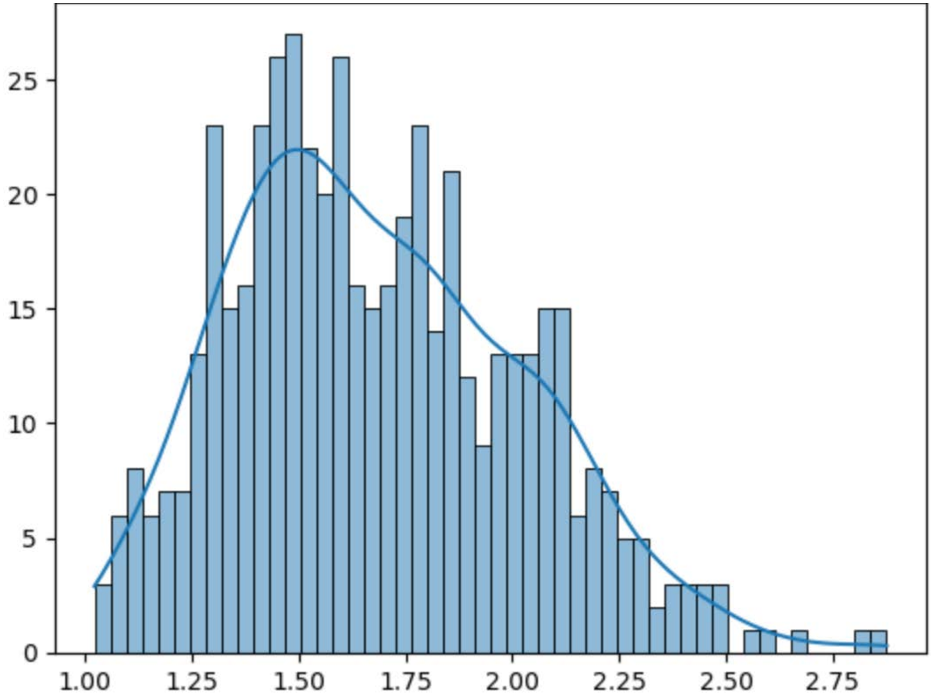
→ dipole scatter → intermittent pattern → front not identifiable



Results, cont'd

PDF of RMS Distance

Prob
Density



RMS Distance

- ⇨ PDF of spreading (vorticity) at given t.
- ⇨ Calculate enstrophy-weighted rms distance for each position X; plot histogram
- ⇨ Note skewed structure.

Summary - 2D Fluid

— Coherent structures - Dipole vortices - mediate spreading of turbulent region

— Mixed region expands as $w \sim t$, consistent with dipoles.

— No discernable “Front”, spreading is strongly intermittent. (space+time)

— Spreading PDF is non-trivial, exhibits tail.

⇨

— Turbulence spreading strongly non-diffusive.

2D MHD + Weak $\underline{B_0}$

2D MHD

- The equations: $$\frac{d}{dt}(\nabla^2\varphi) = \nu\nabla^2\nabla^2\varphi + \nabla A \times \hat{\boldsymbol{z}} \cdot \nabla\nabla^2 A + \tilde{f}$$
$$\frac{d}{dt}A = \eta\nabla^2 A$$
$$\frac{d}{dt} = \partial_t + \nabla\varphi \times \hat{\boldsymbol{z}} \cdot \nabla$$

- Inviscid Invariants: $E = \langle V^2 + B^2\rangle,\ H = \langle A^2\rangle,\ H_c = \langle\vec{V}\cdot\vec{B}\rangle$

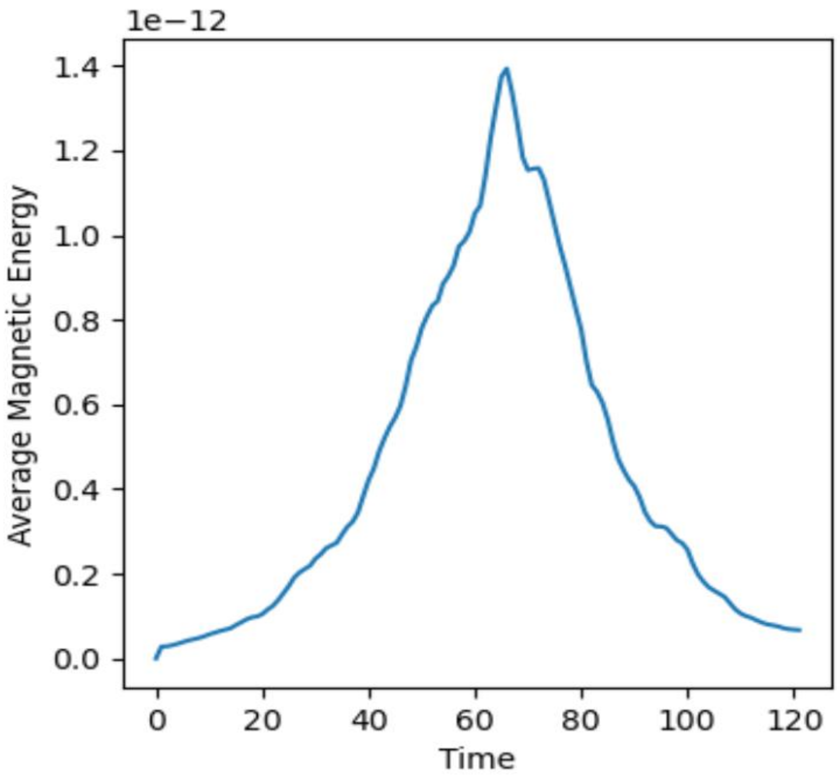
  Conservation of H is Key !

- Consider weak mean magnetic field: $B = B_0(y)\hat{x}$
  $B_0(y) \sim B_0\sin(y)$ ⇒ initial imposed pattern

- As before, localized forcing region, effectively unmagnetized

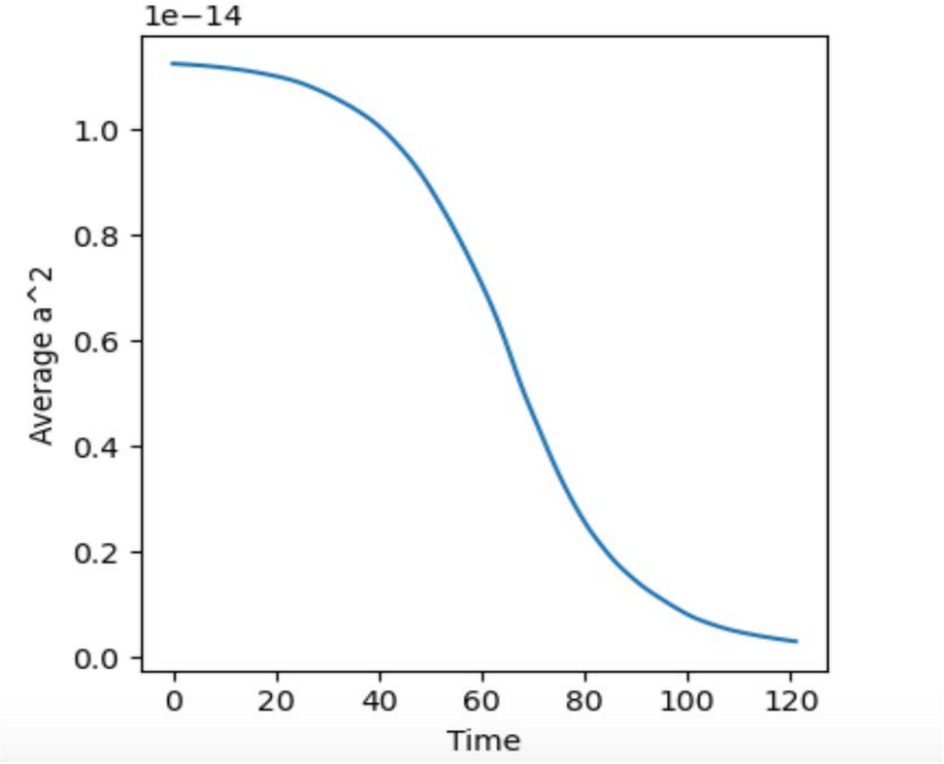
⇒ 2D MHD

- Cowling's Theorem: No dynamo in 2D



⇨ Field ultimately decays

- Consequence of decay $\langle A^2 \rangle$



$$\frac{d}{dt}\langle A^2 \rangle = -\eta \langle B^2 \rangle$$

Key Physics of 2D MHD

- Lorentz force suppresses inverse kinetic energy cascade.
  Inverse cascade $\langle A^2 \rangle$ develops

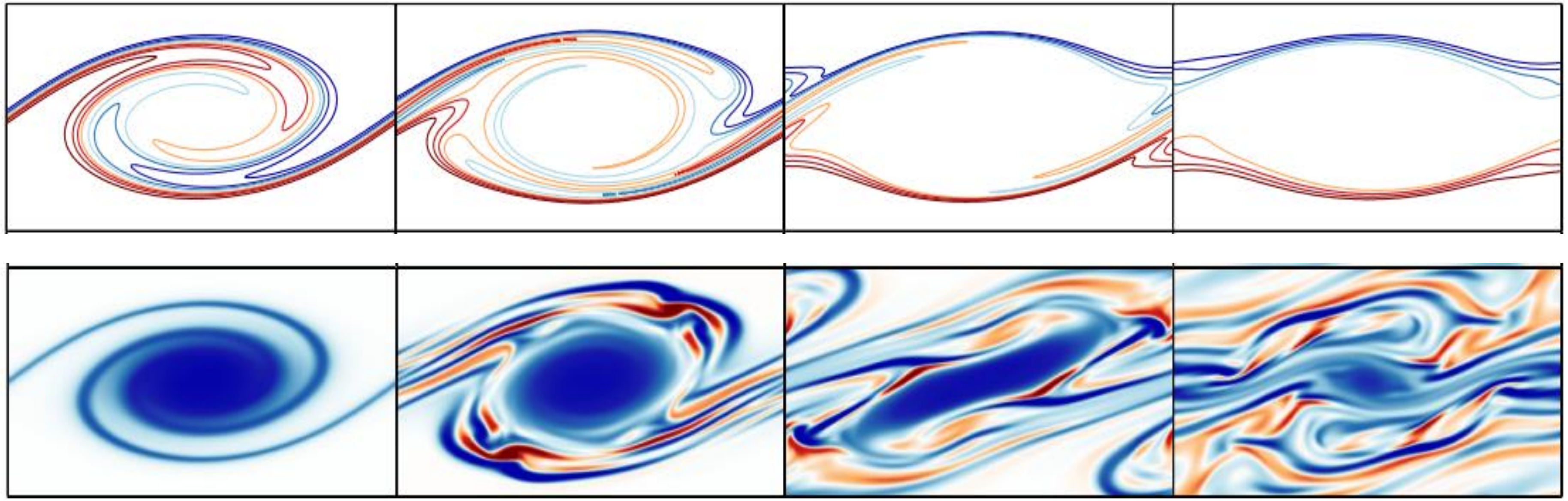
- Single Eddy: Expulsion (Weiss'66) vs. Vortex Bursting (Mak. 2017)

Key Parameter: $Z = Rm \frac{V_{A0}^2}{V_E^2}$

$Z \sim 1$ bounds the two regimes

Expulsion:

Vortex bursting:



Key Physics of 2D MHD, cont'd

- Turbulent Diffusion: (Cattaneo + Vainshtein '92 ; Gruzinov + P.D. '94)

  Closure + $\langle A^2 \rangle$ conservation ⇨ Quenched Diffusion of B - field

  From: $D_t \sim \eta_{anom} \sim \langle \tilde{V}^2 \rangle \tau_c$

  To: $D_t \sim \eta_{anom} \sim \langle \tilde{V}^2 \rangle \tau_c / \left[1 + R_m V_{A0}^2 / \langle \tilde{V}^2 \rangle \right] \sim D_{Kin} / (1 + Z)$

- Once again,

Key Parameter: $Z = R_m \frac{V_{A0}^2}{<\tilde{V}^2>}$

N.B.: - V_{A0} is initial weak mean magnetic field

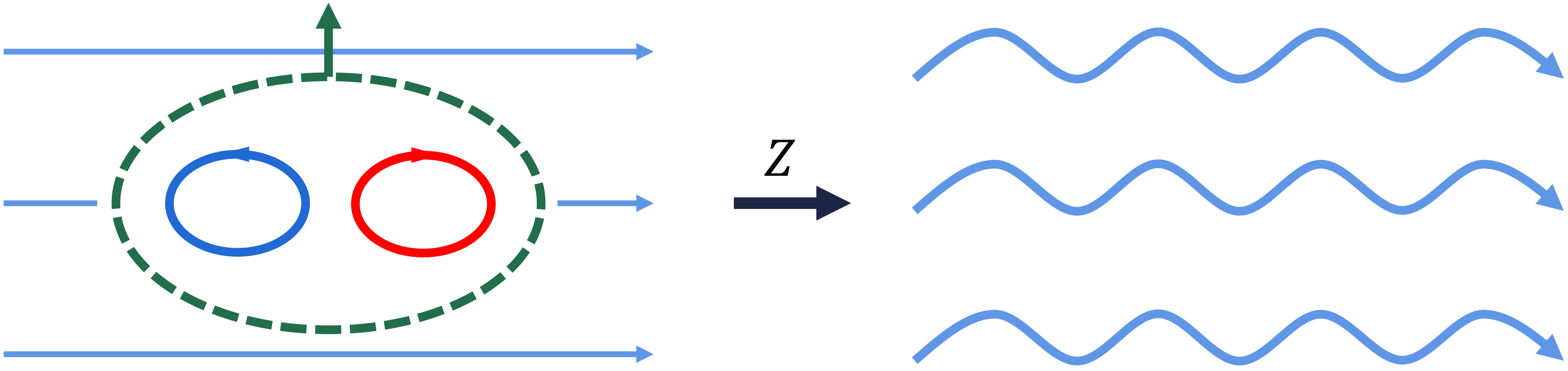
- R_m large...

Crux of the Issue!?

⇨ Hydrodynamics: Dipole vortex ‘Carries’ turbulence energy ⇨ spreading

⇨ But... weak $\underline{B_0}$ can ‘burst’ vortices ⇨

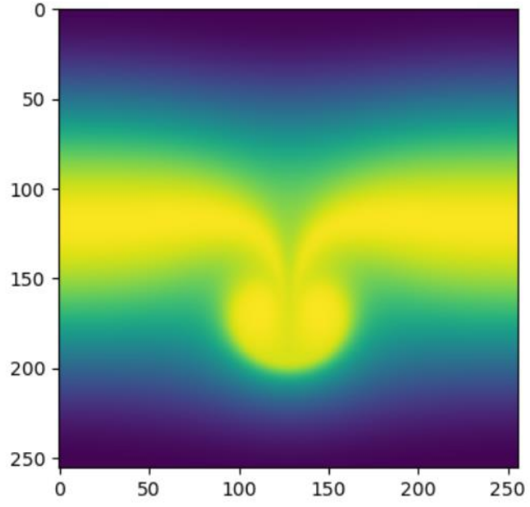
converts dipole kinetic energy to Alfven waves, propagating laterally, and dissipation.



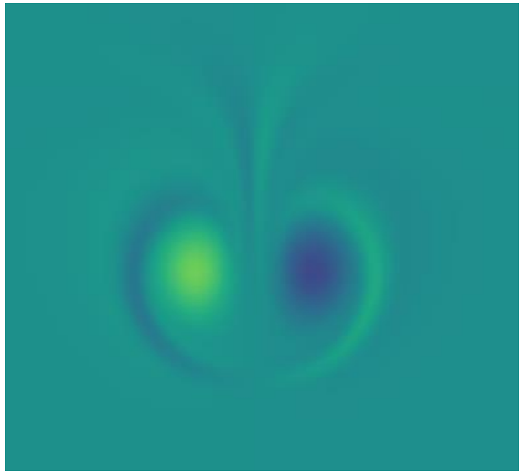
⇨ So, can a weak B_0 block spreading in 2D MHD ! ?

⇒ Single Dipole in weak B_0

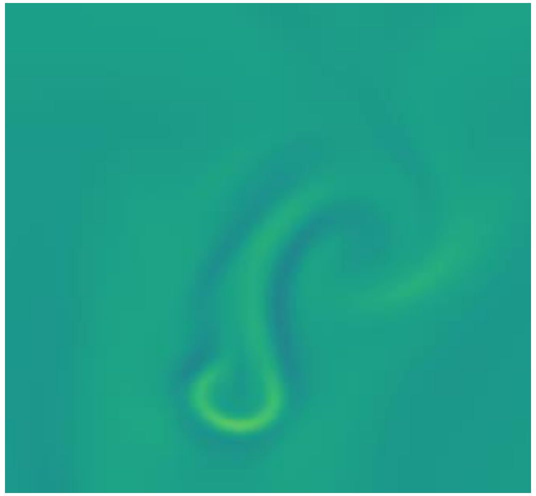
A at $t = 40$



ω at $t = 40$



J at $t = 80$



Note wrapping filament tends to cancel and push on dipole, so it distorts and ultimately bursts

Filament and vortex bursting. Concentration at small scale ⇨ fast dissipation

Connection: vortex busting ↔ MHD cascade singularity?!

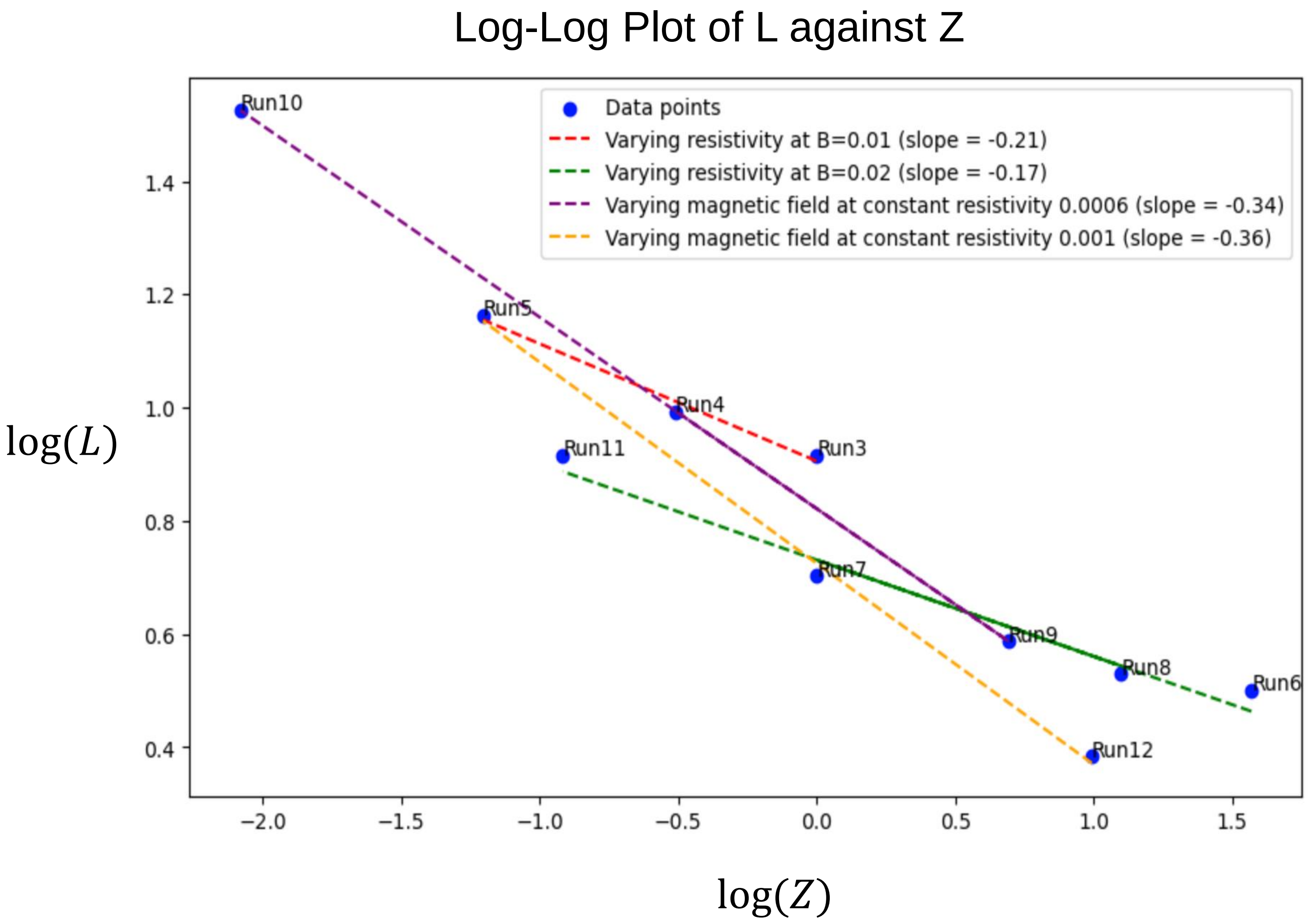
⇨ Single Dipole Penetration

- Dipole penetration decreases with increasing Z

- Evidence that varying B_0 and R_m impact penetration.

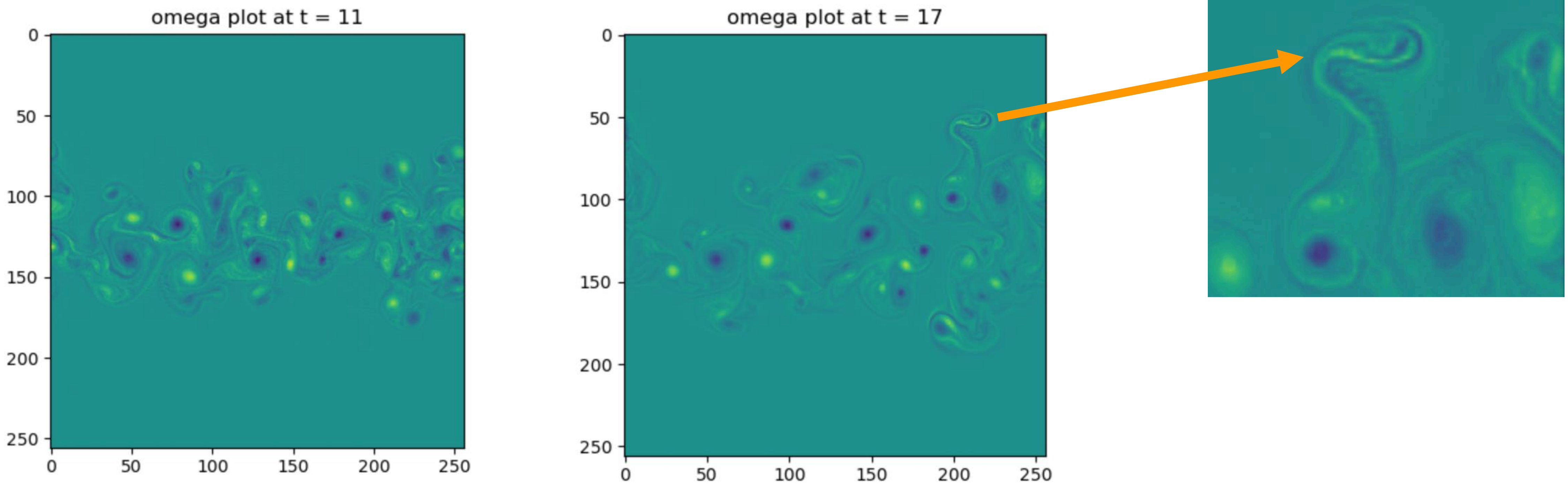
Z is not the full story…

⇨



⇨ Close Look at Vorticity Field

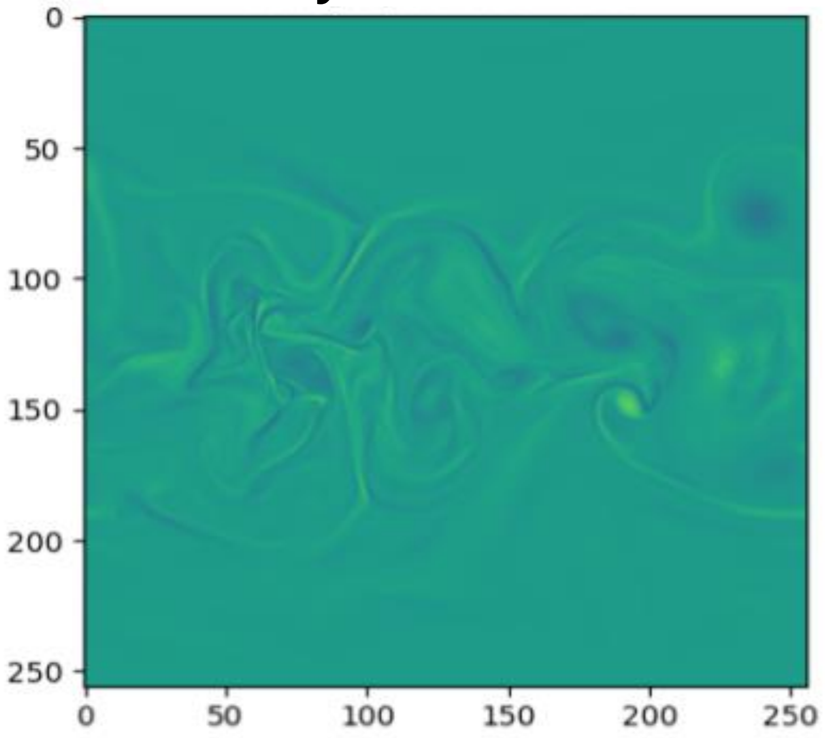
Bursting/Filamentation



- Z=3, Rm≈50, Re≈500, B=0.01
- Dipoles evident at early times, but encounter stronger field as rise/sink
- Vortex bursting occurs at later times ⇨ Spreading halted.

⇨ Vorticity Field for Z>1

Vorticity Plot at t=176



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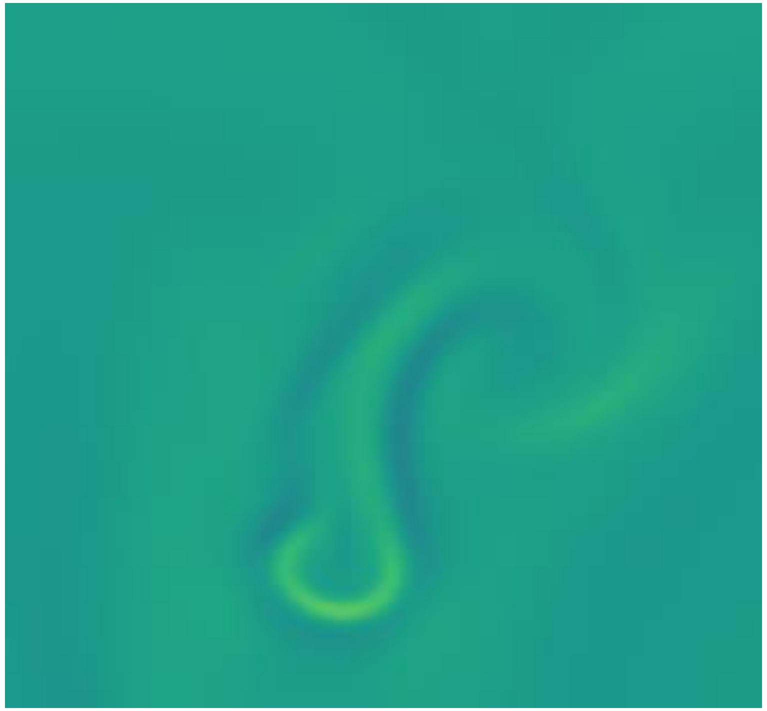
Fate of single Dipole



- “Vortices” barely evident
- Vorticity residual is ~ horizontal filaments, consequence of vortex bursting

⇨ Spreading vs. Z - Turbulence

- Now consider turbulence:

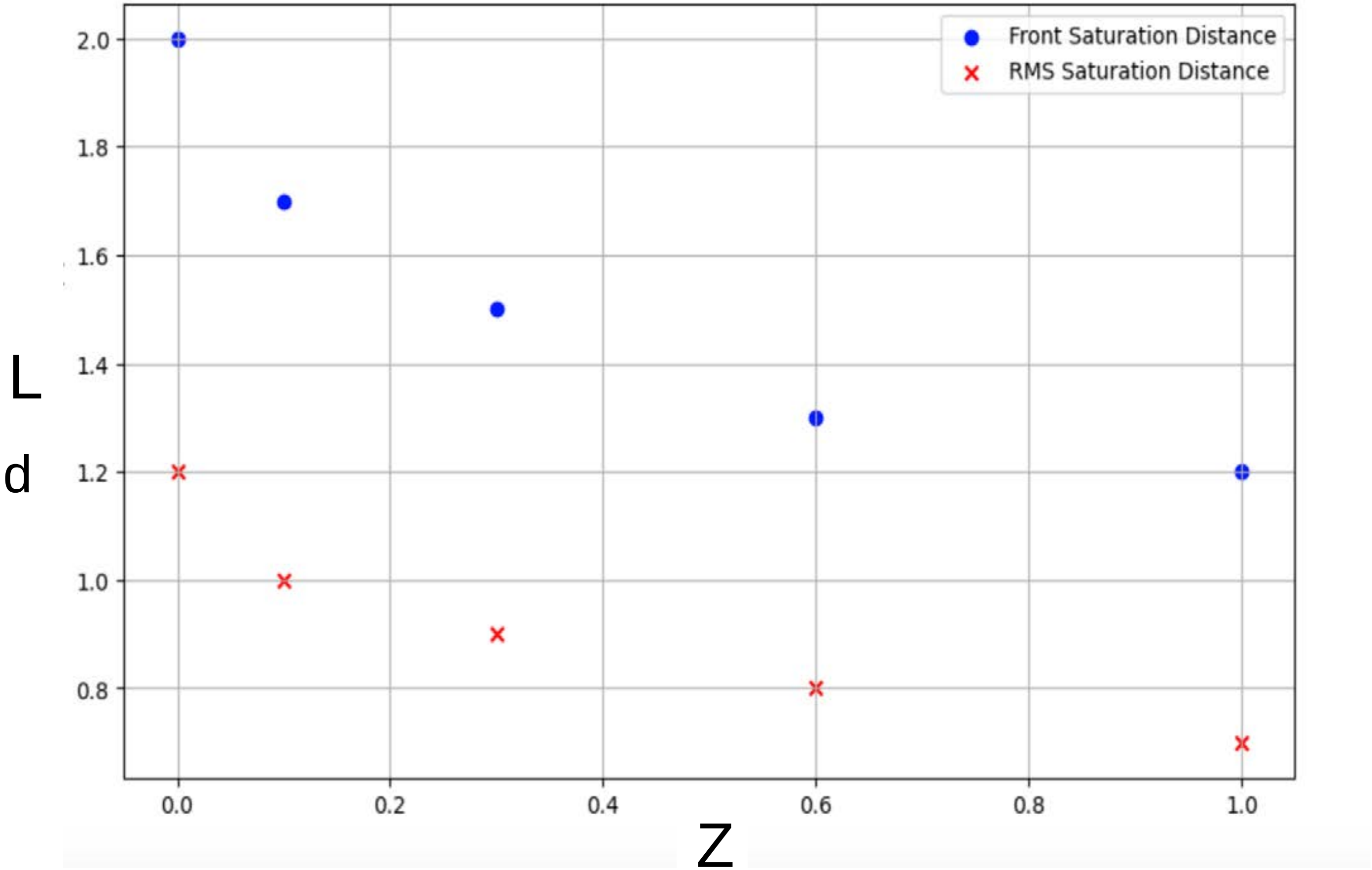
- Kinetic Energy Stopping length decreases with increasing $Z = R_m \frac{V_{A0}^2}{<V_{rms}^2>}$

  N.B. Z reflects both R_m and B_0

- Systematic difference between Front and RMS saturation evident

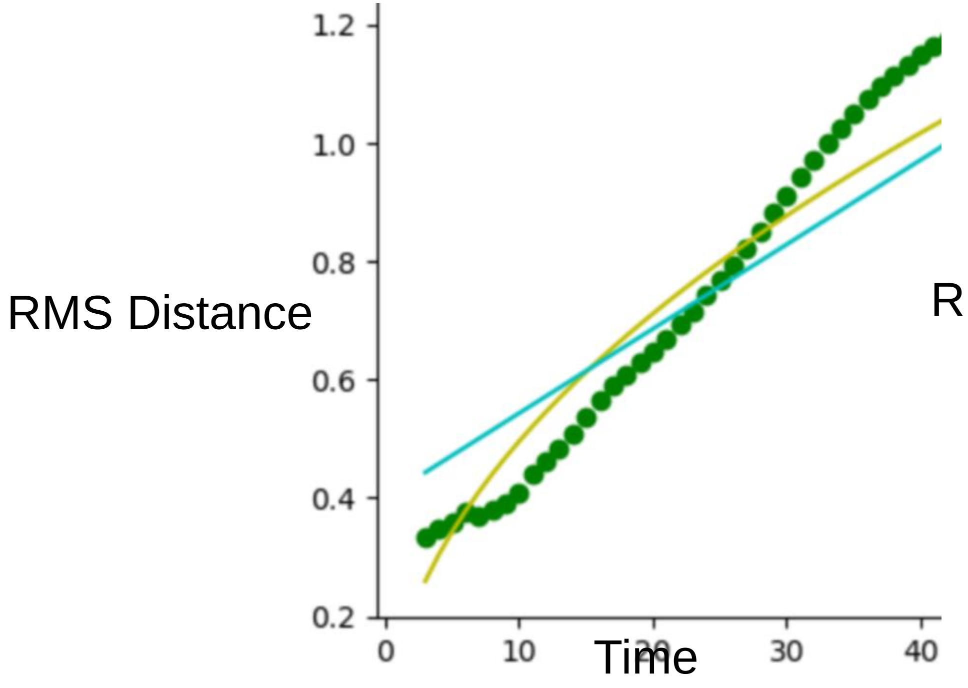
⇨ Insight from vortex studies useful

Saturation distance L vs. Z

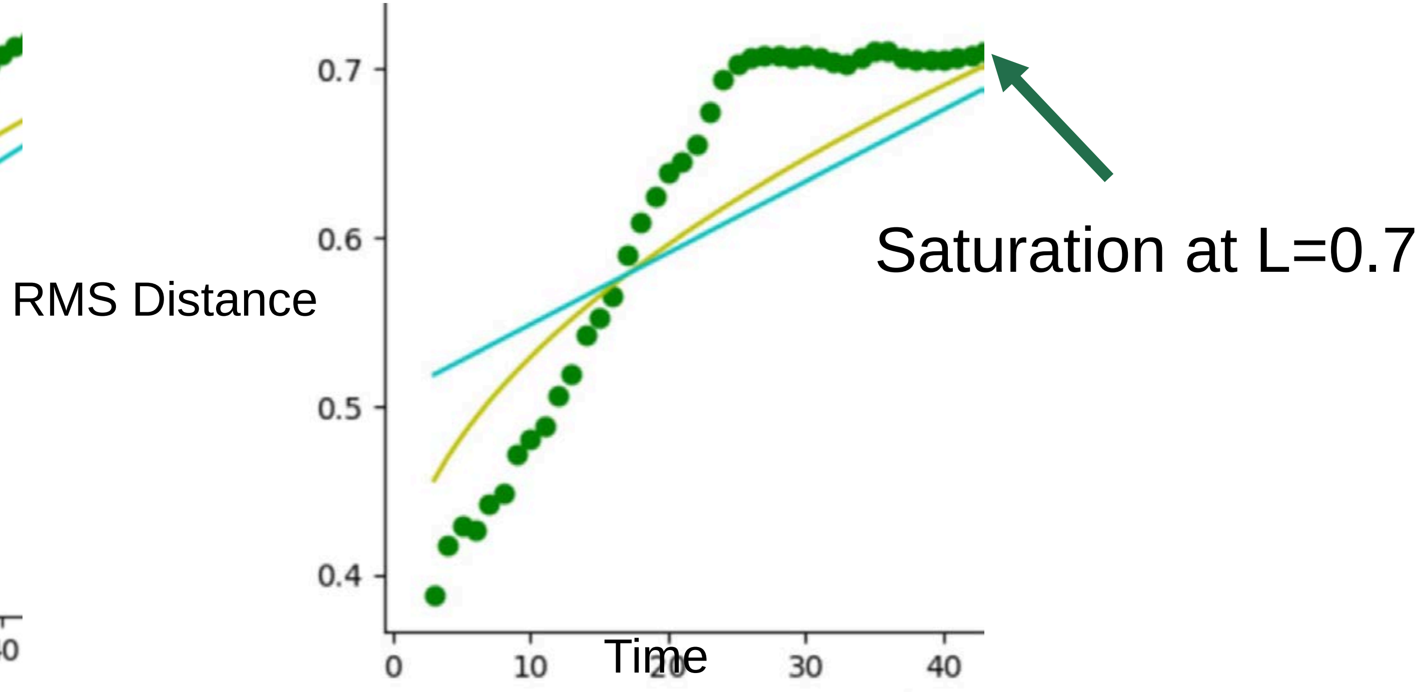


⇨ Time evolution of Spreading

Hydro regime: $Rm = 100, Bo = 0.001, Z = 0.01$



MHD: $Rm = 100, Bo = 0.01, Z = 1$



⇨ Hydro case spreads linearly

⇨ Z=1 Case saturates.

⇨ 2D MHD: Summary

- Weak B_0 allows vortex bursting

⇩

- Conversion dipole KE to Alfven waves, dissipation

- Spreading saturated by weak B_0 i.e. advance of kinetic energy blocked

- $Z = R_m \frac{V_{A0}^2}{<V_{rms}^2>}$

but:

- Bursting dynamics complex ⇨ May introduce additional dependencies on ν, P_m

Drift Wave – Zonal Flow Turbulence

⇨

Hasegawa – Mima + Zonal Flow

H-M + Zonal Flow System

– System:

$$\frac{d}{dt}\left(\tilde{\phi} - \rho_s^2 \nabla_\perp^2 \tilde{\phi}\right) + v_* \frac{\partial \tilde{\phi}}{\partial y} + v_{*u} \frac{\partial \tilde{\phi}}{\partial y} = 0$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \bar{v}_z \frac{\partial}{\partial y} + \nabla \tilde{\phi} \times \hat{\boldsymbol{z}} \cdot \nabla$$

$$\frac{\partial}{\partial t} \nabla_x^2 \phi_z + \frac{\partial}{\partial r} \left\langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \right\rangle + \mu \nabla_x^2 \phi_z = 0$$

– viscosity controls small scales

– drag controls zonal flow

– conserved: Energy $\longrightarrow \left\langle \tilde{\phi}^2 + \rho_s^2 \left(\nabla \tilde{\phi}\right)^2 \right\rangle$ Potential Enstrophy $\longrightarrow \left\langle \left(\tilde{\phi} - \rho_s^2 \nabla^2 \tilde{\phi}\right)^2 \right\rangle$

N.B. : For waves

H-M + Zonal Flow System, cont'd

→ Now: *waves* $\omega = \omega_* / 1 + k_\perp^2 \rho_s^2$, $\underline{v_{gr}}$

eddies $\tilde{v}$

zonal mode (symmetry)

$\left\{ \begin{array}{l} \tilde{v} \text{ vs } v_* \rightarrow \\ \text{mixing length} \end{array} \right.$

i.e. ⇒ Potential Enstrophy Flux: $\left\{ \begin{array}{l} \sum_{\boldsymbol{k}} v_{gr}(\boldsymbol{k}) u_{\boldsymbol{k}} \rightarrow \text{and other} \\ \langle \tilde{v}_r \tilde{u} \rangle \rightarrow 3^{rd} \text{ order} \end{array} \right.$

N.B. 2 channels for "turbulence spreading" → Waves / Turbulent mixing

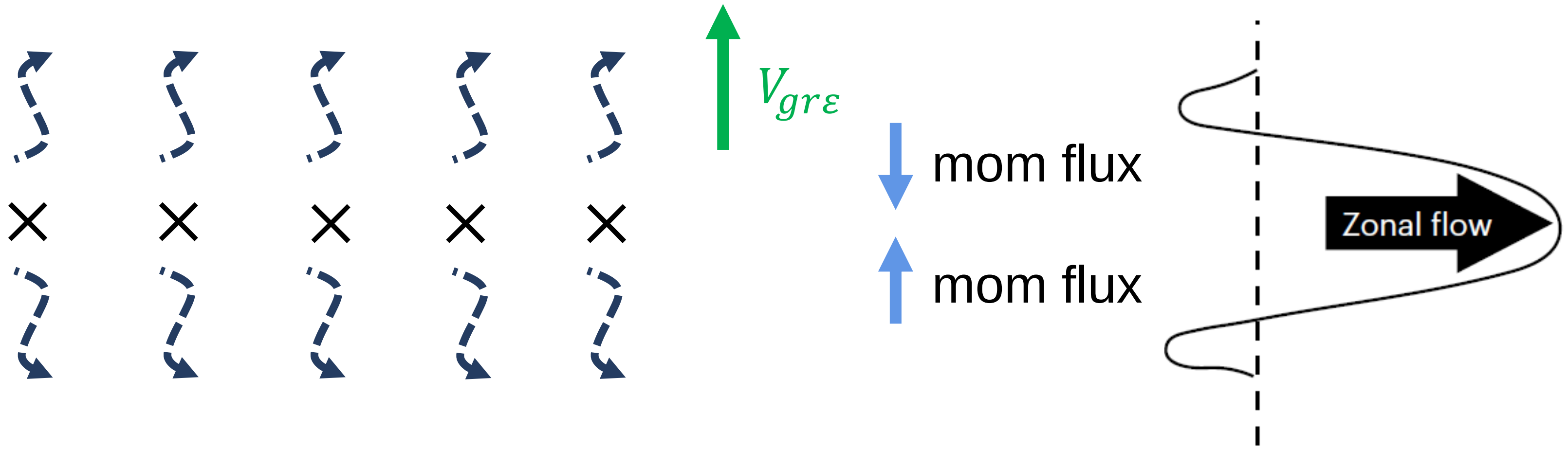
-Branching ratio, vs. Ku number ?

H-M + Zonal Flow System, cont'd

→ Enter the Zonal Flow…

- Multiple channels for NL interaction
- But with ZF⟷ eddy, wave coupling to ZF dominant
- Mode of minimal inertia, damping, transport

⇒ energy coupled to ZF ($\tilde{v}_r = 0$) cannot "spread"



⇨ **For clarity**

Contrast:

⇨ **spreading in presence of fixed, externally prescribed shear layer**

Here: → Forcing → $\left\{\begin{matrix} \text{Waves} \\ \text{Eddies} \end{matrix}\right\}$ → Zonal flow

ZF is self-generated

∴ forcing ($\tilde{v}_{rms}, Re$) + drag ⇒ control parameters

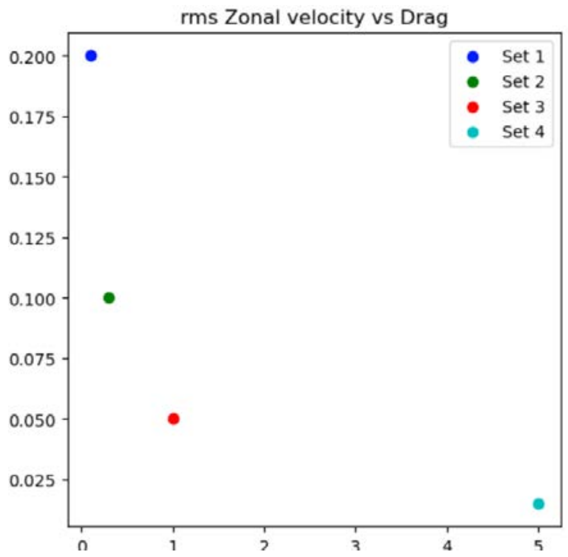
→ "weak" and "strong" Turbulence Regimes

v_{gr} vs v_r

Results (Preliminary / Ongoing)

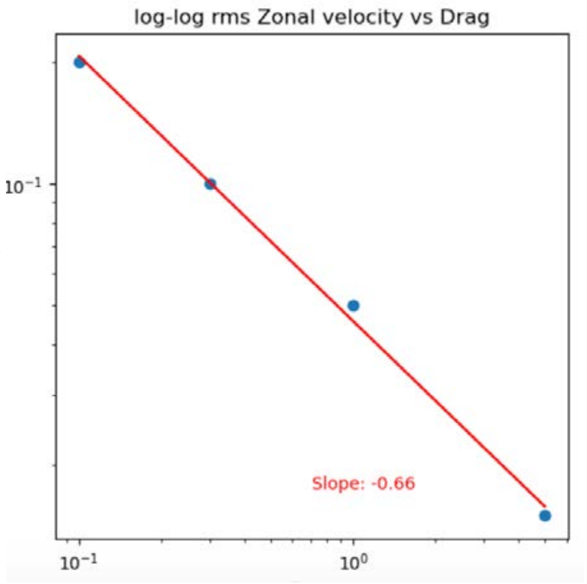
Expected:

V_{Zonal}



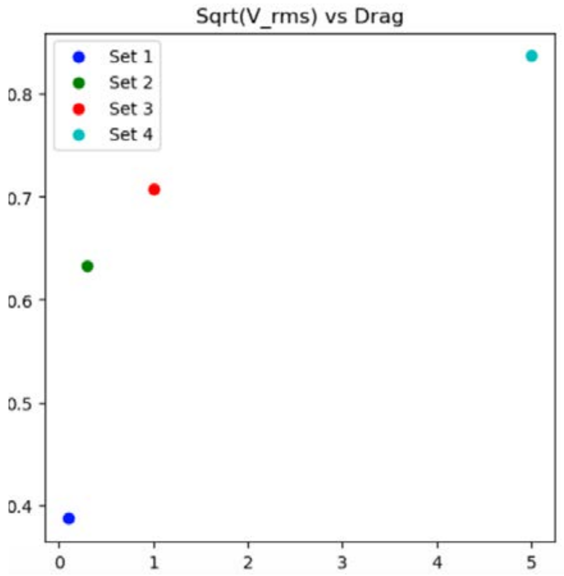
drag

lnV_z



$\ln(drag)$

V_{rms}



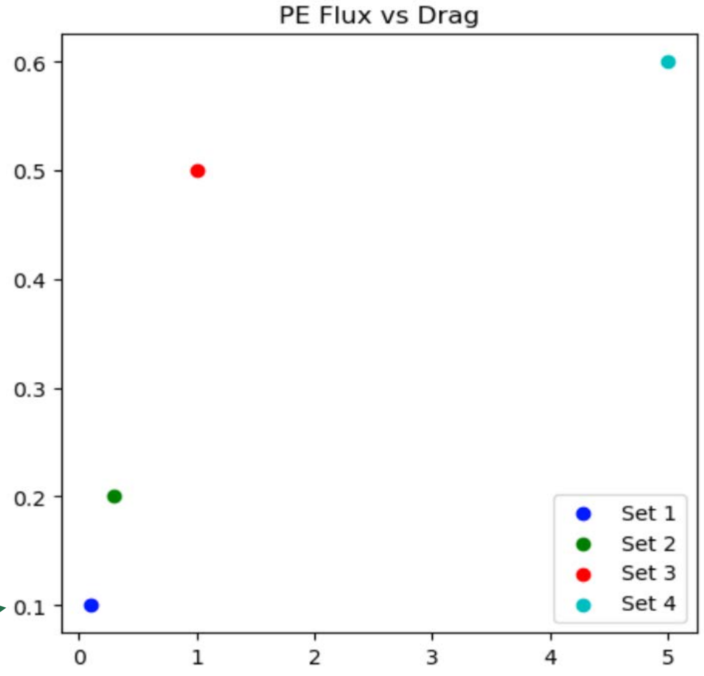
Drag

Zonal velocity decreases with increasing drag

Fluctuation intensity increases as drag increases

<u>Results</u>, Cont'd

Potential
Enstrophy
Flux



Drag

Dimits Regime

⟹ at given x, averaged over y

— Potential enstrophy flux <u>increases</u> as drag increases.

— $\langle \tilde{v}_r \tilde{u} \rangle \to 0$ as $\mu \to 0$ is "Dimits regime" for turbulence spreading. Spreading vanishes as power coupled to Z.F.

— Self-generated barrier to spreading.

Summary - Drift Wave Turbulence

→ Study ongoing, Several questions identified

→ Dimits regime limit for spreading discovered

Wish List:

→ Spreading { Wave Propagation / Turbulent Mixing } vs. Ku ?

→ Spreading Flux vs. { Forcing / Drag } ⇨ 3D plot ?

→ Potential Enstrophy Penetration PDF vs. Ku ⇨ Waves vs. Vortex structures?

→General Summary

→ Spreading dynamics non-diffusive (cf: Ting Long, next talk)
Coherent structures mediate spreading←→ “ballistic scaling”

→ Conventional wisdom of front tracking, diffusion, intensity flux grossly incomplete, or worse.

→ Self-inhibited states manifested

— Weak B_0 blocks spreading, Z≥1 ←→disrupts dipole

—Potential Enstrophy Flux manifests Dimits shift.

→ Wave vs mixing spreading flux of interest.

→Future Plans

→ Complete study of Drift-Wave-Zonal Flow System

→ High resolution studies

Special Interest:

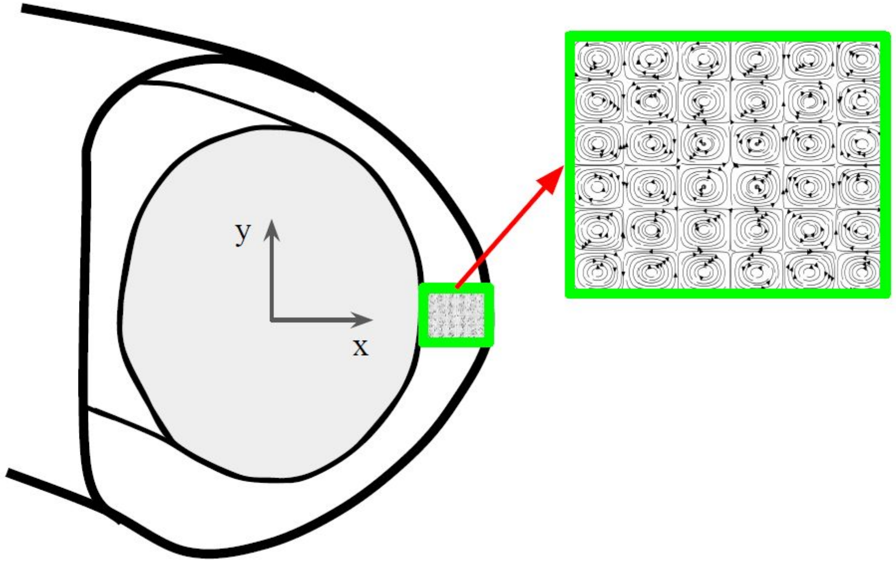
— Is spreading at all “universal”?

— Physics of vortex bursting in 2D MHD

— Vortex bursting ←→ 2D MHD cascade, physics?

— Can KH / “Tertiary” mediate spreading thru Dimits shift regime ? Episodic ?

Reduced Model of Staircase



Fixed Cellular Array Problem

(another way to get a Staircase)

$$Pe = \frac{\tau_D}{\tau_H}$$

Fixed Cellular Array

Consider a **general** case of a system of eddies not overlapping but tangent → **Staircase**

Transport? Answer: Deff ~ D $Pe^{1/2}$ {**Not a simple addition of process!**}

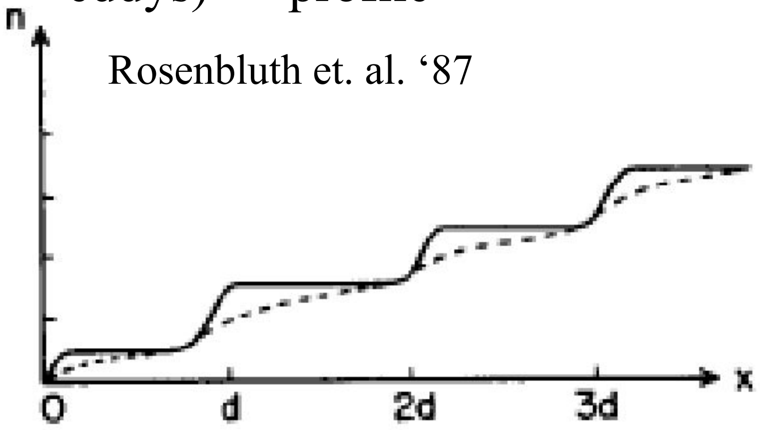
→ Two time rates: v / ℓ, D / ℓ²

→ Pe = v ℓ / D >> 1

$$\frac{\partial n}{\partial t} + \mathbf{u} \cdot \nabla n = D \nabla^2 n,$$

Profile?

Consider concentration of injected dye (passive scalar transport in eddys) → profile

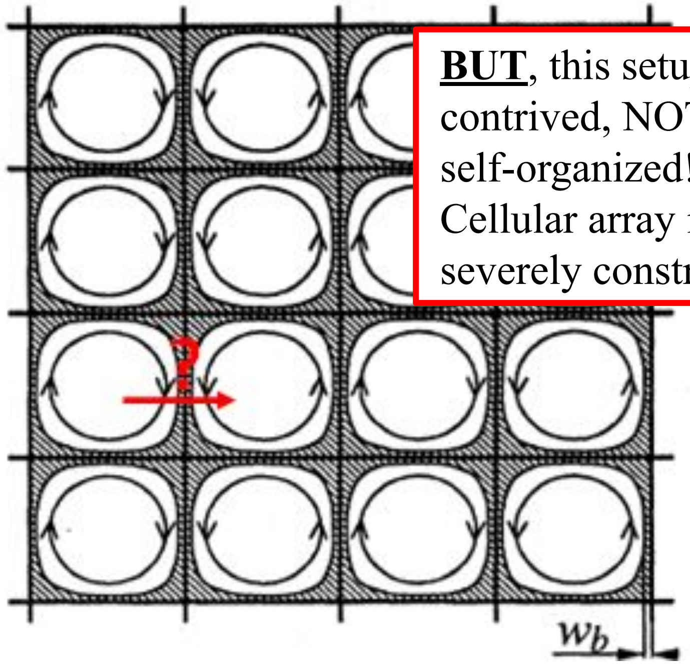


→ Layering!

→ **Simple** consequence of **two rates**

"Steep transitions in the density exist between each cell."

Relevant to key question of "near marginal stability"



BUT, this setup is contrived, NOT self-organized!!! Cellular array is severely constrained!

Staircase arises in an array of stationary eddies!

Important:

- **Staircase** arises in stationary array of passive eddies (Note that there is no FEEDBACK)
- Global transport hybrid:
  - → fast rotation in cell
  - → slow diffusion in boundary layer
- Irreversibility localized to inter-cell boundary.

What about the dynamics of a **less constrained** cell array (i.e., vortex array with fluctuations) ?

Relaxing Fixed Cellular Array with Fluctuating Vortex Array

Consider a Broader Approach

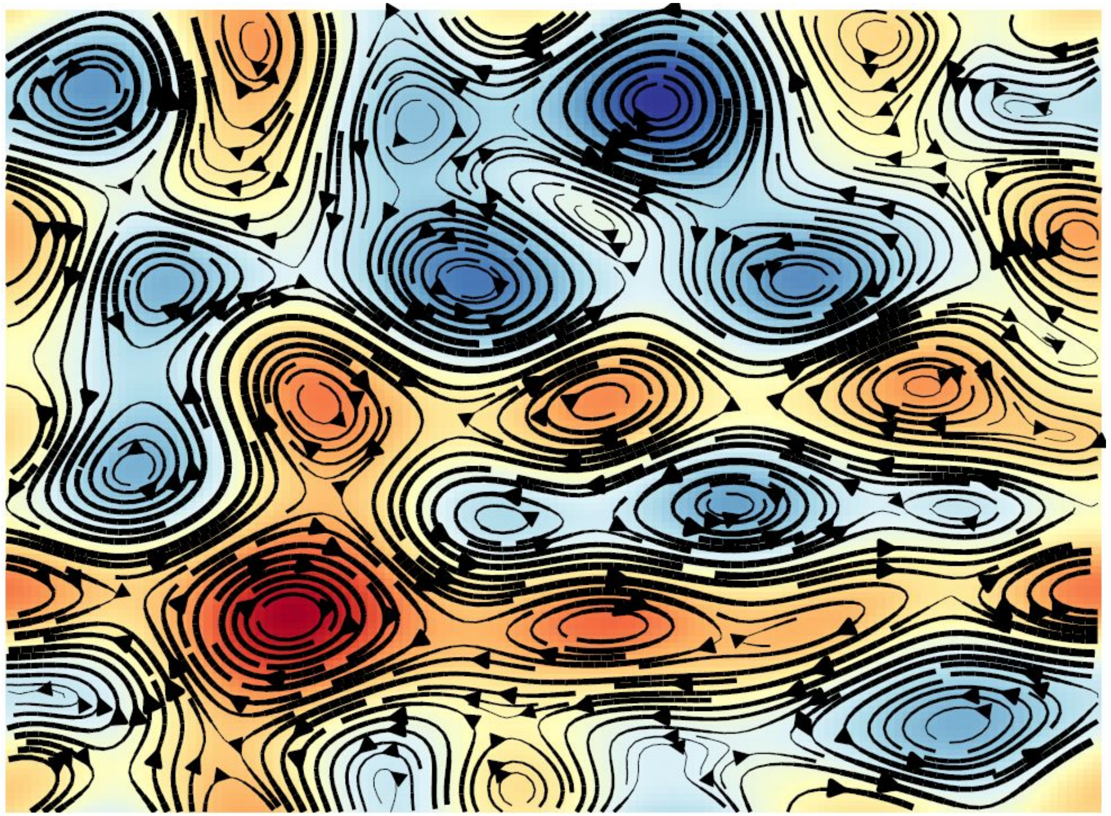
- We want to study a much more **general** and **less constrained** version of the cell array.
  - Consider a vortex array with fluctuations; jitters.
- How **resilient** is the staircase in the presence of these small variations to a fixed vortex array?

In the process of studying the **resilience** of the staircase, we aim to answer the following:

1. What happens to interspersed regions of strong scalar concentration mixing as cells relax? What about general cell interactions/behavior?
2. What is the behavior of the scalar trajectory through the vortex array?
3. How does the increase of scattering in the vortex array affect the transport of scalar concentration?

To answer these questions, we use the idea of a **Melting Vortex Crystal**…

Example of **less constrained** cell array



Fluctuating Vortex Array

Why are we doing this? We know that a system with two disparate time scales forms a staircase!

- Now consider fluctuations… → Will staircase survive?

Vortex array is an alternative way to view convection cells!

→ We begin with the 2D NS equation that can be written in nondimensional form (Perlekar and Pandit 2010),

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\right)\omega = \frac{1}{\Omega}\nabla^2\omega + F_\omega - \alpha\omega, \qquad \nabla^2\psi = \omega.$$

→ The "vortex array" is simply the array of cells and "fluctuation" is related to turbulence induced variability in the structure. The fluctuating vortex array (FVA) allows us to study a **less constrained** version of the array! **Improved model of cells near marginality.**

→ The fluctuating flow structure is created by **slowly increasing the Reynolds number** in the NS equation

$$\Omega = \frac{\tau_\nu}{\tau_H}$$

→ By increasing the Reynolds number this modifies the forcing and drag term, thus, **scattering** the vortex array. The **resilience** of the staircase is studied by **increasing disorder** in the vortex crystal through F_ω

$$F_\omega \equiv -n^3\left[\cos(nx) + \cos(ny)\right]/\Omega$$

The streamfunction, ψ, at different evolutionary stages of the "fluctuating" vortex array is inserted into the passive scalar equation to study the resilience of the staircase structure.

Comparison of Vortex Array model to Drift-wave Turbulence in fusion devices

	Vortex Field	Drift-Wave Turbulence (tokamak)
Inhomogeneity (free energy source)	$\boldsymbol{\nabla} n$	B_0, $\boldsymbol{\nabla} n$, and $\boldsymbol{\nabla} T$
Reynolds number	$\Omega = 0 - 40$	$Re = 10^1 - 10^2$ (Landau Damping)
Flux	Scalar	Heat
Zonal Flow	Boundary layer between cells	$\mathbf{E} \times \mathbf{B}$ shear flow (poloidal)

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\right) \omega = \frac{1}{\Omega} \nabla^2 \omega + F_\omega - \alpha \omega$$

$$F_\omega \equiv -n^3 \left[\cos(nx) + \cos(ny)\right] / \Omega$$

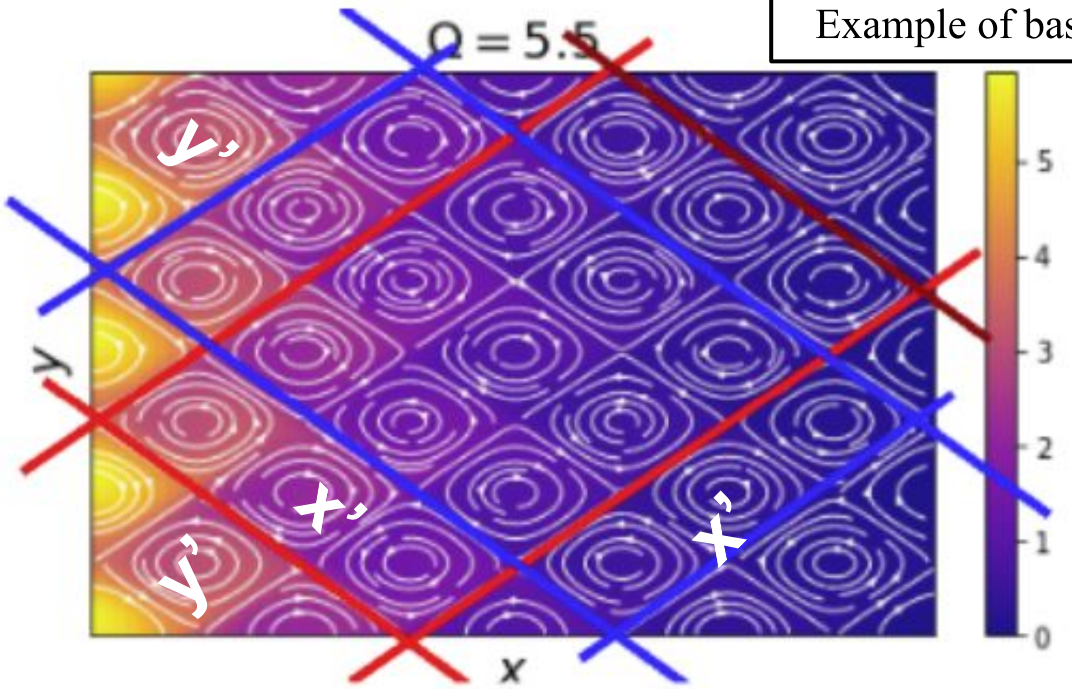
What Happens to Staircase? (Passive Scalar Dynamics)

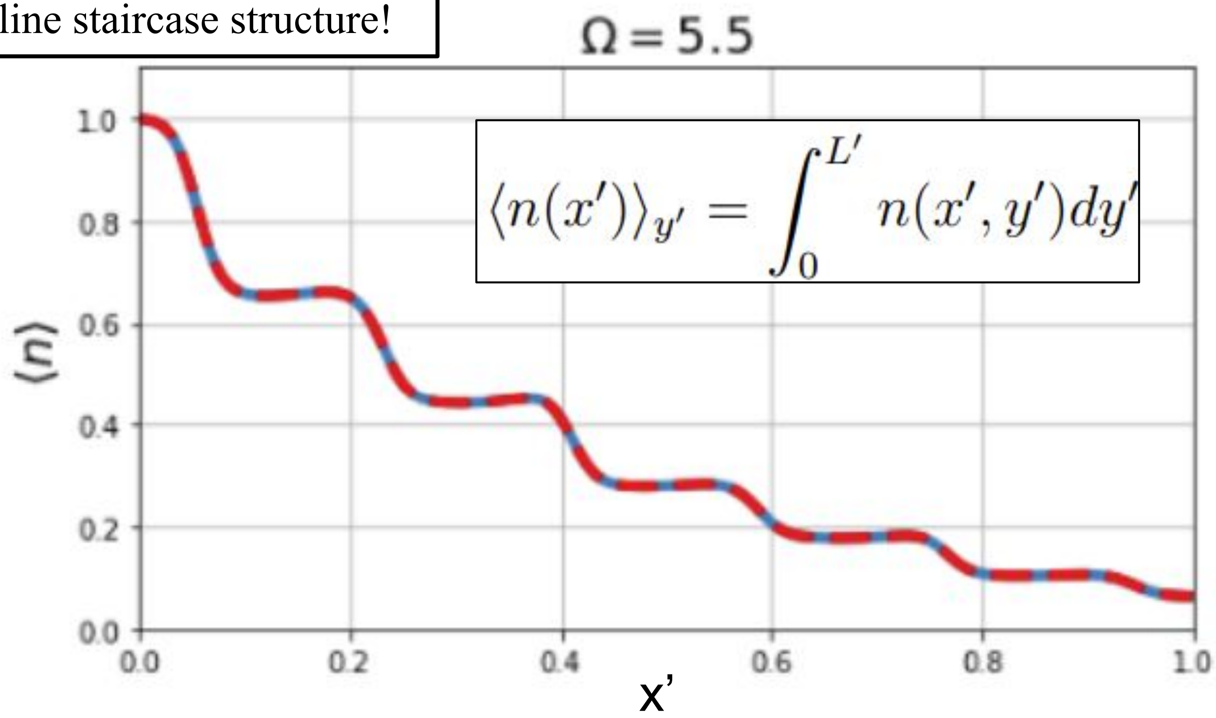
$$\frac{\partial n}{\partial t} + \mathbf{u} \cdot \nabla n = D \nabla^2 n,$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \right) \omega = \frac{1}{\Omega} \nabla^2 \omega + F_\omega - \alpha \omega, \qquad \nabla^2 \psi = \omega.$$

The Staircase

Example of baseline staircase structure!

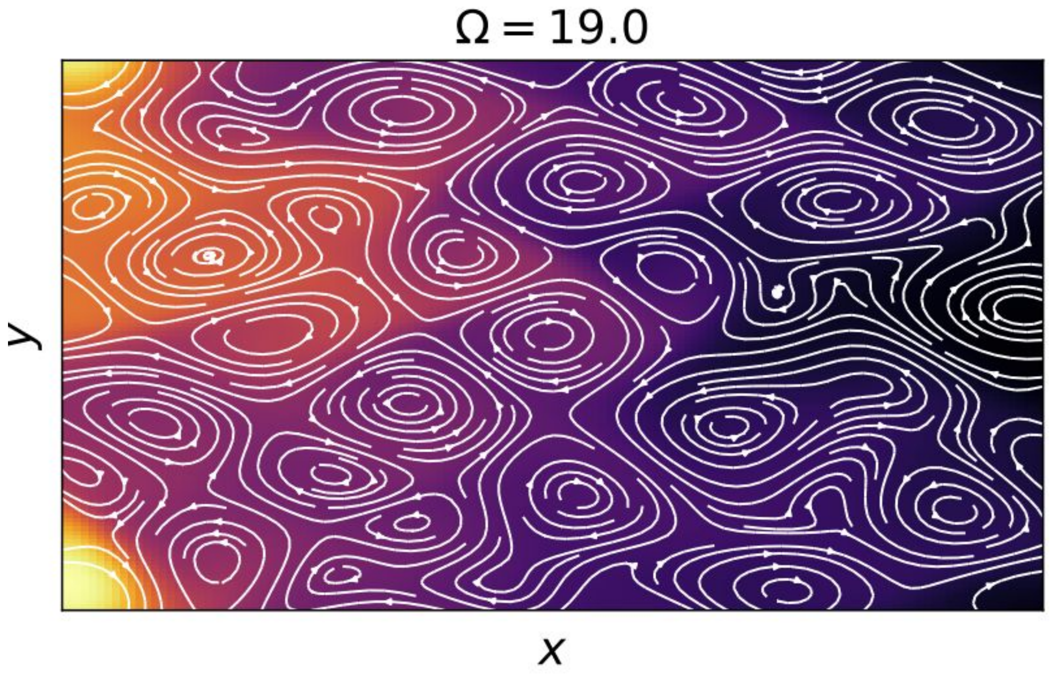


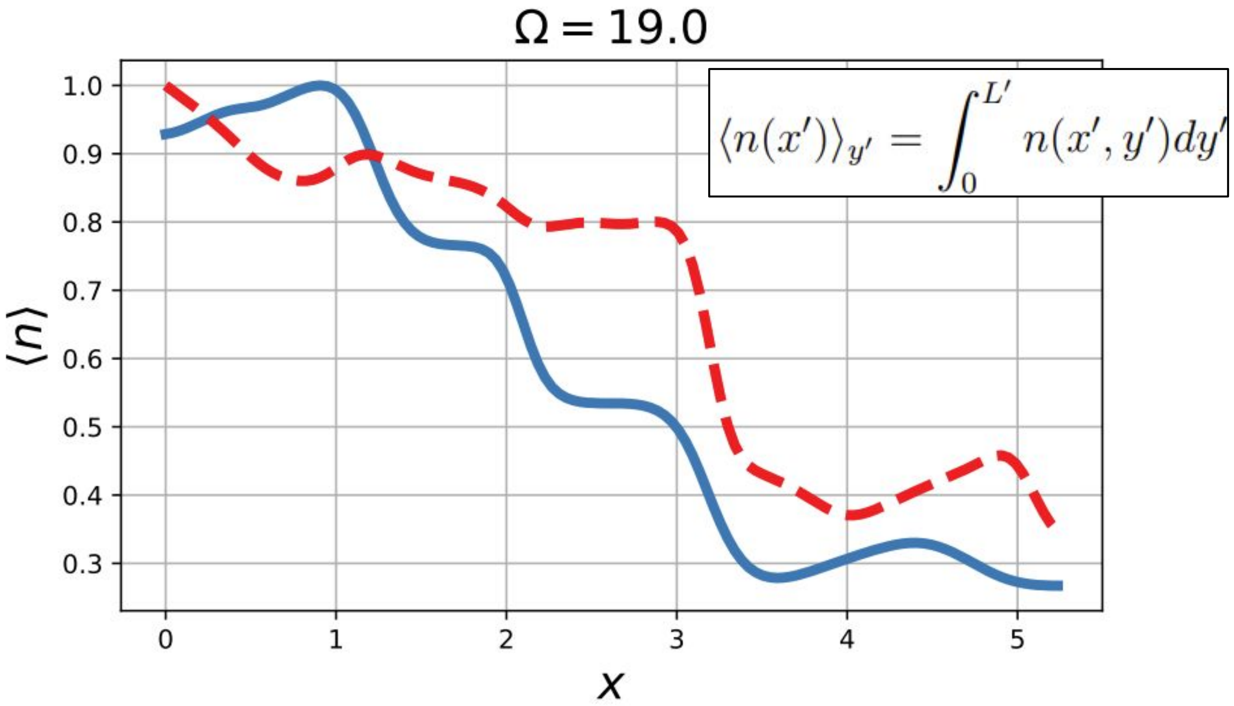


- For a weakly FVA we get a **baseline staircase** structure.
- On the left figure the blue and red box correspond to the blue and red plot line on the right. Note that **steps** are **evenly** spaced!
  - Both blue and red average scalar concentration have the same profile in stable stage.

So what happens to the staircase if we increase the Reynolds number in the VA?

Staircase Resiliency to Fluctuations



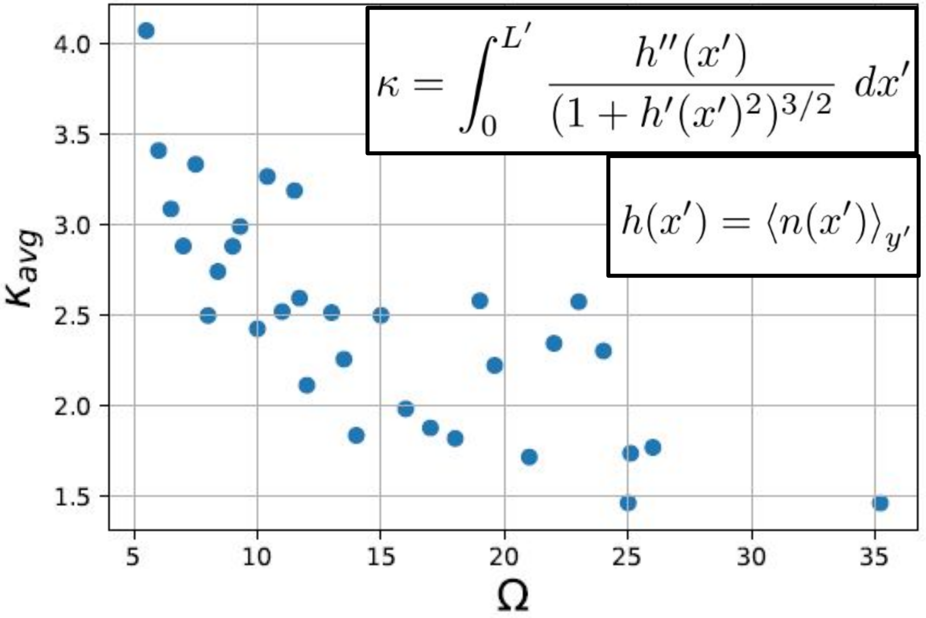


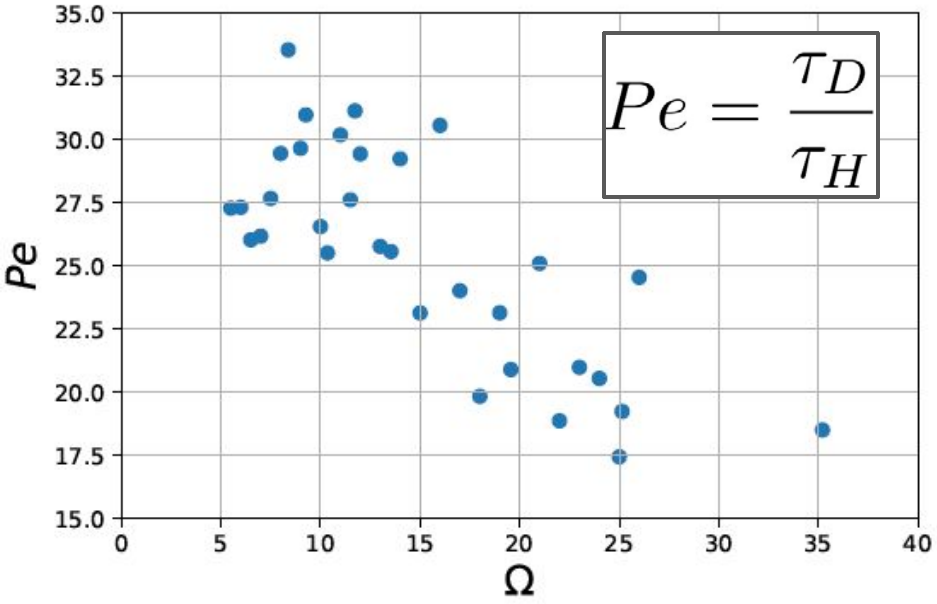
Main Point: Despite that vortex array becoming more turbulent, the staircase structure does not collapse.

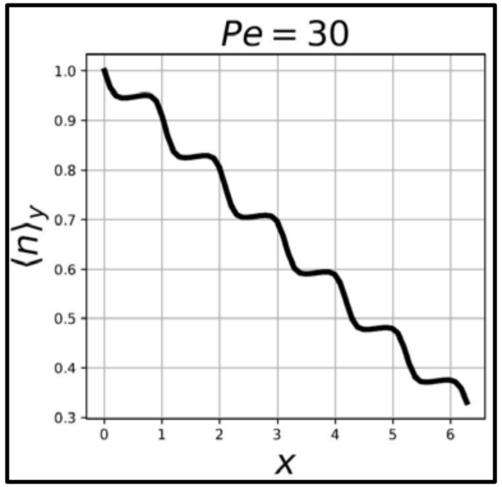
- Staircase steps become **less regular**. They merge into longer steps.

Okay, but how to quantify?

Criteria for Staircase Resiliency





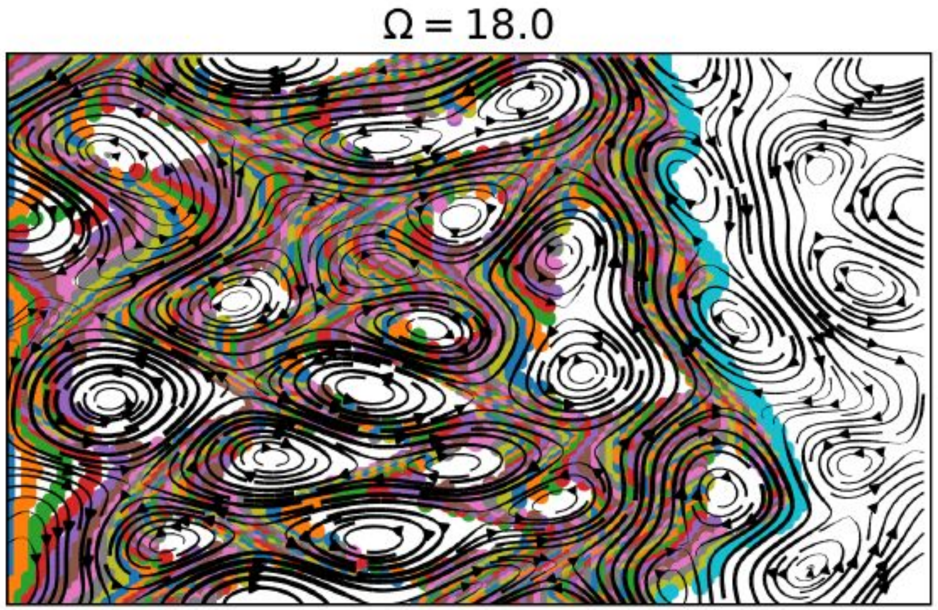


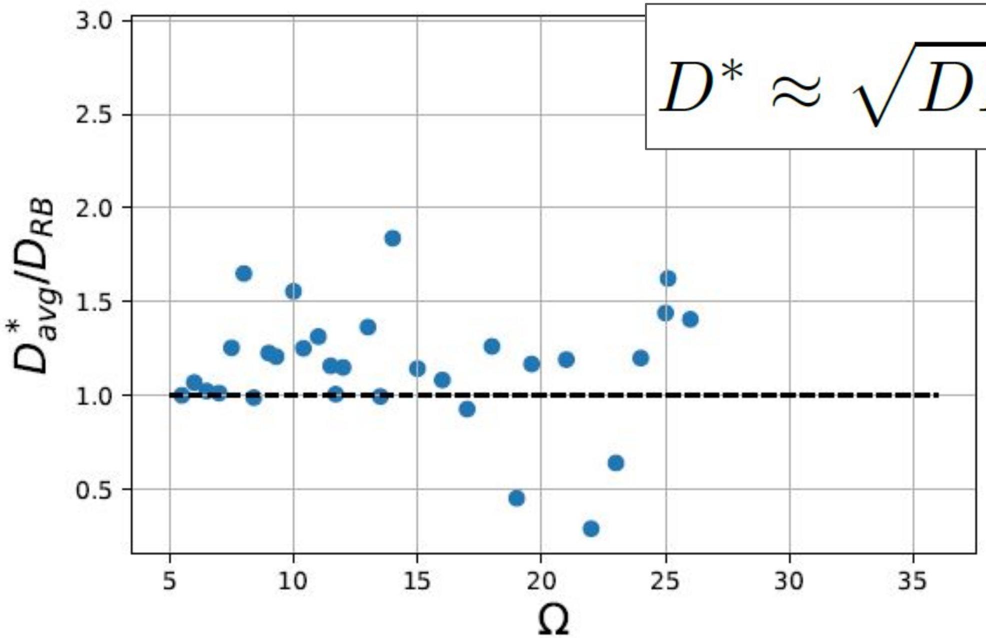
We establish a **set of criteria** to give a precise meaning to the statement of "**resiliency**":

1) $Pe \gg 1$ is a **necessary** condition for the **formation of transport barriers** in the process of scalar mixing (**First principles**). $Pe \gg 1$ criterion is satisfied for the range of $0 < \Omega < 40$.
2) A staircase should **maintain a sufficiently high curvature** (equivalent to sustaining a sufficient number of steps). Our studies suggest that $\kappa \gtrsim 1.5$ is an adequate value for a staircase.

Passive Scalar Transport

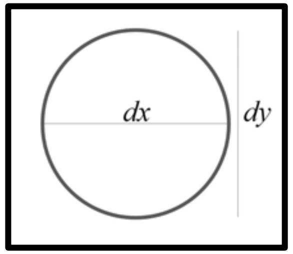
$$Pe = \frac{\tilde{u} d \beta}{D}$$





$$D^* \approx \sqrt{D D_{\text{cell}}} = D\sqrt{Pe}$$

$$D^* \propto \sqrt{d_x \beta}$$



Before the staircase structure forms, scalar flows **quickly in regions of strong shear** and around vortices!

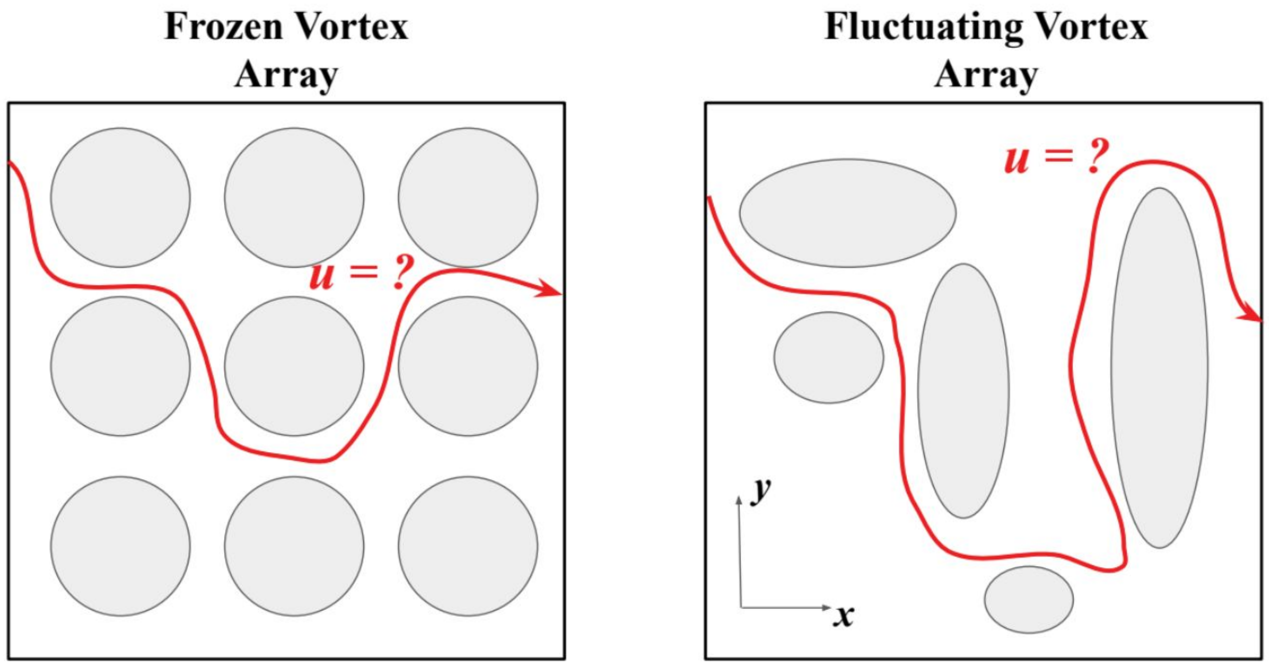
- Staircase **barriers form first!** Scalar travels along cell boundaries.
- Overtime, vortex **entrains** scalar by a kind of "**homogenization**" process via the synergy of differential rotation and diffusion.

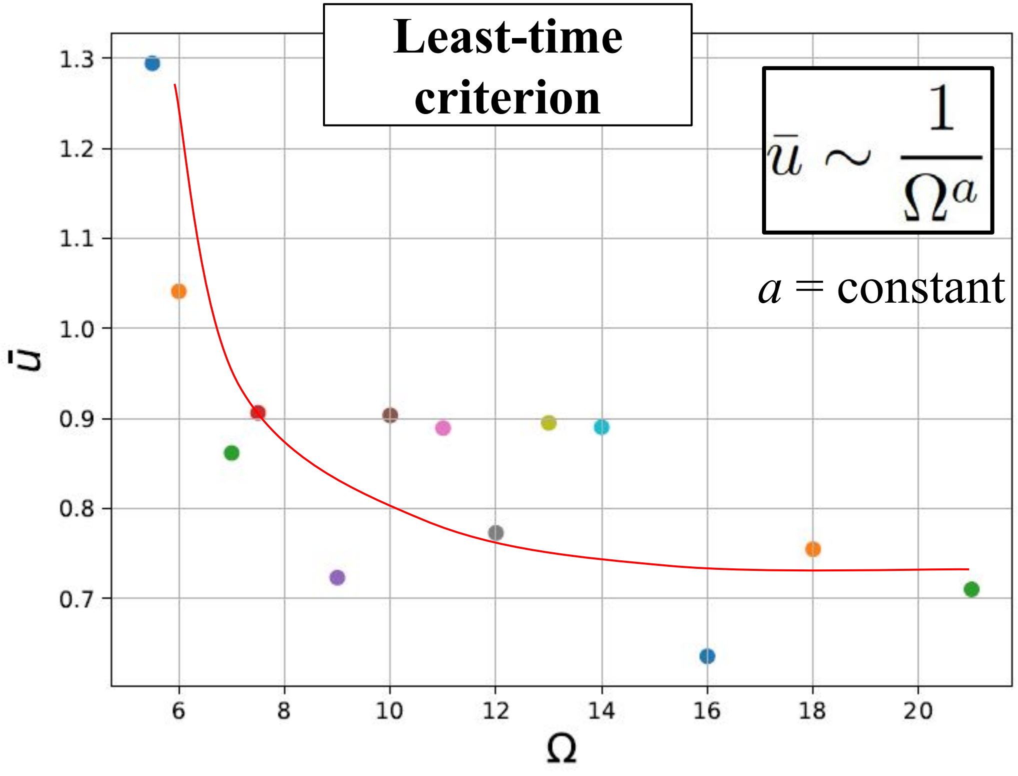
As cells fluctuate, the **effective diffusivity** deviates but **remains close** to the FCA effective diffusivity.

→ We find that as long as the **boundaries** and **speed** of the cells are **maintained**, the effective diffusivity and transport **does not change**.

- Only **dimensions** of cells **affect transport**.

Passive Scalar Transport (cont.d)





The **scattering of vortices** leads to an overall **decrease** in scalar concentration **velocity**! Agrees with least time criterion (similar idea to scattered path of light in atmosphere).

Summary

- Staircase form and are **resilient** and **persistent** to increasing Reynolds number (i.e., fluctuating vortex array).

- Scalar concentration **travels along** regions of **strong shear**.
  - **IMPORTANT**: Staircase barriers form first! Vortex “homogenizes” scalar at a later time!

- The scattering of vortices leads to an overall decrease in scalar concentration velocity.
  - Agrees with **least time criterion**.

- If flow velocity and background diffusion are kept fixed, only **cell geometric properties** affect the effective diffusivity! ($D^* \propto D\, Pe^{1/2}$)
  - Effective diffusivity of the perturbed vortex array **does not deviate** significantly!

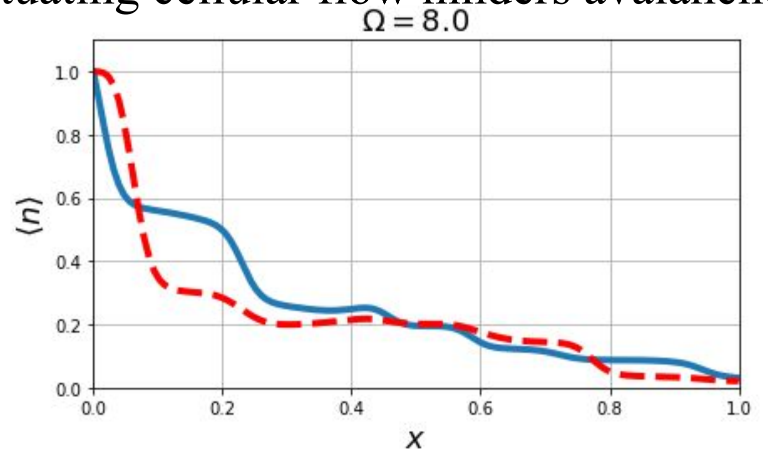
Why would a fusion experimentalist care about this?

These results have interesting implications for experiment and theory:

1. Effective diffusivity derived by Rosenbluth *et al* (for fixed cellular array) is a suitable approximation for the fluctuating cellular array (**not simple addition**: $D^* = D + D_{cell}$).
   - Relevant to cells touching (similar to what we find near-marginal stability).

2. Staircase structure is resilient in the regime of low-modest Reynolds numbers (this regime is relevant to drift-wave turbulence).
   - Structures/Profiles are not exotic.
     - Staircase profile structure does not require special tuning.

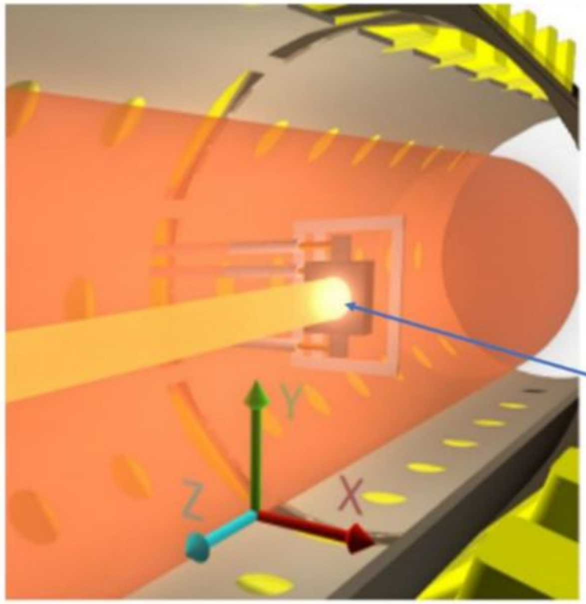
3. Geometry of streamlines is important. If more saddles than close vortices, Heat avalanches will first form the staircase barrier.
   - Fluctuating cellular flow hinders avalanche propagation.

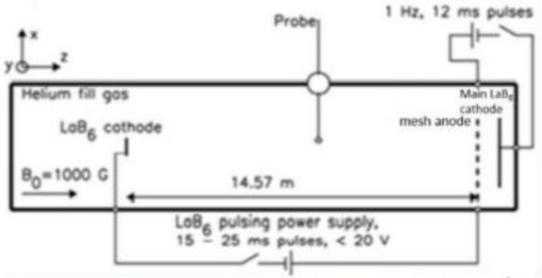


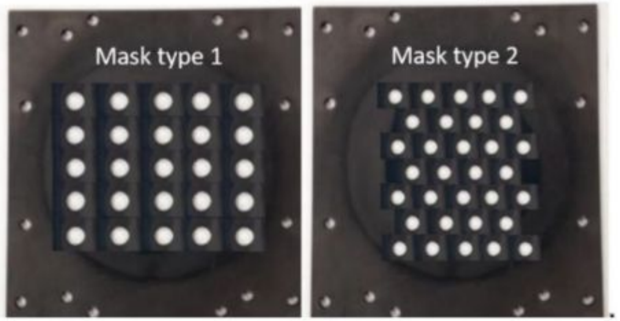
IMPORTANT: We can test the theory presented here with actual experimental data.

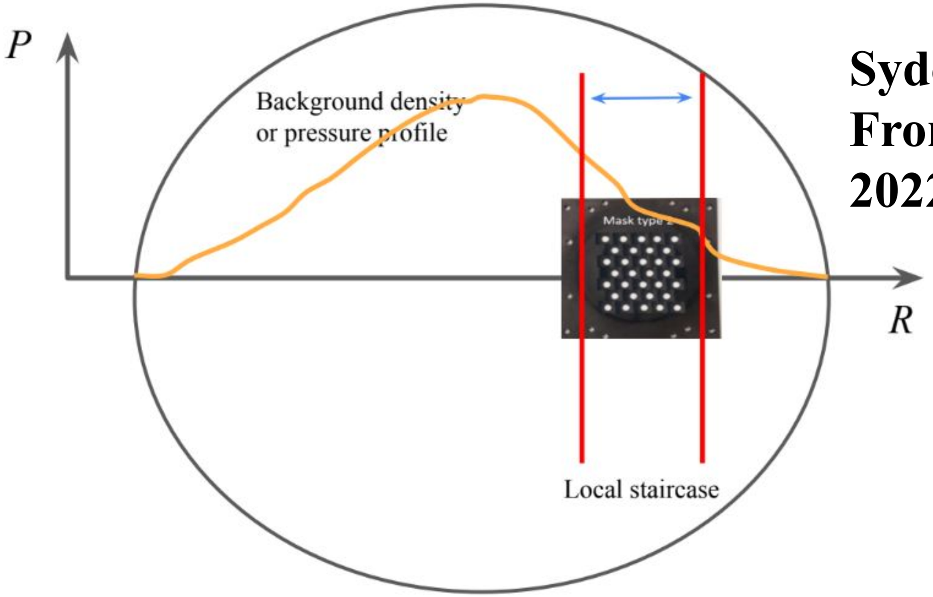
LAPD Experiment

Work in progress!









Sydora,
Frontiers Proposal
2022

A vortex array can be created in the large linear magnetized plasma device (LAPD)

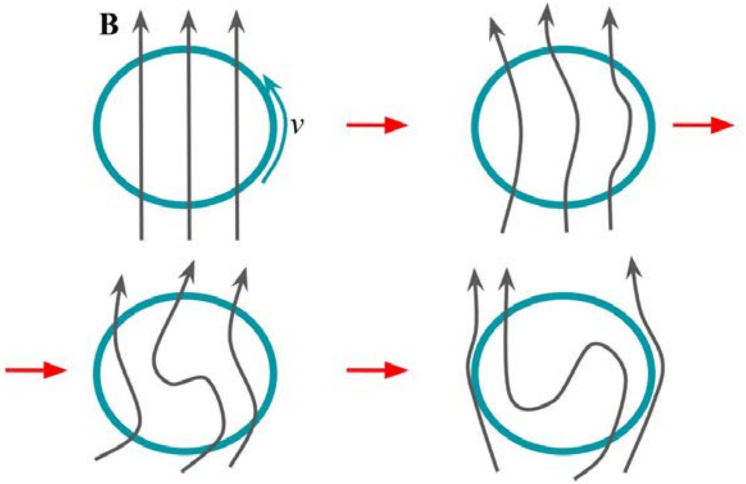
- Modification of a cathode plasma source with designer masks that form multiple current channels in a cellular pattern → form staircase!
  - Experiment will be conducted in the afterglow phase of the main discharge.
- Staircase structure can be subject to controllable amount of of low frequency density fluctuations, which act as a noise source.
  - Allow us to test hypotheses and models of staircase resiliency!

Results of experiment will yield a unique set of observations that can be used to test staircase models.

Active Scalar Dynamics

Active Scalar

$$M = \frac{v_A}{U_0}$$



A logical next step to explore is the effects than an *active* scalar has on the cellular array and inhomogenous mixing.

- Converting passive to active will result in effects such as flux expulsion
  - Flux expulsion is simplest dynamic problem in non-ideal MHD.

Why this model?

- ***B*** expelled to boundaries, thus holds cells together! → Rigid staircase.

We turn passive scalar into an active scalar, creating a feedback between magnetic field and vortices:

Flux expulsion:

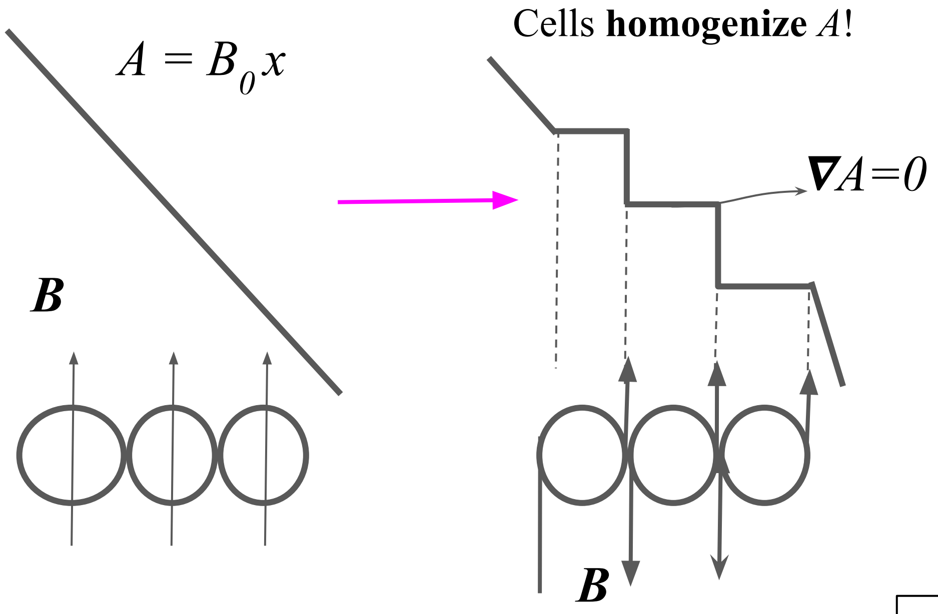
- Background ***B*** is wind up and folded by an eddy → field inside eddy drops → expelled to boundary layer of eddy.
- Time scale for flux expulsion is, $\tau_{fe} = R_m^{1/3}\tau_H$
- **Note:** Larger R_m results in greater expulsion (weaker field in interior).

$$\frac{\partial n}{\partial t} + \mathbf{u}\cdot\nabla n - D\nabla^2 n = 0 \longrightarrow \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)A = \frac{1}{R_m}\nabla^2 A + F_A$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\omega = \frac{1}{\Omega}\nabla^2\omega + M^2\boxed{\left(\mathbf{B}\cdot\nabla\nabla^2 A\right)} + F_\omega$$

Note: Strength of B_o plays an important role!

Kinematic/Dynamic Regime



<u>Important:</u> Flux expulsion only occurs in the **kinematic** regime

- Useful to explore **dynamic** regime (aka Vortex bursting).

Since $v_A \propto B_0$, the strength of the magnetic field will play a role in the dynamics of the cellular array.

- If B_0 is sufficiently small, we get cell strengthening.
- If B_0 is large, vortices will not be allowed to form.

Through scans of B_0, we will address what **occurs** to expulsion of **neighbor cells** and their **interaction**…

$M^2 R_m < 1$ (Flux expulsion)

$M^2 R_m \geq 1$ (Vortex bursting)

$$M = \frac{v_A}{U_0}$$

To be clear, staircase forms in the flux expulsion regime.

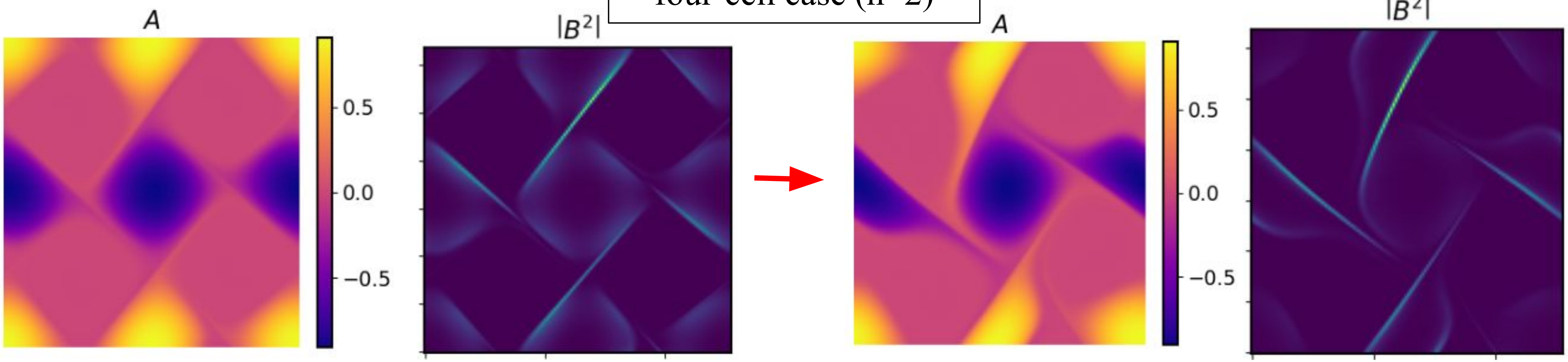
- Unclear if staircase forms in vortex bursting regime (TBD).

Consider a **linear** magnetic potential profile:

- We expect that the vortex array will homogenize ($\nabla A=0$) the profile in areas of vortices.
- Expect that magnetic field will maintain or restore the cell array structure when fluctuations are present (i.e., B_0 will elasticise the cell array).

Formation & Destruction of Barriers ($M^2 R_m = 1$)

four-cell case (n=2)



This problem is **important** and can be related back to the idea of **feedback**!

- We have only address the idea that staircases are resilient and robust in the presence of cell fluctuations.
- But could the scalar affect the dynamics or maintain the cell structure which is responsible for the staircase? Preliminary results show that **magnetic field restores cell structure**!
  - Only a small window where this occurs (i.e., small ***Bo***)...

NOTE: ***B*** eventually **decays** in 2D, so the **structure** is only **temporary**… (need to force magnetic field)

$$F_\omega \equiv -n^3 \left[\cos\left(nx\right) + \cos\left(ny\right)\right] / \Omega$$

Work in progress…