Ergodic, Mixing, Chaotic ... and all that

i.) Ergodicity

The ergodic theorem states that for typical Hamiltonian systems with finite dynamics, the time average *is equal to the* ensemble average.

→ Here time average means:

$$\left\langle F\left(\underline{q},\, \underline{p}\right)\right\rangle_t = \lim_{T \to \infty} \frac{1}{T} \int_0^T F\left(\underline{q}(t), \underline{p}(t)\right) dt$$

and is straightforward. $\underline{q}$, $\underline{p}$ are coordinate and momentum.

→ For ensemble average,

$$\left\langle F\left(\underline{q},\, \underline{p}\right)\right\rangle = \lim \frac{1}{N} \sum_{i=1}^{N} F\left(\underline{q}(t, \Gamma_0), \underline{p}(t, \Gamma_0)\right).$$

Here Γ_0 is the initial $\underline{q}_0$, $\underline{p}_0$ of the trajectory, which serves as an index for that trajectory.

→ So-in PLAIN ENGLISH: A time average is computed for a given trajectory, as time varies, while an ensemble average is computed over a set of N trajectories, labeled by their initial conditions. The ERGODIC THEOREM states these are equal, for *long* time and *large* N. Ergodicity is important to the foundations of statistical mechanics.

ii.) Mixing

Mixing means that correlation decays in time.

→ For correlation, let F be a square integrable function, and the dynamics ergodic. Then self-correlation

$$R(f,t) = \left\langle f\left(\Gamma(t+\tau, \Gamma_0\right) f\left(\Gamma(t, \Gamma_0)\right) - \left\langle f(\Gamma)\right\rangle^2 \right\rangle.$$

Here $\Gamma = \underline{p}$, $\underline{q}$ and Γ_0 as above.

→ Mixing, means R decays, i.e.

$$\lim_{t \to \infty} R(t,f) = 0 .$$

So-in PLAIN ENGLISH, Mixing means that the *self correlation* of a well behaved) function of the phase space coordinates *decays in time*. Note that the *rate* of decay is not specified! In practice, mixing may be thought of by the cartoon:

N.B. There at least two types of mixing, strong (see above) and weak.

Weak mixing means:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^t \left| R^2 \right| dt = 0 .$$

Namely the time average of the *squared* correlation decays. Weak mixing admits the possibility of long-lived, weak fluctuations.

iii.) Chaos

Chaos means the phase space orbits exhibit local instability*, namely that phase space trajectories exiting from nearby points* diverge exponentially. *This is equivalent to the existence of a positive* Lyapunov exponent.

→ At the cartoon level, chaos is suggested by:

where the rate of divergence is *exponential.*

→ Trajectories in an ergodic flow diverge algebraically, in contrast to the exponential divergence of chaos. Thus, 'chaos' is stronger than 'ergodicity'. A chaotic flow is definitely mixing.

→ Chaotic flows undergo *stretching* and (in a finite domain) *folding*.

→ Loosely put, 'chaos' implies:

$$d(t) \sim d(0)\exp(\alpha t)$$

where $d(t)$ is the distance between the trajectories and $d(0)$ is infinitesimal. Furthermore, the above must be satisfied for $d(0) \to 0$, $t \to \infty$ for any initial pair.

$$\alpha = \lim_{d(0)\to 0} \lim_{t\to\infty} \frac{1}{t} \ell n \left| \frac{d(t)}{d(0)} \right|$$

is the growth rate of instability. Note that α is a function of initial condition, so in principle, different regions of the same phase space can be chaotic or integrable. α effectively corresponds to the Lyapunov exponent.

→ The formal definition of Lyapunov exponent is more cumbersome but equivalent conceptually.

So-in PLAIN ENGLISH: Chaotic dynamics and phase space flows exhibit instability-specifically exponentially growing separation of trajectories from neighboring points. Chaotic flows stretch and fold, and are mixing. The rate of exponential divergence is (modulo some details) the Lyapunov exponent.

iv.) Yet More:

Why are ergodicity, mixing and chaos related? Why consider them together?

→ *Statistical Mechanics and, ENTROPY*!

- ERGODICITY is fundamental to ensemble theory in statistical mechanics, linking time and ensemble averages.

- MIXING and CHAOS justify certain key assumptions in the derivation of the Boltzmann equation and the proof of the H-Theorem. Most notable of these is the validity of the Principle of Molecular Chaos, i.e. $f(1,2) \to f(1) f(2)$.

- A measure of the degree of chaos in a dynamical system is the *Kolmogorov-Sinai Entropy*, ultimately expressed in terms of *rates*. Lyapunov exponents (i.e. ~orbit divergence rates). The *K-S* entropy is given by:

$$h_{KS} = \sum_{\alpha_i > 0} \alpha_i$$

where the sum is over the *positive* Lyapunov exponents (i.e. orbit separate rate). The *K-S* entropy does *not* involve kinetics, statistical theory, etc. but *does* require a phase space partition. The partition is needed to relate orbit divergence to information.

There is a ton more to be said. See books by Zaslavsky, or Ott, (used in preparing this), among others.