Elastic Turbulence in Flatland: A Tale of Blobs, Barriers, and Inhomogeneous Mixing

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Outline

• Elastic Fluids:
  
  Elasticity ↔ Memory ↔ Transport

• Active Scalar Transport in 2D MHD:
  
  Conventional Wisdom

• New Development: Blobs and Barriers

• Revisiting Quenching

• Inhomogeneous Mixing and Staircases

• Open Questions
Elastic Fluids

- Internal DOF exerting restoring force on fluid $\rightarrow$ “springiness”

- Examples:
  - MHD $\rightarrow \vec{B}, \vec{j} \times \vec{B}$
    - Polymer hydro $\rightarrow$ Elastic element oldroyd-B
  - Spinodal Decomposition (CHNS) $\rightarrow$ droplet surface tension

- Elasticity $\rightarrow$ Memory $\rightarrow$ Impact on mixing?!
Active Scalar Transport in 2D MHD: Background and Conventional Wisdom
Physics: Active Scalar Transport

- Magnetic diffusion, $A$ transport are cases of active scalar transport
- (Focus: 2D MHD) (Cattaneo, Vainshtein ‘92, Gruzinov, P. D. ’94, ’95)

\[ \partial_t A + \nabla \phi \times \hat{z} \cdot \nabla A = \eta \nabla^2 A \]
\[ \partial_t \nabla^2 \phi + \nabla \phi \times \hat{z} \cdot \nabla \nabla^2 \phi = \nabla A \times \hat{z} \cdot \nabla \nabla^2 A + \nu \nabla^2 \nabla^2 \phi + \tilde{f} \]

- Seek $\langle \nu_x A \rangle = -D_T \frac{\partial \langle A \rangle}{\partial x} - \eta \frac{\partial \langle A \rangle}{\partial x}$
- Point: $D_T \neq \sum_k |v_k^\perp|^2 \tau^K_k$, often substantially less than kinematic
- Why: Memory! $\leftrightarrow$ Freezing-in
- Cross Phase
Two Stage Evolution:

• 1. The **suppression stage**: the (large scale) magnetic field is sufficiently strong so that the diffusion is suppressed.

• 2. The **kinematic decay stage**: the magnetic field is dissipated so the diffusion rate returns to the kinematic rate.

• Suppression is due to the memory induced by the magnetic field.
Conventional Wisdom

- [Cattaneo and Vainshtein 1991]: turbulent transport is suppressed, even for a weak large scale magnetic field is present.

- Starting point: \( \partial_t \langle A^2 \rangle = -2\eta \langle B^2 \rangle \)

- Assumptions:
  - Energy equipartition: \( \frac{1}{\mu_0 \rho} \langle B^2 \rangle \sim \langle v^2 \rangle \)
  - Average B can be estimated by: \( |\langle B \rangle| \sim \sqrt{\langle A^2 \rangle}/L_0 \)

- Define Mach number as: \( M^2 = \langle v_A^2 \rangle / \langle \tilde{v}^2 \rangle = \langle v^2 \rangle / \nu_A^2 = \langle v^2 \rangle / \frac{1}{\mu_0 \rho} \langle B^2 \rangle \)

- Result for suppression stage: \( \eta_T \sim \eta M^2 \)

- Fit together with kinematic stage result: \( \eta_T \sim \frac{ul}{1 + \text{Rm}/M^2} \)

- Physics interpretation of \( \eta_T \)?
Origin of Memory?

• (a) flux advection vs flux coalescence
  • intrinsic to 2D MHD (and CHNS)
  • rooted in inverse cascade of $\langle A^2 \rangle$ - dual cascades

• (b) tendency of (even weak) mean magnetic field to “Alfvenize” turbulence [cf: vortex disruption feedback threshold!]

• Re (a): Basic physics of 2D MHD

Forward transfer: fluid eddies chop up scalar A.
Memory Cont’d

• V.S.

Obvious analogy: straining vs coalescence

Upshot: closure calculation yields:

\[ \Gamma_A = - \sum_{k'} \left[ \tau_c \langle \nu^2 \rangle_{k'} - \tau_c A \langle B^2 \rangle_{k'} \right] \frac{\partial \langle A \rangle}{\partial x} + \cdots \]

N.B.:
- Coalescence
  - Negative diffusion
  - Bifurcation

Inverse transfer: current filaments and A-blobs attract and coagulate.

flux of potential
competition

scalar advection vs. coalescence ("negative resistivity")
(+)
(-)
Conventional Wisdom, Cont’d

• Then calculate \( \langle B^2 \rangle \) in terms of \( \langle v^2 \rangle \) (after Zeldovich)

\[
\partial_t A + \mathbf{v} \cdot \nabla A = -v_x \frac{\partial \langle A \rangle}{\partial x} + \eta \nabla^2 A
\]

• Multiplying by \( A \) and sum over modes:

\[
\frac{1}{2} [\partial_t \langle A^2 \rangle + \langle \nabla \cdot (\mathbf{v} A^2) \rangle] = -\Gamma_A \frac{\partial \langle A \rangle}{\partial x} - \eta \langle B^2 \rangle
\]

\[
\frac{\partial \langle A \rangle}{\partial x} \rightarrow B_0
\]

• Therefore:

\[
\langle B^2 \rangle = -\frac{\Gamma_A}{\eta} \frac{\partial \langle A \rangle}{\partial x} = \frac{\eta}{\eta} B_0^2
\]

• Define Mach number as:

\[
M^2 \equiv \frac{\langle v^2 \rangle}{v_{A0}^2} = \frac{\langle v^2 \rangle}{(\frac{1}{\mu_0 \rho} B_0^2)}
\]

• Result:

\[
\eta_T = \sum_k \tau_c \langle v^2 \rangle_k = \frac{ul}{1 + Rm/M^2}
\]

• This theory is not able to describe \( B_0 \rightarrow 0 \) limited!

Dropped stationary case
Dropped periodic boundary \( \rightarrow \) introduce nonlocality?!

\[\text{Dropped stationary case} \quad \text{Dropped periodic boundary} \rightarrow \text{introduce nonlocality?}!\]
New Wrinkles
New Observations

• With no imposed $B_0$, in suppression stage:

Field Concentrated!

• v.s. same run, in kinematic stage (trivial):
• Nontrivial structure formed in real space during the suppression stage.
• $A$ field is evidently composed of “blobs”.
• The low $A^2$ regions $\leftrightarrow$ 1-dimensional.
• The high $B^2$ regions are strongly correlated with low $A^2$ regions, and also $\leftrightarrow$ 1-dimensional.
• 1-dimensional high $B^2$ regions “barriers”. There, mixing is sharply reduced, relative to $\eta_K$.

→ Story one of ‘blobs and barriers’
Evolution of PDF of A

- Probability Density Function (PDF) in two stage:
  - Time evolution: horizontal “Y”.
  - The PDF changes from double peak to single peak as the system evolves from the suppression stage to the kinematic stage.
Unimodal Initial Condition

- One may question whether the bimodal PDF feature is purely due to the initial condition. The answer is **No**.
- Two non-zero peaks in PDF of A still arise, even if the initial condition is unimodal.
The problem of the mean field $\langle B \rangle$

→ What does “Mean” mean?

• $\langle B \rangle$ depends on the averaging window.

• With no imposed external field, $B$ is highly intermittent, therefore $\langle B \rangle$ is not well defined.

$|\langle B \rangle| \sim \sqrt{\langle A^2 \rangle}/L_0 \checkmark$ vs.

Reality

$\langle B \rangle$ not well defined
Revisiting Quenching
New Understanding

• Summary of important length scales: \( l < L_{stir} < L_{env} < L_0 \)
  • System size \( L_0 \)
  • Envelope size \( L_{env} \rightarrow \) emergent (blob)
  • Stirring length scale \( L_{stir} \)
  • Turbulence length scale \( l \), here we use Taylor microscale \( \lambda \)
  • Barrier width \( W \rightarrow \) emergent

• Quench is not uniform. Transport coefficients differ in different regions.

• In the regions where magnetic fields are strong, \( Rm/M^2 \) is dominant. They are regions of barriers.
• In other regions, i.e. Inside blobs, \( Rm/M'^2 \) is what remains. \( M'^2 \equiv \langle V^2 \rangle / \left( \frac{1}{\rho} \langle A^2 \rangle / L_{env}^2 \right) \)
New Understanding, cont’d

• From \[ \partial_t \langle A^2 \rangle = -\langle vA \rangle \cdot \nabla \langle A \rangle - \nabla \cdot \langle v A^2 \rangle - \eta \langle B^2 \rangle \]

• Retain 2nd term on RHS. Average taken over an envelope/blob scale.

• Define diffusion (closure):

\[ \langle vA \rangle = -\eta_{T1} \nabla \langle A \rangle \]
\[ \langle vA^2 \rangle = -\eta_{T2} \nabla \langle A^2 \rangle \]

• Plugging in:

\[ \partial_t \langle A^2 \rangle = \eta_{T1} (\nabla \langle A \rangle)^2 + \nabla \eta_{T2} \cdot \nabla \langle A^2 \rangle - \eta \langle B^2 \rangle \]

• For simplicity:

\[ \langle B^2 \rangle \sim \frac{\eta T}{\eta} (\langle B \rangle^2 + \langle A^2 \rangle / L^2_{env}) \]

• \( L_{env} \) is the envelope size. Scale of variation of \( \langle A^2 \rangle \).

• Define new strength parameter:

\[ M^{r2} \equiv \langle v^2 \rangle / (\frac{1}{\mu_0 \rho} \langle A^2 \rangle / L^2_{env}) \]

• Result:

\[ \eta_T = \frac{ul}{1 + Rm/M^2 + Rm/M^{r2}} = \frac{ul}{1 + Rm \frac{1}{\mu_0 \rho} \langle B \rangle^2 / \langle v^2 \rangle + Rm \frac{1}{\mu_0 \rho} \langle A^2 \rangle / L^2_{env} \langle v^2 \rangle} \]
\[ \eta_T = V l / \left[ 1 + \frac{R_m}{M^2} + \frac{R_m}{M' r^2} \right] \]

• **Barriers:**

\[ \eta_T \approx V l / \left[ 1 + R_m \frac{\langle B \rangle^2}{\rho \langle \tilde{V}^2 \rangle} \right] \]

Strong field

• **Blobs:**

\[ \eta_T \approx V l / \left[ 1 + R_m \frac{\langle A^2 \rangle}{\rho L_{env}^2 \langle \tilde{V}^2 \rangle} \right] \]

Weak effective field

• **Quench stronger in barriers, highly non-uniform**
Formation of Barriers

• How do the barriers form?

\[ \eta_T = \sum_k \tau_c [\langle v^2 \rangle_k - \frac{1}{\mu_0 \rho} \langle B^2 \rangle_k] \]

• From above, strong B regions can support negative incremental

\[ \eta_T = \frac{\delta \Gamma_A}{\delta (-\nabla A)} < 0, \text{ suggesting clustering} \]

• \( \langle \eta_T \rangle > 0 \)

• Positive feedback: a twist on a familiar theme

B is strong in a specific region \( \rightarrow \) diffusion of A is negative

B in that region increases \( \rightarrow \) \( \nabla A \) increases
Formation of Barriers, Cont’d

- Negative resistivity leads to barrier formation.
- The S-curve due to the nonlinear dependence of $\Gamma_A$ on $B$.
- When slope negative $\Rightarrow$ negative (incremental) resistivity.

Bistability of $\Gamma_A$ vs $\nabla A$

→ a familiar theme

Barriers $\leftrightarrow$

Landscape unknown

Quenched $\eta_T$
Describing the Barriers

• How to measure the barrier width $W$.

• Starting point: $W \sim \Delta A / B_b$

• Use $\sqrt{\langle A^2 \rangle}$ to calculate $\Delta A$

• Define the barrier regions as:

• Define barrier packing fraction: $P = \frac{\# \text{ of grid points for barrier regions}}{\# \text{ of total grid points}}$

• Use the magnetic fields in the barrier regions to calculate the magnetic energy:

\[ \sum_{\text{barriers}} B_b^2 \sim \sum_{\text{system}} B^2 \]

• Thus $\langle B_b^2 \rangle \sim \langle B^2 \rangle / P$

• So barrier width can be estimated by:

\[ W^2 \equiv \frac{\langle A^2 \rangle}{\langle \langle B^2 \rangle / P \rangle} \]

N.B. All magnetic energy in the barriers
Describing the Barriers

- Time evolution of $P$ and $W$:
  - $P$, $W$ collapse in decay
  - $M'$ rises
- Sensitivity of $W$:
  - $A_0$ or $1/\mu_0\rho$ greater $\rightarrow W$ greater;
  - $f_0$ greater, $W$ smaller; (ala’ Hinze)
  - $W$ not sensitive to $\eta$ or $\nu$. 

(a) (b) (c) (d) (e)
Active Scalar Staircases
Staircase (inhomogeneous Mixing, Bistability)

• Staircases emerge spontaneously! – **Barrier lattices**
• Initial condition is the usual cos function (bimodal)
• The only major sensitive parameter (from runs above) is the forcing scale $k=32$ (for all runs above $k=5$).
• Resembles the PV staircase

![Graphs and images](1) (2) (3) (4)
Conclusions / Summary

• Magnetic fields suppress turbulent diffusion in 2D MHD by: formation of intermittent transport barriers.

• Magnetic structures:
  - Barriers – thin, 1D strong field regions
  - Blobs – 2D, weak field regions

• Quench not uniform:

\[ \eta_T = \frac{ul}{1 + Rm \frac{1}{\mu_0 \rho} \langle B \rangle^2 / \langle v^2 \rangle + Rm \frac{1}{\mu_0 \rho} \langle A^2 \rangle / L_{env}^2 \langle v^2 \rangle} \]

  - barriers, strong B
  - blobs, weak B, $\nabla^2 \langle A^2 \rangle$ remains

• Barriers form due to negative resistivity:

\[ \eta_T = \sum_k \tau_c [\langle v^2 \rangle_k - \frac{1}{\mu_0 \rho} \langle B^2 \rangle_k] \]

  - flux coalescence

• Formation of “magnetic staircases” observed for some stirring scale
General Conclusions (MHD and CHNS)

• Dual (or multiple) cascades can interact with each other, and can modify one another. N.B.: Focus on $\langle A^2 \rangle$ transfer.

• We show how a length scale, e.g. the Hinze scale in 2D CHNS, emerges from the balance of kinetic energy and elastic energy in blobby turbulence. $\rightarrow$ blob scale in MHD?!

• Negative incremental resistivity can exist in a simple system such as 2D MHD. This results in the formation of nontrivial real space structure.
Future Works

• Extension of the transport study in MHD:
  • Numerical tests of the new $\eta_T$ expression?
  • What determines the barrier width and packing fraction?
  • Why does layering appear when the forcing scale is small?
  • What determines the step width, in the case of layering?

• Other similar systems can also be studied in this spirit. e.g. Oldroyd-B model for polymer solutions. (drag reduction)

• Reduced Model of Magnetic Staircase

See Also:

Reading

Fan, P.D., Chacon:

• PRE Rap Comm 99, 041201 (2019)
• PoP 25, 055702 (2018)
• PRE Rap Comm 96, 041101 (2017)
• Phys Rev Fluids 1, 054403 (2016)

Thank you!
Back-Up
2D CHNS and 2D MHD

• 2D CHNS Equations:

\[
\frac{\partial t}{\partial t} \psi + \vec{v} \cdot \nabla \psi = D \nabla^2 (-\psi + \psi^3 - \xi^2 \nabla^2 \psi)
\]

\[
\frac{\partial t}{\partial t} \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega
\]

With \( \vec{v} = \hat{z} \times \nabla \phi \), \( \omega = \nabla^2 \phi \), \( \vec{B}_\psi = \hat{z} \times \nabla \psi \), \( j_\psi = \xi^2 \nabla^2 \psi \). \( \psi \in [-1, 1] \).

• 2D MHD Equations:

\[
\frac{\partial t}{\partial t} A + \vec{v} \cdot \nabla A = \eta \nabla^2 A
\]

\[
\frac{\partial t}{\partial t} \omega + \vec{v} \cdot \nabla \omega = \frac{1}{\mu_0 \rho} \vec{B} \cdot \nabla \nabla^2 A + \nu \nabla^2 \omega
\]

With \( \vec{v} = \hat{z} \times \nabla \phi \), \( \omega = \nabla^2 \phi \), \( \vec{B} = \hat{z} \times \nabla A \), \( j = \frac{1}{\mu_0} \nabla^2 A \)

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See [Fan et al. 2016] for more about CHNS.
2D CHNS and 2D MHD

- The $A$ field in 2D MHD in suppression stage is strikingly similar to the $\psi$ field in 2D CHNS (Cahn-Hilliard Navier-Stokes) system:

 psi field in 2D CHNS

v.s.

A field in 2D MHD
Constitutive Relations → Deborah Number

➢ J. C. Maxwell:

\[
(stress) + \tau_R \frac{d(stress)}{dt} = \eta \frac{d}{dt}(strain)
\]

➢ If \( \tau_R/\tau \ll 1 \), stress = \( \eta \frac{d}{dt}(strain) \)

\[
\Pi = -\eta \nabla \hat{v} \quad \text{viscous}
\]

➢ If \( \tau_R/\tau \gg 1 \), stress \( \cong \frac{\eta}{\tau_R} \)(strain)

\[
\sim \mathcal{E}(strain) \quad \text{elastic}
\]

➢ Limit of “freezing-in”: \( D \gg 1 \) is criterion.
• $D \sim \text{Deborah Number} \sim |\nabla V|/\omega_Z \sim \tau_{relax}/\tau_{dyn}$

• Limit for elasticity: $D \gg 1 \rightarrow$ limit for elasticity

• Why “Deborah”? →

  Hebrew Prophetess Deborah:

  “The mountains flowed before the Lord.” (Judges)

∴

• Revisit Heraclitus (1500 years later):

  “All things flow” – if you can wait long enough
Simulation Setup

- PIXIE2D: a DNS code solving 2D MHD equations in real space:

\[ \partial_t A + \mathbf{v} \cdot \nabla A = \eta \nabla^2 A \]
\[ \partial_t \omega + \mathbf{v} \cdot \nabla \omega = \frac{1}{\mu_0 \rho} \mathbf{B} \cdot \nabla^2 A + \nu \nabla^2 \omega + f \]

- $1024^2$ resolution.
- External forcing $f$ is isotropic homogeneous.
- Periodic boundary conditions (both).
- Initial conditions:
  - (1) bimodal:
    \[ A_I(x, y) = A_0 \cos 2\pi x \]
  - (2) unimodal:
    \[ A_I(x, y) = A_0 \cdot \begin{cases} 
    -(x - 0.25)^3 & 0 \leq x < 1/2 \\
    (x - 0.75)^3 & 1/2 \leq x < 1 
    \end{cases} \]