Potential Vorticity (PV) Dynamics and Models of Zonal Flow Formation

Pei-Chun Hsu

Advisor: Patrick Diamond

CASS, UCSD


Outline

- **Introduction**
  - Physical systems: geophysical fluids and fusion plasmas
  - Importance of zonal flows & wave-flow connection
  - Physics issues: zonal flow formation — how?

- **Thesis research**
  - Key mechanism: PV mixing
    - Representation of anisotropic PV mixing (MFT and beyond)
    - PV mixing in the presence of mean shear
    - Non-perturbative constrained relaxation models: structure
      - (1) (2)
    - Perturbation Theory: coefficients
      - (1)

- **Summary**: What did we learn?

Geophysical fluids

- Phenomena: weather, waves, large scale atmospheric and oceanic circulations, water circulation, jets...

- Geophysical fluid dynamics (GFD): low frequency \( \omega < \Omega \)
  
  \[ \text{"We might say that the atmosphere is a musical instrument on which one can play many tunes. High notes are sound waves, low notes are long inertial waves, and nature is a musician of the Beethoven than the Chopin type. He much prefers the low notes and only occasionally plays arpeggios in the treble and then only with a light hand." – J.G. Charney} \]

- Geostrophic motion: balance between the Coriolis force and pressure gradient

\[
R_0 = V / (2\Omega L) << 1 \\
\rightarrow \mathbf{v} = -\nabla P \times \hat{z} / 2 \Omega \\
\mathbf{P} \leftrightarrow \text{stream function} \\
\rightarrow \omega = \hat{z} \cdot (\nabla \times \mathbf{v}) = \nabla^2 \psi
\]
**Physical systems II**

**Kelvin’s theorem** – unifying principle throughout

- Kelvin’s circulation theorem for rotating system
  \[ \oint \mathbf{v} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{v} + 2\Omega \times \hat{z}) \cdot \hat{z} dS \equiv C \]
  \[ \dot{C} = 0 \]
  \[ \text{relative} \text{ planetary} \]

- Displacement on beta-plane
  \[ \dot{C} = 0 \to \frac{d}{dt} \nabla^2 \psi = -2\Omega \cos \theta \frac{d\theta}{dt} = -\beta \psi \]
  \[ \beta = \frac{2\Omega \cos \theta_0}{R_0} \]

- Quasi-geostrophic eq
  \[ \frac{d}{dt} (\nabla^2 \psi + \beta y) = 0 \]
  \[ \text{PV conservation} \]

\[ \rightarrow \text{Rossby wave} \]

![Rossby wave diagram]

\[ t=0 \]

\[ \omega<0 \]

\[ t>0 \]

\[ \omega>0 \]

G. Vallis 06

**Physical systems III**

**Magnetically confined plasma**

- Nuclear fusion: option for generating large amounts of carbon-free energy

- Challenge: ignition -- reaction release more energy than the input energy
  Lawson criterion:
  \[ n_i \tau_E T_i > 3 \times 10^{21} \text{m}^{-3} \text{s keV} \]

\[ \rightarrow \text{confinement} \]

\[ \rightarrow \text{turbulent transport} \]

- Turbulence: instabilities and collective oscillations
  \[ \rightarrow \text{lowest frequency modes dominate the transport} \]
  \[ \rightarrow \text{drift wave} \]

![Magnetically confined plasma diagram]

DIII-D

![ITER diagram]
Drift wave model – Fundamental prototype

• Hasegawa-Wakatani: simplest model incorporating instability

\[ V = \frac{c}{B} \hat{z} \times \nabla \phi + V_{\text{pol}} \]

\[ n J_{\perp} = n |e| V_{\text{pol}} \]

\[ \nabla \cdot J_{\perp} + \nabla \cdot \nabla J_{\parallel} = 0 \rightarrow \text{vorticity: } \rho_s^2 \frac{d}{dt} \nabla^2 \phi = -D_s \nabla^2 (\phi - n) + \nu \nabla^2 \nabla^2 \phi \]

\[ \frac{dn}{dt} + \nabla \cdot J_{\parallel} = 0 \rightarrow \text{density: } \frac{d}{dt} n = -D_s \nabla^2 (\phi - n) + D_0 \nabla^2 n \]

\[ \rightarrow \text{PV conservation in inviscid theory } \frac{d}{dt} (n - \nabla^2 \phi) = 0 \]

\[ \rightarrow \text{PV flux = particle flux + vorticity flux} \]

\[ \rightarrow \text{zonal flow being a counterpart of particle flux} \]

• Hasegawa-Mima ( \( D_s k_i^2 / \omega >> 1 \rightarrow n \sim \phi \) )

\[ \frac{d}{dt} (\phi - \rho_s^2 \nabla^2 \phi) + \nu \partial_y \phi = 0 \]

PV conservation

• PV conservation \( dq/dt=0 \)

<table>
<thead>
<tr>
<th>GFD: Quasi-geostrophic system</th>
<th>Plasma: Hasegawa-Wakatani system</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q = \nabla^2 \psi + \beta y )</td>
<td>( q = n - \nabla^2 \phi )</td>
</tr>
<tr>
<td>relative vorticity</td>
<td>density</td>
</tr>
<tr>
<td>planetary vorticity</td>
<td>(guiding center)</td>
</tr>
<tr>
<td>Physics: ( \Delta y \rightarrow \Delta (\nabla^2 \psi) )</td>
<td>Ion vorticity (polarization)</td>
</tr>
<tr>
<td>Physics: ( \Delta r \rightarrow \Delta n \rightarrow \Delta (\nabla^2 \phi) ) ZF!</td>
<td></td>
</tr>
</tbody>
</table>

• Charney-Hasrgawa-Mima equation

\[ n = n_0 + \tilde{n} \]

\[ \tilde{n} \sim \frac{\phi}{\hat{T}} \]

\[ \text{H-W } \rightarrow \text{H-M: } \frac{1}{\omega_{ci}} \frac{\partial}{\partial t} (\nabla^2 \phi - \rho_s^2 \phi) - \frac{1}{L_n} \frac{\partial}{\partial y} \phi + \frac{\rho_s}{L_m} J(\phi, \nabla^2 \phi) = 0 \]

\[ \text{Q-G: } \frac{\partial}{\partial t} (\nabla^2 \psi - L_i^2 \psi) + \beta \frac{\partial}{\partial x} \psi + J(\psi, \nabla^2 \psi) = 0 \]
### Outline

- **Introduction**
  - Physical systems: geophysical fluids and fusion plasmas
  - Key feature: zonal flows
  - Importance of zonal flows
    - Atmospheric phenomena
    - Confinement of fusion plasma
  - Physics issues: zonal flow formation

- **Thesis research**
  - Representation of anisotropic PV mixing (MFT and beyond)
  - Constrained relaxation models
    - (1) (2)
  - Perturbation Theory
    - (1)
  - PV mixing in the presence of mean shear
    - (3)

- **Summary**: What did we learn?
**Atmospheric phenomena**

- Phenomena: jet streams, Jovian zonal bands and zones, ... → zonal flows
- Zonal flows have a crucial influence on turbulent mixing and formation of transport barriers.
  - inhibiting the transport of vortex eddies across the flows
  - ozone hole problem ↔ transport barrier (shear layer insulation)

---

**Mid-latitude circulation**

- Stirring generates Rossby waves: \( \omega_k = -\beta k_x / k^2 \), \( v_x = 2\beta k_x k_y / k^4 \), \( v_y = -\beta k_x / k^2 \) \( \Rightarrow \) opposite
- Waves propagate away from the disturbance
  - energy density flux: \( v_x (\nabla \psi_k)^2 / 2 = \beta k_x k_y k^2 |\psi_k|^2 \) \( \Rightarrow \) opposite
  - eddy zonal-momentum flux: \( \langle \tilde{v}_x \tilde{v}_y \rangle = -k_x k_y |\tilde{\psi}_k|^2 \)
- Momentum converges in the region of stirring
Wave radiation in DW turbulence

- Localized instability drives drift waves
  \[ \omega_k = \frac{k_y v_r}{1 + k_z^2 \rho_s^2}, \quad v_r < 0 \]
  \[ v_{gr} v_{phr} < 0 \]

- Outgoing wave energy flux
  \[ \rightarrow \text{incoming wave momentum flux} \]
  \[ v_{gr} \left( \nabla \phi_k \right)^2 = -\rho_s^2 \frac{k_y k_z v_r}{(1 + k_z^2 \rho_s^2)^2} k^2 |\psi_k|^2 \]
  \[ \langle \ddot{v}_r, \ddot{v}_\theta \rangle = -\frac{c^2}{B^2} k_y k_z |\phi_k|^2 \]
  \[ \rightarrow \text{Zonal flow layers form at excitation regions} \]

- Essential elements in zonal flow generation
  -- inhomogeneous mixing in space
  -- one direction of symmetry (nature of torus and planet)

Confinement of fusion plasma

- Zonal flows:
  -- sheared E×B flows
    \[ \text{poloidally symmetric (n = 0)} \]
    \[ \text{toroidally symmetric (m = 0)} \]
    \[ \rightarrow \text{cannot tap free energy} \]
    \[ \rightarrow \text{driven nonlinearly by drift waves} \]
  -- modes of
    \[ \text{minimum inertia} \]
    \[ \text{minimal Landau damping} \]
    \[ \text{no radial transport} \]
    \[ \rightarrow \text{important for plasma confinement} \]
L-H transition

L-mode
- Increase of heating power -> heat flux
- edge turbulence -> Reynolds stress
- development of zonal flows

→ LCO
- regulation of turbulence and transport
- (self-regulation → oscillation)
- buildup of a steep pressure gradient
- growth of the mean shear

→ H-mode transition (onset: G. Tynan et al 2013)
- damping of both turbulence and zonal flows by mean shear

ZF and MS play different roles
ZF: extracts from turbulence
MS: locks in transition

Outline

• Introduction
  Physical systems: geophysical fluids and fusion plasmas
  Importance of zonal flows
  Understanding turbulence-zonal flow interaction

• Thesis research
  Representation of anisotropic PV mixing (MFT and beyond)
  Constrained relaxation models (1) (2)
  Perturbation Theory (1)
  PV mixing in the presence of mean shear (3)

• Summary: What did we learn?
Zonal flow formation

- Zonal flows are generated by nonlinear interactions between wave turbulence and zonal flow.
- In x space, zonal flows are driven by Reynolds stress $\frac{\partial}{\partial t} \langle u_x \rangle = -\frac{\partial}{\partial y} \langle \tilde{u}_y \tilde{u}_x \rangle - \mu \langle u_x \rangle$

Taylor’s Identity $\langle \tilde{u}_y \tilde{q} \rangle = -\frac{\partial}{\partial y} \langle \tilde{u}_y \tilde{v}_x \rangle \rightarrow \text{PV flux fundamental to zonal flow formation}$

- Inhomogeneous PV mixing, not momentum mixing (dq/dt=0)
  $\rightarrow$ up-gradient momentum transport (negative-viscosity) not an enigma

Inhomogeneous PV mixing

- PV mixing is the fundamental mechanism for zonal flow formation

\[ \delta(PV) \rightarrow \delta(\nabla^2 \psi) \rightarrow \delta(\psi) \rightarrow \nu = \nabla \times \psi \]

McIntyre 1982

PV mixing & Flows I
Outline

- **Introduction**
  - Physical systems: geophysical fluids and fusion plasmas
  - Importance of zonal flows
  - Physics issues: zonal flow formation

- **Thesis research**
  - Representation of anisotropic PV mixing (MFT and beyond)
  - Key mechanism: PV mixing
  - Non-perturbative constrained relaxation models: structure
  - Perturbation theory: coefficients

- **Summary**
  - PV mixing in the presence of mean shear

---

**Non-perturbative approaches**

- PV mixing in space is essential in ZF generation.
  - Taylor identity: \( \langle \tilde{u}_j \nabla^2 \tilde{\phi} \rangle = -\partial_x \langle \tilde{u}_j \tilde{u}_j \rangle \)
    - vorticity flux
    - Reynolds force

**Key:**

- How represent inhomogeneous PV mixing?

**General structure of PV flux?**

- Relaxation principles!

**non-perturb model 1:** use selective decay principle

- What form must the PV flux have so as to dissipate enstrophy while conserving energy?

**non-perturb model 2:** use joint reflection symmetry

- What form must the PV flux have so as to satisfy the joint reflection symmetry principle for PV transport/mixing?
General principle: selective decay

- 2D turbulence conservation of energy and potential enstrophy
  → dual cascade
  → Minimum enstrophy state

- eddy turnover rate and Rossby wave frequency mismatch are comparable
  \[ \frac{\partial \omega}{\partial t} + \bar{u} \cdot \nabla \omega + \beta v = 0 \]
  \[ \frac{U^r}{LT} \frac{U'}{E'} \frac{\beta U}{\beta} \]
  → Rhines scale \( L_r \sim \frac{U}{\sqrt{\beta}} \)

Using selective decay for flux

<table>
<thead>
<tr>
<th>non-perturb model 1</th>
<th>minimum enstrophy relaxation</th>
<th>analogy</th>
<th>Taylor relaxation</th>
</tr>
</thead>
<tbody>
<tr>
<td>turbulence</td>
<td>2D hydro</td>
<td>3D MHD</td>
<td></td>
</tr>
<tr>
<td>conserved quantity (constraint)</td>
<td>total kinetic energy</td>
<td>global magnetic helicity</td>
<td></td>
</tr>
<tr>
<td>dissipated quantity (minimized)</td>
<td>fluctuation potential enstrophy</td>
<td>magnetic energy</td>
<td></td>
</tr>
<tr>
<td>final state</td>
<td>minimum enstrophy state</td>
<td>Taylor state</td>
<td></td>
</tr>
<tr>
<td></td>
<td>flow structure emergent</td>
<td>force free B field configuration</td>
<td></td>
</tr>
</tbody>
</table>

- flux? what can be said about dynamics?
- structural approach (this work): What form must the PV flux have so as to dissipate enstrophy while conserving energy?

General principle based on general physical ideas → useful for dynamical model
PV flux

\[ \frac{\partial \langle q \rangle}{\partial t} + \partial_j \langle \nabla_j q \rangle = \nu_0 \partial^2_j \langle q \rangle \]

\[ \Gamma_q : \text{mean field PV flux} \]

Key Point: what form does PV flux have s/t dissipate enstrophy, conserve energy

selective decay

\[ \rightarrow \text{energy conserved} \quad E = \int \frac{\langle \partial_j \langle \phi \rangle \rangle^2}{2} \]

\[ \frac{\partial E}{\partial t} = \int \langle \phi \rangle \partial_j \Gamma_q = - \int \partial_j \langle \phi \rangle \Gamma_q \quad \Rightarrow \Gamma_q = \frac{\partial_j \Gamma_E}{\partial_j \langle \phi \rangle} \]

\[ \rightarrow \text{enstrophy minimized} \quad \Omega = \int \frac{\langle q \rangle^2}{2} \]

\[ \frac{\partial \Omega}{\partial t} = - \int \langle q \rangle \partial_j \Gamma_q = - \int \partial_j \left[ \frac{\partial_j \langle q \rangle}{\partial_j \langle \phi \rangle} \right] \Gamma_E \]

\[ \frac{\partial \Omega}{\partial t} < 0 \Rightarrow \Gamma_E = \mu_0 \partial_j \left[ \frac{\partial_j \langle q \rangle}{\partial_j \langle \phi \rangle} \right] \]

\[ \Rightarrow \Gamma_q = \frac{1}{\partial_j \langle \phi \rangle} \partial_j \left[ \mu_0 \partial_j \left( \frac{\partial_j \langle q \rangle}{\partial_j \langle \phi \rangle} \right) \right] \]

general form of PV flux

parameter TBD

non-perturb model 1

Structure of PV flux

\[ \Gamma_q = \frac{1}{\langle v_\perp \rangle} \partial_j \left[ \mu_0 \partial_j \left( \frac{\partial_j \langle q \rangle}{\partial_j \langle v_\perp \rangle} \right) \right] = \frac{1}{\langle q \rangle} \partial_j \left[ \mu_0 \left( \frac{\langle q \rangle \partial_j \langle q \rangle}{\langle v_\perp \rangle^2} + \partial_j^2 \langle q \rangle \right) \right] \]

diffusion parameter calculated by perturbation theory, numerics...

diffusion and hyper diffusion of PV

\[ \Rightarrow \text{usual story: Fick's diffusion} \]

relaxed state:

Homogenization of \[ \frac{\partial_j \langle q \rangle}{\langle v_\perp \rangle} \] allows staircase

characteristic scale \[ \ell_c = \sqrt{\frac{\langle v_\perp \rangle}{\partial_j \langle q \rangle}} \]

\[ \ell > \ell_c : \text{zonal flow growth} \]

\[ \ell < \ell_c : \text{zonal flow damping (hyper viscosity-dominated)} \]

Rhines scale \[ L_R \sim \sqrt{\frac{U}{\beta}} \]

\[ \ell > L_R : \text{wave-dominated} \]

\[ \ell < L_R : \text{eddy-dominated} \]
PV staircase

relaxed state: homogenization of \( \frac{\partial_q \langle q \rangle}{\langle v_z \rangle} \) → PV gradient large where zonal flow large

→ Zonal flows track the PV gradient → PV staircase

- Highly structured profile of the staircase is reconciled with the homogenization or mixing process required to produce it.
- Staircase may arise naturally as a consequence of minimum enstrophy relaxation.

What sets the “minimum enstrophy”

- Decay drives relaxation. The relaxation rate can be derived by linear perturbation theory about the minimum enstrophy state

\[
\begin{align*}
\langle q \rangle &= q_m(y) + \delta q(y, t) \\
\langle \phi \rangle &= \phi_m(y) + \delta \phi(y, t) \\
\partial_y q_m &= \lambda \partial_y \phi_m \\
\delta q(y, t) &= \delta q_0 \exp(-y_{rel,t} - i\omega t + iky)
\end{align*}
\]

\[
\begin{align*}
\gamma_{rel} &= \mu \left( \frac{k^4 + 4\lambda k^2 + 3\lambda^2}{\langle v_z \rangle^2} - \frac{8q_m^2(k^2 + \lambda)}{\langle v_z \rangle^4} \right) \\
\omega_k &= \mu \left( -\frac{4q_m^2 k^3 + 10q_m k \lambda}{\langle v_z \rangle^3} - \frac{8q_m^2 k}{\langle v_z \rangle^4} \right)
\end{align*}
\]

- The condition of relaxation (modes are damped):

\[
\begin{align*}
\gamma_{rel} > 0 &\quad \Rightarrow k^2 > \frac{8q_m^2}{\langle v_z \rangle^2} > 3\lambda \\
k^2 > 0 &\quad \Rightarrow \frac{8q_m^2}{\langle v_z \rangle^2} > 3\lambda \quad \Rightarrow \text{Relates } q_m^2 \text{ with ZF and scale factor}
\end{align*}
\]

- ZF cannot grow arbitrarily large, and is constrained by the enstrophy
- To sustain a zonal flow in the minimum enstrophy state, a critical residual enstrophy density is needed.

→ \( q_m^2 \): the ‘minimum enstrophy’ of relaxation, related to scale
Role of turbulence spreading

- Turbulence spreading: tendency of turbulence to self-scatter and entrain stable regime

- Turbulence spreading is closely related to PV mixing because the transport/mixing of turbulence intensity has influence on Reynolds stresses and so on flow dynamics.

- PV mixing is related to turbulence spreading
  \[ \frac{\partial E}{\partial t} = \int \langle \phi \rangle \partial_y \Gamma_q = -\int \partial_y \langle \phi \rangle \Gamma_q \Rightarrow \Gamma_q = \frac{\partial_y \Gamma_E}{\partial_y \langle \phi \rangle} \]

- The effective spreading flux of turbulence kinetic energy
  \[ \Gamma_E = -\int \Gamma_q \langle u_x \rangle dy = -\int \frac{1}{\langle u_x \rangle} \partial_y \left[ \mu \partial_y \left( \frac{\partial_y \langle q \rangle}{\langle u_x \rangle} \right) \right] \langle u_x \rangle dy = \mu \partial_y \left( \frac{\partial_y \langle q \rangle}{\langle u_x \rangle} \right) \]

  \( \Rightarrow \) the gradient of the \( \partial_y \langle q \rangle / \langle u_x \rangle \), drives spreading

  \( \Rightarrow \) the spreading flux vanishes when \( \partial_y \langle q \rangle / \langle u_x \rangle \) is homogenized

Discussion

- PV mixing \( \leftrightarrow \) forward enstrophy cascade \( \leftrightarrow \) hyper-viscosity
  \( \Rightarrow \) How to reconcile effective negative viscosity with the picture of diffusive mixing of PV in real space?

- A possible explanation of up-gradient transport of PV due to turbulence spreading
**PV-avalanche model – beyond diffusion**

- avalanching: tendency of excitation to propagate in space via local gradient change
- Joint-reflection symmetry: \( \Gamma[\delta q] \) invariant under \( y \rightarrow -y \) and \( \delta q \rightarrow -\delta q \)

\[
\Gamma[\delta q] = \sum_j \alpha_j (\delta q_j)^2 + \sum_m \beta_m (\partial_y \delta q_m) + \sum_n \gamma_n (\partial_y^3 \delta q)^n + ... 
\]

- large-scale properties: higher-order derivatives neglected
  - small deviations: higher-order terms in \( \delta q \) neglected
  - Simplest approximation: \( \Gamma[\delta q] = \frac{\alpha}{2} (\delta q)^2 + \beta \partial_y \delta q + \gamma \partial_y^3 \delta q \)

**Key Point:** what form does PV flux have s/t satisfy joint-reflection symmetry principle

**non-perturb model 2**

\[
\Gamma[\delta q] = \frac{\alpha}{2} (\delta q)^2 + \beta \partial_y \delta q + \gamma \partial_y^3 \delta q
\]

- PV equation:
  \[
  \partial_t \delta q + a \partial_y \delta q + \beta \partial_y^3 \delta q + \gamma \partial_y^5 \delta q = 0 
  \]

Kuramoto-Sivashinsky type equation

**Non-linear convection of \( \delta q \)**

- Avalanche-like transport is triggered by deviation of PV gradient
  - PV deviation implicitly related to the local PV gradient \( \delta q \rightarrow \partial_y \delta q \)
  - transport coefficients (functions of \( \delta q \)) related to the gradient \( D(\delta q) \rightarrow D(\partial_y \delta q) \)
  - gradient-dependent effective diffusion \( \Gamma_y \sim -D(\partial_y \delta q) \partial_y \delta q \rightarrow -D(\delta q) \delta q \)

- Convective component of the PV flux can be related to a gradient-dependent effective diffusivity
  \[
  \Gamma[\delta q] \sim \delta q^2 \rightarrow -D(\delta q) \delta q \\
  D(\delta q) \rightarrow D_0 \delta q 
  \]
Perturbation theory

- The evolution of perturbation (seed ZF) as a way to look at PV transport

Revisiting modulational instability

ZF evolution determined by Reynolds force

\[ \frac{\partial}{\partial t} \delta V_x = - \frac{\partial}{\partial y} \left( \bar{u}_x \bar{v}_y \right) = \frac{\partial}{\partial y} \sum_{k<} \frac{k^2}{k^4} \tilde{N}_k \]

\( N_k = k^2 \left| \psi_k \right|^2 / \omega_k \) is wave action density, for Rossby wave and drift wave, it is proportional to the enstrophy density. \( N_k \) is determined by WKE:

\[ \frac{\partial \tilde{N}}{\partial t} + \vec{u}_g \cdot \nabla \tilde{N} + \delta \omega \tilde{N} = \frac{\partial \left( k \omega \delta V_x \right)}{\partial y} \frac{\partial N_0}{\partial k_y} \]

Turbulent vorticity flux derived

\[ \frac{\partial}{\partial t} \delta V_q = \frac{\partial^2}{\partial y^2} \delta V_q \sum \left( \frac{k^2 k_x}{k^4} \right) \frac{\partial \omega_k}{\partial k_x} \frac{\partial N_0}{\partial k_y} \]

\( \kappa(q) \neq \text{const} \)

- scale dependence of PV flux
- non-Fickian turbulent PV flux
• A simple model from which to view $\kappa(q)$:
  - Defining MFP of wave packets as the critical scale $q^{-1}_c = v_c \delta \omega_k^{-1}$
  - Keeping next order term in expansion of response
    $$q^{-1} >> q^{-1}_c \Rightarrow \frac{\delta \omega_k}{(qv_c)^2 + \delta \omega_k^2} = \frac{1 - q^2}{q^2}$$

→ zonal growth evolution:

$$\partial_t \delta V_x = D \partial_y^2 \delta V_x - H \partial_y^4 \delta V_x$$

\textbf{Transport coefficients (viscosity and hyper-viscosity) for relaxation models:}

$$\frac{\partial \langle \mathbf{v}_s \rangle}{\partial t} = \Gamma_q = \frac{1}{\delta \langle \phi \rangle} \delta \left[ \mu \left( \frac{\partial \langle \mathbf{q} \rangle}{\partial \langle \phi \rangle} \frac{\partial \langle \mathbf{q} \rangle}{\partial \langle \phi \rangle} + \frac{\partial^2 \langle \mathbf{q} \rangle}{\partial \langle \phi \rangle^2} \right) \right]$$

$$\Gamma[\delta q] = \frac{\alpha}{2} (\delta q)^2 + \beta \partial_q \delta q + \gamma \partial_y^2 \delta q\right)$$

**Discussion of $D$ and $H$**

• Roles of negative viscosity and positive hyper-viscosity (Real space)

$$\partial_t \delta V_x = D \partial_y^2 \delta V_x - H \partial_y^4 \delta V_x$$

$$\frac{\partial}{\partial t} \int \frac{1}{2} \delta V_x^2 \, d^2 x = -D \int \left( \partial_y \delta V_x \right)^2 \, d^2 x - H \int \left( \partial_y^2 \delta V_x \right)^2 \, d^2 x$$

$D < 0 \Rightarrow \gamma_{q,D} > 0$ ZF growth (Pumper D)

$H > 0 \Rightarrow \gamma_{q,H} < 0$ ZF suppression (Damper H)

→ $D$, $H$ as model of spatial PV flux beyond over-simplified negative viscosity

$D = Hq^2$ sets the cut-off scale

$$\ell^2_c = \left[ \frac{H}{|D|} \right]$$

$\ell > \ell_c$ : ZF energy growth \→ D process dominates at large scale

$\ell < \ell_c$ : ZF energy damping \→ H process dominates at small scale

**Minimum enstrophy model**

$$\Gamma_q = \frac{1}{\delta \langle \phi \rangle} \partial_t \left[ \mu \partial_{\langle \phi \rangle} \left( \frac{\partial \langle \mathbf{q} \rangle}{\partial_{\langle \phi \rangle}} \right) \right] \Rightarrow \ell_c = \left[ \frac{\langle \mathbf{v}_s \rangle}{\partial_{\langle \phi \rangle}} \right]$$
### Parametric instability

<table>
<thead>
<tr>
<th></th>
<th>pseudo-fluid</th>
<th>plasma fluid</th>
</tr>
</thead>
<tbody>
<tr>
<td>elements</td>
<td>wave-packets</td>
<td>charged particles (species $\alpha$)</td>
</tr>
<tr>
<td>distribution function</td>
<td>$N_k(k, \omega_k)$</td>
<td>$f_\alpha(r, v, t)$</td>
</tr>
<tr>
<td>mean free path</td>
<td>$</td>
<td>v_g</td>
</tr>
<tr>
<td>density</td>
<td>$n_w = \int N_k dk$</td>
<td>$n_\alpha = \int f_\alpha dv$</td>
</tr>
<tr>
<td>momentum</td>
<td>$P^w = \int k N_k dk$</td>
<td>$p_\alpha = \int m_\alpha v f_\alpha dv$</td>
</tr>
<tr>
<td>velocity</td>
<td>$V^w = \int v_g N_k dk / \int N_k dk$</td>
<td>$u_\alpha = \int v f_\alpha dv / \int f_\alpha dv = \frac{p_\alpha}{m_\alpha n_\alpha}$</td>
</tr>
</tbody>
</table>


---

**perturbation theory 2** – narrowband

---

**-- pseudo-fluid evolution:**

- multiplying the WKE by $v_y$ and integrating over $k$
- normalizing by pseudo-density $n_w$

\[ \frac{\partial}{\partial t} V^w_y + V^w_y \frac{\partial}{\partial y} V^w_y = -a \langle v_x \rangle \]

---

**-- ZF evolution:**

\[ \langle \tilde{v}_x, \tilde{v}_y \rangle = \int v_y N_k dk \rightarrow V^w_y P^w_x \]

\[ \frac{\partial}{\partial t} \langle v_x \rangle = -\frac{\partial}{\partial y} V^w_y P^w_x \]

---

**ZF growth rate in monochromatic limit:**

- linearizing the above two eqs.

\[ \gamma_q = \sqrt{q^2 k_z^2 \left| q_z \right|^2 \left( 1 - \frac{4k_z^2}{k^2} \right)} \]

The reality of $\gamma_q$ requires $k_z > 3k_y^2$

\[ \gamma_q \propto |q| \] indicates convective instability
<table>
<thead>
<tr>
<th>PV flux</th>
<th>convective</th>
<th>viscous</th>
<th>hyper-viscous</th>
<th>coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>(non-perturb.)</td>
<td>Min. enstrophy relaxation</td>
<td>•</td>
<td>•</td>
<td></td>
</tr>
<tr>
<td></td>
<td>PV-avalanche relaxation</td>
<td>•</td>
<td>•</td>
<td></td>
</tr>
<tr>
<td>(perturbative)</td>
<td>Modulational instability</td>
<td>•</td>
<td>•</td>
<td>$D_i(&lt;0), H_i(&gt;0)$</td>
</tr>
<tr>
<td></td>
<td>Parametric instability</td>
<td>•</td>
<td></td>
<td>$\gamma_q(\sim</td>
</tr>
</tbody>
</table>

• Minimum enstrophy
  \[
  \frac{\partial \langle v_i \rangle}{\partial t} = \Gamma_q = \frac{1}{\alpha} \partial \left[ \mu \left( \frac{\langle q \rangle \partial \langle \phi \rangle}{\partial \langle \phi \rangle} + \frac{\partial^2 \langle q \rangle}{\partial \langle \phi \rangle^2} \right) \right]
  \]

• PV-avalanche
  \[
  \Gamma[\delta q] = \frac{\alpha}{2} (\delta q)^2 + \beta \partial \delta q + \gamma \partial^3 \delta q
  \]

• Modulational instab.
  \[
  \partial_t \delta V_x = -q^2 D \delta V_x + q^4 H \delta V_x
  \]

• Parametric instab.
  \[
  \gamma_q = \sqrt{q^2 k_x^2 \left| \psi \right|^2 \left( 1 - \frac{4k_y^2}{k^2} \right)}
  \]

III) Multi-scale shearing effects

• Motivation
  - L-H transition intermediate phase: coexistence of mean shear and zonal flow

• Generic problem:
  - coupling/interaction between different scale shearing fields

• Important issue
  - how mean shear flows affect the PV flux and zonal flow generation

Modulational instab. w/ mean shear

Momentum flux -> Reynolds stress -> wave action \(<\nu,\nu> = \sum_k k_j^2 N_k\)

- mean shear in WKE: \(\partial_t \tilde{N}_k + \nu_{\gamma_k} \partial_j \tilde{N}_k - k_j \langle V_s \rangle \partial_j \tilde{N}_k + \delta \omega_k \tilde{N}_k - k_j \delta V_s \partial_j \partial_k N_0\)
  \[ \nu_{\gamma_k} = \frac{2\beta k_j k_k}{k^2} \Omega^{3/2} \]
  Non-linear diffusion
  Seed ZF

- Ray trajectory refraction:
  \[ \frac{d\tilde{k}_j}{dt} = -\partial_j (\omega + k_j V_s) ; \tilde{V}_s = \langle V_s \rangle + \tilde{V}_s \]
  \[ \frac{dy}{dt} = \nu_{\gamma_k} = \frac{2\beta k_j k_k}{k^2} \Omega^{3/2} \]

\[ k_j(t) = k_j(0) + k_j \Omega t \]

\[ y(t) = y(0) + e(t), \quad e(t) = \beta \frac{1}{\Omega} \left( \frac{1}{k_0^2} - \frac{1}{k_0^2 + (k_0 + k_0 \Omega^2)^2} \right) \]

Modulational instab. w/ mean shear

- characteristic method (shearing frame)

  original frame \[ \left\{ \begin{array}{l} k_j = k_j(0) + k_j \Omega t \\ y = y(0) + e(t) \end{array} \right. \]

  shearing frame \[ \left\{ \begin{array}{l} \tilde{k}_j = k_j(0) + k_j \Omega t \\ \tilde{y} = y - e(t) \end{array} \right. \]

  \[ \partial_t \tilde{N}_k + \nu_{\gamma_k} \partial_j \tilde{N}_k - k_j \langle V_s \rangle \partial_j \tilde{N}_k + \delta \omega_k \tilde{N}_k - k_j \delta V_s \partial_j \partial_k N_0 \]

  - Solving Green’s function in shearing frame
  - Changing variables back to original frame

  \[ \tilde{N}_k(\Omega t >> 1) = \int_0^\infty d\tau e^{-i\omega k \tau} d\Omega \frac{e^{i\gamma_0}}{i2k} \]

- strong mean shear limit \( (\Omega >> 3/\delta \omega_k) \)

  \[ \tilde{N}_k = \left( \frac{3}{\delta \omega_k \Omega^2} \right)^{1/3} k_j \delta \tilde{V} \partial_j N_0 + O(\Omega^{-2}) \]

\[ \gamma_q \cdot \langle \delta \tilde{V}, \tilde{V} \rangle \propto \left( \frac{3}{\delta \omega_k \Omega^2} \right)^{1/3} \]

\[ \rightarrow \text{Mean shear reduces ZF growth } \sim \Omega^{2/3} \]

\[ \rightarrow \text{scaling of PV flux in strong mean shear} \]
Summary

- Inhomogeneous PV mixing is identified as the fundamental mechanism for zonal flow formation. This study offered new perspectives and approaches to calculating spatial flux of PV.

- The general structure of PV flux is studied by two non-perturbative relaxation models.
  1. Selective decay model
     \[ \Gamma_\phi = \frac{1}{\langle \phi \rangle} \partial_y \left[ \mu \left( \frac{\langle q \rangle \partial_y \langle q \rangle}{\langle \phi \rangle} + \frac{\partial^2 \langle q \rangle}{\partial_y \langle \phi \rangle} \right) \right] \]
     - PV flux contains diffusion and higher order diffusion terms. The homogenized quantity in the relaxed state is the ratio of PV gradient to zonal flow velocity. This is consistent with the structure of the PV staircase.
  2. PV-avalanche model
     \[ \Gamma[\delta q] = \frac{\alpha}{2} (\delta q)^2 + \beta \partial_y \delta q + \gamma \delta q^3 \]
     - PV flux is constrained by the joint reflection symmetry condition, and contains diffusive, hyper-diffusive, and convective terms. The convective transport of PV can be generalized to an effective diffusive transport.

- The transport coefficients are derived using perturbative analyses of wave kinetic equation. Relaxation models and perturbative analyses are synergetic and complementary approaches.

- In modulational instability analysis for a broadband spectrum, a negative viscosity and a positive hyper-viscosity, which represents ZF saturation mechanism, are derived. In parametric instability analysis for a narrow spectrum, a convective transport coefficient is obtained.

- Important issues addressed in our models includes PV staircase, turbulence spreading, avalanche-like transport, characteristic scales.

- The effect of the mean shear on PV flux and zonal flow formation is studied. ZF growth rate and the PV flux are shown to decreases with mean shearing rate as \( \Omega^{-2/3} \). Framework of PV transport for systems with multi-scale shearing fields is established.
Extension of work

• What’s the right relaxation principle?
  minimum enstrophy, maximum entropy, PV homogenization...

• Numerical simulation test:
  form of the PV flux during relaxation
  profile of the homogenized quantity for the relaxed state
  the minimum enstrophy in the relaxed state
  PV flux spectrum -1/f (?)
  staircase formation during relaxation

• Including the magnetic field
  β-plane MHD model of PV mixing processes (the effect of magnetic
  field on the cross phase of the Reynolds stress)

• Use PV flux expression to improve ZF dynamics representation in
  reduced L-H transition models

Onset of transition into H-mode

• Transition criterion

  turbulence power coupled to flow ≥ turbulence power increase

  \[ R_f = \frac{\langle \vec{D}, \vec{D} \rangle \langle \nu_x \rangle}{\langle \vec{D}^2 \rangle (\gamma_{eff} - \gamma_{decorr})} \geq 1 \]

  G.R. Tynan et al 2013

  criterion for turbulence collapse and transition onset
  \( \Rightarrow \) zonal flow role critical