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Transport of radial heat flux and second sound in fusion plasmas

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Simple flux-gradient relations that involve time delay and radial coupling are discussed. Such a formulation leads to a rather simple description of avalanches and may explain breaking of gyroBohm transport scaling. The generalization of the flux-gradient relation (i.e., constitutive relation), which involve both time delay and spatial coupling, is derived from drift-kinetic equation, leading to kinetic definitions of constitutive elements such as the flux of radial heat flux. This allows numerical simulations to compute these cubic quantities directly. The formulation introduced here can be viewed as an extension of turbulence spreading to include the effect of spreading of cross-phase as well as turbulence intensity, combined in such a way to give the flux. The link between turbulence spreading and entropy production is highlighted. An extension of this formulation to general quasi-linear theory for the distribution function in the phase space of radial position and parallel velocity is also discussed.
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I. INTRODUCTION

In modern day fusion devices, collisions are rare, and therefore heat and particle transport driven by them are feeble. This permits, by heating the core of the plasma, to enforce temperature profiles, where the plasma is very hot in the center and relatively much colder where the plasma touches material surfaces. These temperature profiles provide large free energy sources in the thermodynamical sense. The plasma tries to find ways of getting rid of this excess free energy, which is continually supplied to the system. It achieves this, by generating collective fluctuating electric and magnetic fields that do so in such a way that a net flux of particles and heat is driven towards plasma walls. For instance a fluctuating radial $E \times B$ velocity that is in phase with a fluctuating temperature, such that the radial velocity is outward when temperature is higher and inward when the temperature is lower, can transport large amounts of heat. Unfortunately, the plasma can excite such oscillations.

On the other hand, these fluctuations saturate via nonlinear interactions involving mode coupling to nearby scales,[1,2] turbulent cascades to smaller scales,[3,4] and interactions with large scale flow structures such as zonal flows.[5] In all these saturation scenarios, the efficiency of the fluctuations to transport heat and particles towards the wall is reduced as compared to the case where the fluctuations consist of a single linearly most unstable mode. This state of interactions between fluctuations driven by the available free energies and zonal flows is a prominent example of what is commonly called the *plasma turbulence* in the context of fusion devices.[6]

The level of fluctuations in this saturated turbulent state (along with the cross-phase between velocity and density or temperature fluctuations) is what determines the turbulent flux. In the standard formulation of plasma transport (even at the level of the so called first-principle gyrokinetic simulations), the local instantaneous approach is surprisingly common. In this approach there is a linear relation between the fluxes and the gradients[7] (e.g., with cross-terms as well as pinches and off-diagonal terms such as residual stress) so that if we consider the heat flux alone (i.e., adiabatic electrons and zero rotation with symmetric $k_\parallel$ spectrum), there is a direct relation between the heat flux and the temperature gradient

$$Q = -\chi_i \frac{\partial T}{\partial r}. \tag{1}$$

This relation is called the Fick's law (or the Fourier law of heat conduction in the case of solids), and it implies that the flux responds *infinitely rapidly* and *perfectly locally* to the gradients (see Figs. 1 and 2). In modern day thermodynamics, the Fick's law itself is seen as an asymptotic solution of a set of more general constitutive relations. For instance in the study of heat transport in liquid helium, which leads to the phenomenon commonly known as "the second sound,"[8] in solids at very fast time scales,[9] and in reactive fluids, a commonly used relation is the following (e.g., Ref. 10):

$$\tau \frac{\partial Q}{\partial t} + \chi_i \nabla_r T + Q = 0, \tag{2}$$

where τ can be considered as a new coefficient of transport physics, which physically corresponds to the response time of the flux to the gradient. This relation, which is called the Maxwell-Cattaneo relation, incorporates the fact that *if the gradient is suddenly changed, the heat flux would respond in some finite response time (i.e., τ)*. Note that the steady

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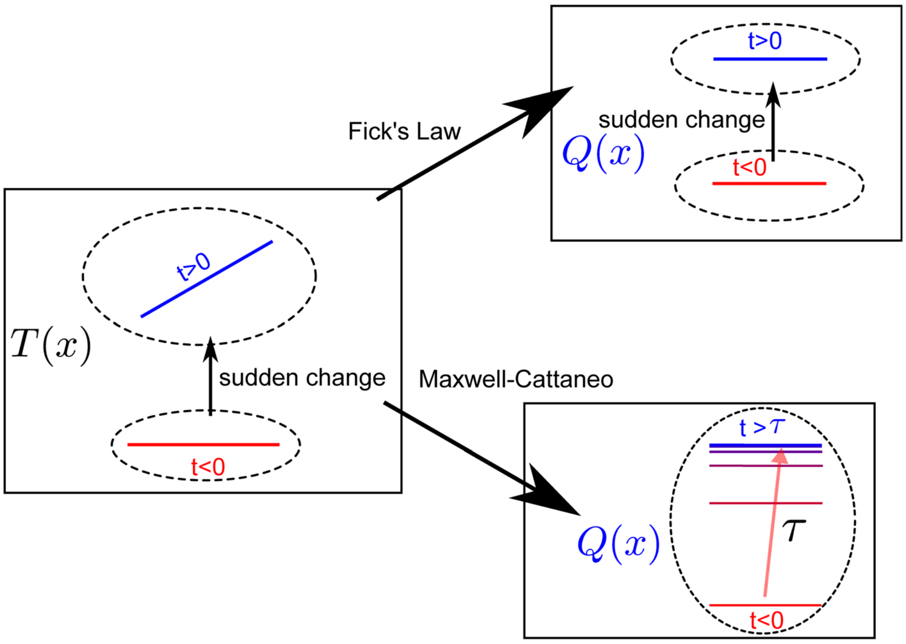

FIG. 1. Schematic representation of the different flux-gradient relations. If the gradient is changed suddenly, the Fick's law responds by changing the flux suddenly, whereas with the Maxwell-Cattaneo relation it takes a time of the order of the relaxation time τ for the flux to relax to its new value.

state limit of Eq. (2) is indeed Eq. (1). In order to understand "transient" transport, we actually need both τ and χ_i even though τ may be as fast as a few turbulent decorrelation times.

However since the plasma transport is driven mainly by turbulence, the response of the flux is not a simple "transient," because by the time the flux is saturated to its mixing length level, the gradient is also changed via the transport equation for heat

$$\frac{\partial T_i}{\partial t} + \nabla_r \cdot Q_i = H(r,t), \tag{3}$$

where $H(r,t)$ is the net heating, and this allows a dynamical coupling between the flux and the gradient. For example expanding around an equilibrium [i.e., $\nabla_r \cdot Q_i = H(r,t)$], it is easy to see that the excess temperature obeys the equation

$$\frac{\partial^2 \delta T_i}{\partial t^2} + \frac{1}{\tau}\frac{\partial}{\partial t}\delta T_i - \nabla_r \cdot \left[\frac{\chi_i}{\tau}\nabla_r \delta T\right] = 0, \tag{4}$$

which is actually *the telegraph equation*. This equation is well known to describe the time evolution of the transport in reactive fluids and in problems of population growth, more accurately than a simple diffusion equation.[11,12] It is

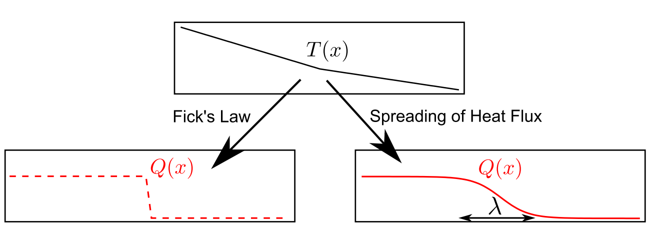

FIG. 2. Schematic representation of the different flux-gradient relations. If the gradient has a discontinuity (but T is nonetheless continuous), the Fick's law gives discontinuous flux. In contrast a relation with spreading of heat flux (such as the Guyer-Krumhansl relation) smears out the discontinuity. The coefficient λ is the spatial scale corresponding to this smearing.

the simplest formulation that describes the phenomenon of diffusion in a medium, which describes not only the positions but also the momenta of the basic elements[13] that constitute that medium. It is clear that as $\tau \to 0$, the first term in Eq. (4) becomes negligible, and we recover the usual diffusion equation. This is a singular limit where the highest order derivative with respect to time drops.

In contrast, for finite τ, the wave character of the equation suggests a radial propagation speed $v_r = \sqrt{\chi_i/\tau}$, which is consistent with the spreading phenomenology $v_{spr} \sim \sqrt{2\gamma D_{\mathcal{E}}}$ (where γ is the growth rate), except that the diffusion coefficient for turbulence intensity $D_{\mathcal{E}}$ is replaced in this coupled formulation by the heat diffusivity χ_i.[14–17]

Note that the heat equation above (i.e., Eq. (3)) is used for its simplicity, which can be justified for plasmas rigorously only if the plasma density is constant. It should in fact be replaced by an equation for pressure in the more general case and accompanied by a second equation for plasma density, as is usually done in transport models (e.g., Ref. 18).

If the flux response is not perfectly local, but rather smoothed out (i.e., a discontinuity in the gradients does not generate a step function flux), we can write

$$\tau\frac{\partial Q}{\partial t} + \tau\nabla_r \cdot \Gamma_Q + \chi_i \nabla_r T + Q = 0, \tag{5}$$

where a local closure for the flux of heat flux

$$\Gamma_Q = -D\frac{\partial Q}{\partial r} \tag{6}$$

can be proposed, which gives the so-called Guyer-Krumhansl constitutive relation,[19] which is sometimes referred to as a "weakly non-local" relation. Obviously, if the radial group velocity of the waves that are responsible for transport is finite, it should be added to Γ_Q (i.e., as $v_{gr}Q$). One may also imagine to add a pinch term for the heat flux here, but this is out of the scope of this paper.

Note that this formulation of the transport now has 3 coefficients, namely, the heat diffusivity χ_i, flux response time τ, and the flux penetration length $\lambda = \sqrt{D\tau}$. Note that Eq. (5) is simply the Cattaneo relation with the added diffusion of heat flux. However, since the transport is anomalous in fusion plasmas, the coefficients χ, τ, and D are functions of turbulence intensity, rendering the constitutive relation nonlinear as will be discussed.

Obviously, the local limit ($\lambda \to 0$) of the Guyer-Krumhansl gives the Cattaneo relation. We will therefore show a derivation of this more general relation for fusion plasmas using the drift-kinetic equation. We suggest this as a transport model that basically reproduces avalanches and other complex seemingly "non-local" phenomena simply because of the time delay in the flux response to the changes in the gradient. Note that the telegraph equation as given in Eq. (4) gives radial propagation of turbulence (i.e., heat flux) coupled with mesoscale fluctuations of temperature *without any explicit non-locality in the equations or turbulence spreading*—even though this would probably be called turbulence spreading from an experimental point of view since the little meso-scale fluctuation of temperature that accompany the propagation of turbulent flux may be undetectable in profile measurements.

However there are methods such as electron cyclotron emission (ECE) imaging,[20,21] which can provide access to temperature fluctuations with reasonable spatio-temporal resolution. The phenomenon that is described here, may be detectable by detailed ECE measurements analyzed using advanced transport modeling, which includes time variation of heat flux (instead of "heat pinch" as a first approximation).[22–24] One can perform something like a "quasi-linear theory" over the above coupled system of equations (computing the average flux driven by these heat waves) in order to obtain an effective "pinch" term. However a transport analysis that includes the fast response of the flux can easily be developed, and τ and λ may be computed by fitting the heat modulation results to the above model. Another idea would be to look at density instead of temperature, whose flux may be measured directly using probes (at least near the edge) and the profile corrugations can be seen with fast sweep microwave or beam emission spectroscopy techniques. The ideal case where this phenomenon may be observed seems to be a sudden H-L transition where the response of the flux to the change in the gradient may be observed. One advantage of such probe measurements is that they could also decouple spreading of turbulence intensity, from a propagation of the cross-phase that is responsible for the transport.

We should probably note that the suggestion that the plasma turbulence is inherently non-local is not a new idea.[14,25–27] There is a school of thought that advocates for the use of fractional diffusion formulations,[28] based on the existence of Lévy flights and non-Gaussian statistics in plasma turbulence using the continuous time random walk framework.[29,30] These formulations use the self-similar structure of the underlying physical mechanisms to capture the scale-independent behaviour of the plasma turbulence.[31] Fractional diffusion models can be formulated in the form of integral equations, usually both in space and in time, replacing standard transport equations for density, pressure, angular momentum, etc. by those using fractional derivatives (hence integrals over space and time). Such models are more advanced but inevitably more complicated than what we propose here. The additional parameters they introduce are similarity indices, whose determination requires a detailed knowledge of the turbulence that drive them. It is suggested that the transport becomes more fractional in character (as opposed to diffusive) when the coupling with self-consistently generated sheared flows are taken into account.[32,33]

In contrast the formulation we propose here adds one more reaction-diffusion type equation (for the heat flux) to the standard transport formulation and two parameters, which define that equation. This is a weakly nonlocal formulation in which an added perturbation has to actually propagate via a local differential equation in order to have an effect elsewhere in the system. We argue that this formulation is suitable for plasma turbulence, already introduces important non-local physics, and may be adopted more easily in existing state-of-the art transport models that are regularly used by experimentalists.

The rest of the paper is organized as follows. In the remainder of Sec. I, we will discuss the physical meanings of the two transport coefficients that are proposed. In Sec. II, we will show that a relation that is equivalent to Eq. (5) can be derived for plasma turbulence using the drift-kinetic formulation. We discuss the implications of this, for entropy production and give a basic expression for the entropy production rate due to turbulence spreading in Sec. III. Section IV discusses an extension of the general quasi-linear theory for the velocity space distribution function. In Sec. V, we conclude and discuss our results.

A. Physical meanings of τ and λ

As explained graphically in Figure 1, τ is the mean response time of the heat flux to the changes in temperature gradient. One interesting analogy, which can be made to clarify its meaning, is between the radial transport of heat in tokamak and the well known problem of traffic flow. Of course the traffic flow is hindered rather than driven by gradients; nonetheless, the changes in the car density may lead to traffic jams or rarefaction waves. In this problem the response time τ is the response time of the drivers to the changes on the local traffic conditions.[34,35]

The physical meaning of λ is explained in Figure 2. It is basically the radial cross-correlation length between the gradient and the flux. In order to see this, one can consider the Green's function solution of the Guyer-Krumhansl relation as given by Eqs. (5) and (6).

$$Q(r,t) = -\int K(t,t';r,r')\nabla_{r'}T(r',t')dr'dt',$$

where $K(t,t',r,r') = \chi_i R(t,t';r,r')$ and the Green's function (for $t > t'$)

$$R(x,x';t,t') = \frac{\sqrt{\tau}}{2\lambda\sqrt{\pi(t-t')}}e^{-\left[\frac{\tau}{\lambda^2}\frac{(x-x')^2}{4(t-t')}+(t-t')/\tau\right]}. \tag{7}$$

Note that while the formulation using the kernel $K(t,t';r,r')$ relates to a generic non-local transport description (e.g., as discussed in Refs. 36 and 37), its particular form is based on the idea that the nonlocality remains "weak" so that a description using differential equations instead of the more general integro-differential equations is possible at the "transport" level.

B. Further studies of avalanches

The telegraph equation (4) can describe avalanche-like dynamics. Similar to the wave-equation, it can describe radially propagating waves or single pulses depending on how it is perturbed. It is a linear equation and gives a radial propagation speed $v_r = \sqrt{\chi_i/\tau}$. While the coefficients themselves are in general non-linear via their dependence on turbulence intensity for the plasma transport problem, since the telegraph equation is driven using an expansion around an equilibrium, it still is linear for δT. While the equation gives a speed of propagation, the direction of propagation is not determined by this equation itself. This is due to the fact that an important physical element is indeed missing in this formulation and that is the formation of the avalanche.

One way to describe avalanches as propagating nonlinear structures has been explored in the context of self organized criticality (SOC),[38,39] in the form of a nonlinear correction to the flux-gradient relation coming from a symmetry argument based on the fact that while blobs move down the gradients, holes tend to move up. This nonlinear effect is discussed in Ref. 34 for a dynamical formulation for the heat flux as is the case in the current paper. The advantage of a formulation in terms of dynamics of flux is that in such a formulation, the time evolution of the sheared flow, which is a mesoscale phenomenon, can be coupled to the time evolution of the flux. Since the $E\times B$ shear reduces the flux, and a reduced flux increases the gradient, and an increased gradient gives an increased $E\times B$ shear, a simple of flux/gradient/$E\times B$ shear relation may conclude that the final state would be the extinction of the anomalous flux. While this is a possible state, since all these couplings (i.e., shear suppression, change of the gradient due to change of the flux, and change of the flux due to change of the gradient, or even the radial force balance that describe the change of the $E\times B$ shear in response to a change of the gradients) take finite time and has finite extent of effectiveness; in reality, mesoscale dynamics give rise to rich dynamical behavior rather than simply going to the fixed point and sitting there. It is sometimes argued based on numerical observations that avalanches of heat flux move against the $E\times B$ velocity shear direction[40] and that staircases in the form of multiple shear layers[36] form from the edge towards the core.[41] Capturing of these effects require, in the least, a dynamical equation for the evolution of $E\times B$ shear and inclusion of the well known effect of $E\times B$ shear on turbulent decorrelation.[42] Similarly, in order to capture the non-gaussian statistics associated with avalanches,[43,44] one has to include the nonlinear terms associated with the SOC structure and the coupling to $E\times B$ shear since the fractional transport is linked to this self-consistent nonlinear dynamical evolution.[32]

II. DRIFT KINETIC DERIVATION OF GENERAL CONSTITUTIVE RELATION

Equations (2) and (5) are justified in a generic way based on the idea that the heat-flux responds to changes in the gradient in a finite response time, and it has an effect over a correlation length. This gives an evolution equation for the heat flux (i.e., an ordinary or a partial differential equation instead of an algebraic relation) that is proposed as an addition to the classical transport formulation. The transport equations for heat, particles, and momenta can in general be obtained by taking the corresponding moments of the drift-kinetic equation (e.g., Ref. 45). Here we show a more rigorous derivation of the evolution equation for the heat flux (i.e., Eq. (5)) using the corresponding moment of a drift-kinetic equation where fluctuations and mean are separated and fluctuation driven, anomalous heat flux is considered.

Consider the drift kinetic equation

$$\left(\frac{\partial}{\partial t}+v_\parallel\nabla_\parallel+\mathbf{v}_d(v)\cdot\nabla+\mathbf{v}_E\cdot\nabla+\frac{q}{m}E_\parallel\frac{\partial}{\partial v_\parallel}\right)f=c(f). \tag{8}$$

Let us separate the mean and fluctuating parts of the distribution function as $f=\bar{f}+\tilde{f}$ (note that $\bar{f}$ evolves in time and its functional form is not limited to a Maxwellian), the $E\times B$ velocity as $\mathbf{v}_E=\bar{\mathbf{v}}_E+\tilde{\mathbf{v}}_E$, and the parallel electric field as $E_\parallel=\bar{E}_\parallel+\tilde{E}_\parallel$.

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+v_\parallel\nabla_\parallel+\mathbf{v}_d(v)\cdot\nabla+\bar{\mathbf{v}}_E\cdot\nabla+\frac{q}{m}\bar{E}_\parallel\frac{\partial}{\partial v_\parallel}\right)\tilde{f}\\&\quad+\tilde{\mathbf{v}}_E\cdot\nabla\bar{f}+\frac{q}{m}\tilde{E}_\parallel\frac{\partial}{\partial v_\parallel}\bar{f}+\delta\tilde{\mathbf{v}}_E\cdot\nabla\tilde{f}+\delta\frac{q}{m}\tilde{E}_\parallel\frac{\partial}{\partial v_\parallel}\tilde{f}=0,\end{aligned} \tag{9}$$

where the notation is $\delta\tilde{a}\tilde{b}=\tilde{a}\tilde{b}-\langle\tilde{a}\tilde{b}\rangle$.

The evolution of the two-point correlation function $\langle\tilde{f}_1\tilde{f}_2\rangle\equiv\langle\tilde{f}(x_1,v_1,t)\tilde{f}(x_2,v_2,t)\rangle$ can be obtained by writing Eq. (9) at x_1,v_1, multiplying it with $\tilde{f}(x_2,v_2,t)$, writing it again at x_2,v_2 and multiplying that with $\tilde{f}(x_1,v_1,t)$, adding the two and ensemble averaging. This gives a complicated expression, which can be written as

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\mathscr{L}_1+\mathscr{L}_2\right)\langle\tilde{f}_2\tilde{f}_1\rangle+\mathscr{P}_1\bar{f}_1+\mathscr{P}_2\bar{f}_2\\&\quad=\nabla_1\cdot\Gamma_{\mathcal{Q}_1}+\nabla_2\cdot\Gamma_{\mathcal{Q}_2}+\partial_{v_{\parallel1}}\mathcal{K}_1+\partial_{v_{\parallel2}}\mathcal{K}_2,\end{aligned} \tag{10}$$

where

$$\begin{aligned}\mathscr{L}_{1,2}&=v_{1,2\parallel}\nabla_{\parallel1,2}+\frac{q}{m}\bar{E}_{\parallel1,2}\frac{\partial}{\partial v_{\parallel1,2}},\\\mathscr{P}_{1,2}&=\langle\tilde{f}_{2,1}\tilde{\mathbf{v}}_{E1,2}\rangle\cdot\nabla_{1,2}+\frac{q}{m}\langle\tilde{f}_{2,1}\tilde{E}_{\parallel1,2}\rangle\frac{\partial}{\partial v_{\parallel1,2}},\\\Gamma_{\mathcal{Q}_{1,2}}&=\langle\tilde{\mathbf{v}}_{E1,2}\tilde{f}_2\tilde{f}_1\rangle,\\\mathcal{K}_{1,2}&=\langle\tilde{E}_{\parallel1,2}\tilde{f}_2\tilde{f}_1\rangle,\end{aligned}$$

where we also dropped the curvature term for simplicity, whose effect is well-known in transport formulation and can be described using the turbulent equipartition ideas.[46–48]

If we further assume that the electrons are adiabatic, we can write the heat flux $Q = Q_{12}|_{2\to 1}$, using the corresponding two-point cross-correlation

$$Q_{12} \equiv -\frac{m}{2}\int v_1^2 \frac{v_{ti}}{n_0}\rho_i \partial_{y_2}\langle \tilde{f}_1 \tilde{f}_2\rangle d^3v_1 d^3v_2. \quad (11)$$

The evolution equation for this two-point cross-correlation can be obtained simply by taking the corresponding moment [i.e., Eq. (11)] of the two-point drift-kinetic equation [i.e., Eq. (10)]. This gives Eq. (A1). One can then take the $2 \to 1$ limit, as discussed in the appendix to obtain the equation for the heat flux

$$\left(\frac{\partial}{\partial t} + \bar{\mathbf{v}}_E\cdot\nabla\right)Q + \langle \tilde{v}_{Er}\nabla_\parallel \tilde{q}_\parallel\rangle - \rho_i v_{ti}\langle \tilde{P}\partial_y \nabla_\parallel \tilde{v}_\parallel\rangle$$
$$- q\bar{E}_\parallel \langle \tilde{v}_{Er}\tilde{v}_\parallel\rangle + \langle \tilde{v}_{Er}\tilde{\mathbf{v}}_E\rangle\cdot\nabla\bar{P} - \frac{v_{ti}}{n_0}\rho_i\langle \tilde{P}\partial_y \tilde{\mathbf{v}}_E\rangle\cdot\nabla\bar{n}$$
$$+ \nabla\cdot\langle \tilde{\mathbf{v}}_E \tilde{v}_{Er}\tilde{P}\rangle - q\int v_1\langle \tilde{E}_{\parallel 1}\tilde{v}_{Er,2}\tilde{f}_1\rangle d^3v_1 = 0,$$

which we can reduce, using a 1-D transport formulation (assuming flat density in order to focus on heat flux evolution alone)

$$\frac{\partial}{\partial t}Q + n_0\langle \tilde{v}_{Er}^2\rangle \nabla_r \bar{T} + \nabla_r\cdot\Gamma_Q + \gamma_{nl}\langle \tilde{v}_{Er}^2\rangle Q = 0,$$

where

$$Q = \frac{m}{2}\int v^2\langle \tilde{v}_{Er}\tilde{f}(v)\rangle d^3v,$$
$$\Gamma_Q = \frac{m}{2}\int v^2\langle \tilde{v}_{Er}^2\tilde{f}(v)\rangle d^3v.$$

Note that

$$\frac{\partial}{\partial t} \to \left(\frac{\partial}{\partial t} + \bar{v}_{Ey}(r)\frac{\partial}{\partial y}\right)$$

can be used for including the effect of a sheared mean radial electric field in this equation. However the way this term should be treated is out of scope of this paper and is considered elsewhere.

If a quasi-linear model similar to that commonly used in turbulence spreading for the "flux of heat flux" is used

$$\Gamma_Q \sim -D(\mathcal{E})\nabla_r Q,$$

with $D(\mathcal{E}) \approx D_0\mathcal{E}^\alpha$, and $\alpha = 1$ for weak and 1/2 for strong turbulence (i.e., $\mathcal{E} \sim \langle \tilde{v}_{Er}^2\rangle$) we obtain

$$\tau\frac{\partial}{\partial t}Q + \chi\mathcal{E}\nabla_r\bar{T} - \nabla_r\cdot(\tau D(\mathcal{E})\nabla_r Q) + \mathcal{E}Q = 0. \quad (12)$$

Note that we multiplied the equation by $\tau \sim \gamma_{\rm nl}^{-1}$ so that $\chi \propto \gamma_0/\gamma_{\rm nl}$ with $\gamma = \gamma_0\nabla_r\bar{T}$ being a linear growth rate and that the steady state solution takes the form of a Fick's law. Note that the steady state solution of turbulence intensity is $\mathcal{E} = \gamma_0/\gamma_{\rm nl}$, so that in steady state we also have $\chi \propto \mathcal{E}$.

In the above equation, the first term is simply the delayed response of the flux. The second term corresponds to the linear growth rate of the fluctuation intensity and thus is proportional to the temperature gradient since $\gamma \sim \gamma_0\nabla_r T$ for ITG (e.g., a critical gradient model may be used to justify this). It is also the term that gives the Fick's law in the local stationary limit by balancing the nonlinear term. The third term is the flux of heat flux, which is modeled using a simple quasi-linear expression, and the final term is simply the turbulent eddy damping which maybe argued to come from $-\langle \nabla_y\cdot(D^y(\mathcal{E})\nabla_y Q)\rangle \to D_0^y k_y^2\mathcal{E}Q = \gamma_{\rm nl}\mathcal{E}Q$ where k_y is roughly the wavelength of the maximum of the growth rate. The constitutive relation (12) that we find above is equivalent to a nonlinear version of the Guyer-Krumhansl relation.

III. ENTROPY PRODUCTION DUE TO SPREADING AND RELATION TO EXTENDED IRREVERSIBLE THERMODYNAMICS

The usual definition of the entropy production due to heat flux

$$\sigma = -\frac{Q}{T^2}\nabla_r T \quad (13)$$

is positive definite when one models the heat flux using Fick's law

$$Q = -\chi\nabla_r T$$

as long as χ is positive. However, Eq. (2) gives

$$Q = -\chi\nabla_r T - \tau\frac{\partial}{\partial t}Q, \quad (14)$$

and when this is substituted into Eq. (13), we obtain

$$\sigma = \frac{\chi}{T^2}(\nabla_r T)^2 + \frac{\tau}{T^2}(\nabla_r T)\frac{\partial}{\partial t}Q,$$

and *there is no guarantee that the second term remains positive*. In fact for propagating heat waves, this term oscillates between positive and negative values, and depending on the values of τ and χ may lead to "negative entropy production."

This is due to the fact that the definition of entropy that is used in this classical formulation corresponds to a "local equilibrium" definition and does not include any deviation from the local equilibrium, which contradicts the fact that heat flux varies in time. In order to remedy this, the formulation of extended irreversible thermodynamics (EIT)[49] was proposed, using

$$\sigma_{EIT} = -\frac{Q}{T^2}\left(\nabla_r T + \frac{\tau}{\chi}\frac{\partial}{\partial t}Q\right),$$

which gives positive definite entropy production together with Eq. (14). This extension is proposed to make σ_{EIT} independent of the time dependent term. It relies on the idea that in the nonequilibrium steady state with a well defined heat flux Q, the entropy is a function of both T and Q, i.e.,

$$\frac{\partial}{\partial t}S(T,Q)=\frac{\partial T}{\partial t}\frac{\partial S}{\partial T}+\frac{\partial Q}{\partial t}\frac{\partial S}{\partial Q}=\frac{\partial S_{eq}}{\partial t}+\frac{\partial Q}{\partial t}\frac{\partial S}{\partial Q}$$

with $\frac{\partial S}{\partial Q}=-\frac{\tau Q}{\chi T^2}$. We can actually use this form to compute the entropy production in the case of Guyer-Krumhansl as

$$\frac{\partial S(T,Q)}{\partial t}=\frac{\partial}{\partial t}\ln T-\frac{\tau Q}{\chi T^2}\frac{\partial Q}{\partial t},$$

where the first term can be interpreted as coming from a simplified Sackur-Tetrode formula where only T changes (i.e., N, V, etc. are constants), and the second term is the extension due to heat flux dependence of entropy. Above combination can be obtained by choosing a region where H is not important, multiplying Eq. (3) by $1/T$ and Eq. (5) by $Q/(\chi T^2)$ and subtracting the latter from the former. This gives

$$\frac{\partial S(T,Q)}{\partial t}=-\frac{1}{T}\nabla_r\cdot Q+\frac{Q}{T^2}\nabla_r T+\frac{\tau Q}{\chi T^2}\nabla_r\cdot\Gamma_Q+\frac{Q^2}{\chi T^2},$$

which gives

$$\sigma_{EIT}=\frac{Q^2}{\chi T^2}-\Gamma_Q\nabla_r\left(\frac{\tau Q}{\chi T^2}\right)$$

with an entropy flux defined as $J_s\equiv Q/T-(\tau Q/\chi T^2)\Gamma_Q$. This form of extended entropy production has the advantage of including the contribution of turbulence spreading to entropy production explicitly. Using the coefficients as they appear in Eq. (12), this becomes

$$\sigma_{spr}=-\Gamma_Q\nabla_r\left(\frac{\tau Q}{\chi\mathcal{E}T^2}\right).$$

Note that the basic form of the entropy production due to non-local heat transport corresponds in a sense to consider the gradient of the heat flux as a "thermodynamic force" and its flux as the conjugate thermodynamic "flux" in accordance with the above formulation.

IV. QUASI-LINEAR PHASE-SPACE EVOLUTION OF $\bar{f}$

We have seen in Secs. I and II, when the gradients are allowed to dynamically couple to fluxes, the gradient-flux relations are no longer algebraic. While this is a generic argument, it also applies equally to quasi-linear transport theories.

In the same spirit, quasi-linear theories of phase space diffusion (which include diffusion in velocity space $D_{v,v}$ as well as cross terms $D_{v,r}$ and $D_{r,v}$ in addition to the usual spatial diffusion $D_{r,r}$ of transport—see Appendix B) can also be extended, if the time evolution of $\bar{f}$ is such that a temporal scale separation with $\tilde{f}$ is not fully justified. In this limit instead of constant diffusion coefficients (or algebraic flux-gradient relations in phase space), we have a dynamically coupled system of three equations (see Appendix B for the derivation)

$$\left(\frac{\partial}{\partial t}+R^{-1}\right)Q_f(v)+\langle\tilde{v}_{Er}^2\rangle\nabla_r\bar{f}+\frac{q}{m}\langle\tilde{v}_{Er}\tilde{E}_\parallel\rangle\frac{\partial}{\partial v_\parallel}\bar{f}=0, \quad (15)$$

$$\left(\frac{\partial}{\partial t}+R^{-1}\right)K_{\parallel f}(v)+\langle\tilde{v}_{Er}\tilde{E}_\parallel\rangle\nabla_r\bar{f}+\frac{q}{m}\left\langle\tilde{E}_\parallel^2\right\rangle\frac{\partial}{\partial v_\parallel}\bar{f}=0, \quad (16)$$

$$\left(\frac{\partial}{\partial t}+v_\parallel\nabla_\parallel+\omega_D\right)\bar{f}+\nabla_r Q_f(v)+\frac{q}{m}\frac{\partial}{\partial v_\parallel}K_{\parallel f}(v)=C(\bar{f}). \quad (17)$$

This is a generalization of the Maxwell-Catteneo idea to the phase space consisting of parallel velocity and radial space coordinate. Here the form of the response function can be taken as usual as $R_k=i(\omega_k-v_\parallel k_\parallel-\omega_D+\Delta\omega_k)^{-1}$ consistent with the linear problem. Note that the existence of the $\Delta\omega_k$ in the response function explains why we do not have an explicit eddy damping term in Eqs. (15) and (16). In steady state, the solution of Eqs. (15) and (16) are Eqs. (B1) and (B2), respectively, which gives Eq. (B3) when substituted into Eq. (17). The more general dynamical system (15)–(17) also includes the phenomenon of fast radial propagation. Note that turbulence spreading and an analogous term for the radial flux of turbulent parallel pumping $\langle\tilde{E}_\parallel\tilde{f}\rangle$ can easily be added to above formulation (in other words this is more of an analogue to Maxwell-Cattaneo than Guyer-Krumhansl). Note also that simply by taking the v^2 moments of Eqs. (15) and (17), we recover the nonlinear Cattaneo constitutive relation (12) and the corresponding heat transport equation (3).

V. CONCLUSION

We introduced a formulation of nearly local (or weakly nonlocal) dynamics of turbulent transport by modifying the flux-gradient relations to incorporate spatial coupling, and response times. The resulting constitutive relation is shown to be equivalent to the Guyer-Krumhansl constitutive relation that is widely used in the study of transport in different disciplines such as heat transport in liquid helium and is related to the phenomenon known as second sound. It is found that the avalanches, in the form of coupled propagation of temperature and heat flux modulations, appear as a result of the time response τ (i.e., not only as a result of the spatial coupling λ) and has a radial propagation speed $v=\sqrt{\chi\mathcal{E}/\tau}$. This suggests that avalanches that appear via a coupling between profile and flux modulations would be unphysically damped in "fixed gradient" or globally "pinned profile" formulations. This is due to the fact that in these formulations a large deviation of the profile from its "pinned" form is unphysically damped rather than allowed to propagate radially as the coupled system tends to do.

The formulation introduces two new transport coefficients, τ and λ, which can be measured in experiment or determined in the gyrokinetic framework as an input to enhanced transport models that include heat flux as a dynamic transport variable. Physical meanings of these coefficients are discussed. It is noted that τ is analogous to the average response time of the individual drivers to the changes in the road conditions in a analogous traffic flow problem and λ is the spatial scale associated with the spatial diffusion of heat flux as clarified by the explicit analytical form given in Eq. (7) of the Green's function.

We have further shown that the constitutive relation that we propose can indeed be derived from the drift-kinetic equation. The derivation clarifies the microscopic meaning of both the response time τ and the radial flux diffusion length λ. It also shows that indeed the coefficients in our Guyer-Krumhansl relation are in fact functions of the turbulent intensity, making the equation effectively nonlinear.

We have also discussed the link to entropy production. Note that as the fluxes evolve in time, the classical thermodynamical formulation in terms of fluxes and forces becomes inadequate (since now, in principle the flux can be against the thermodynamical force for a short period of time as the system evolves). Extended irreversible thermodynamics was introduced exactly to consider this anomaly. Applying it to the constitutive relation that we introduced, we find that the production of entropy associated with turbulence spreading is $\sigma_{spr} = -\Gamma_Q \nabla_r\left(\frac{\tau Q}{\chi \mathcal{E} T^2}\right)$.

Furthermore, we proposed a dynamic quasi-linear drift kinetic model, which is similar in spirit, and reduces in the steady state limit to Dupree's renormalized drift-kinetic equation. When the v^2 moment of these equations are computed, we recover the Cattaneo constitutive relation and the corresponding heat transport equation, which give the telegraph equation, and propagating wave solutions, which may be related to avalanches.

We also suggest that the phenomena described in this paper, such as a finite response time τ or a finite radial response size λ of the flux to the gradient could possibly be observed during modulation experiments for temperature using ECE for detailed temperature measurements and advanced transport modeling including the time evolution of heat flux (replacing a heat pinch). Another possibility is to look at particle transport, and measure fluxes directly using probes, and density corrugations using fast sweep reflectometry or BES, and trying to get the response time and depth directly. An H-L transition may provide the sudden change in plasma conditions that may be required to see the relaxation.

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APPENDIX A: TWO-POINT FLUX

The equation for the evolution of the two-point cross-correlation defined in Eq. (11), which is obtained by taking the corresponding moment of Eq. (10), is

$$\left(\frac{\partial}{\partial t} + \bar{\mathbf{v}}_E \cdot (\nabla_1 + \nabla_2)\right) Q_{12} + \nabla_{\parallel 1}\langle \tilde{v}_{Er,2}\tilde{q}_{\parallel 1}\rangle - \rho_i v_{ti} \nabla_{\parallel 2}\partial_{y_2}(\tilde{P}_1 \tilde{v}_{2\parallel}) - q\bar{E}_{\parallel 1}\langle \tilde{v}_{Er,2}\tilde{v}_{\parallel 1}\rangle + \langle \tilde{v}_{Er,2}\tilde{\mathbf{v}}_{E1}\rangle \cdot \nabla_1 \bar{P}_1$$
$$- \frac{v_{ti}}{n_0}\rho_i \partial_{y_2}\langle \tilde{P}_1 \tilde{\mathbf{v}}_{E2}\rangle \cdot \nabla_2 \bar{n}_2 + \nabla_2 \cdot \langle \tilde{\mathbf{v}}_{E2}\tilde{v}_{Er,2}\tilde{P}_1\rangle + \nabla_1 \cdot \langle \tilde{\mathbf{v}}_{E2}\tilde{v}_{Er,2}\tilde{P}_1\rangle - q\int v_1 \langle \tilde{E}_{\parallel 1}\tilde{v}_{Er,2}\tilde{f}_1\rangle d^3 v_1, \tag{A1}$$

where

$$\tilde{q}_\parallel \equiv \frac{m}{2}\int v^2 v_\parallel \tilde{f} d^3 v,$$
$$\tilde{P} = \frac{m}{2}\int v^2 \tilde{f} d^3 v.$$

Note that the use of fluid quantities in Eq. (A1) is simply for notational simplicity, and otherwise the expression makes no assumption about the underlying turbulence (i.e., the fluctuations can be fully kinetic).

APPENDIX B: QUASI-LINEAR PHASE SPACE DIFFUSION

In the spirit of WKB, we write

$$(-i\omega + iv_\parallel k_\parallel + i\omega_D)\tilde{f}_k \approx -\left(\tilde{v}_{Er,k}\cdot \nabla_r + \frac{q}{m}\tilde{E}_\parallel \frac{\partial}{\partial v_\parallel}\right)\bar{f},$$

which is linearized so that it has the coupling between mean and fluctuating quantities, but not the nonlinear terms.

We can still define an instantaneous linear dispersion relation via

$$\frac{e}{T}\tilde{\Phi}_k = \frac{1}{n}\int \tilde{f}_k d^3 v = \frac{1}{n}\tilde{\Phi}_k \int \frac{\left(k_y \frac{c}{B}\nabla_r \bar{f} + \frac{q}{m}k_\parallel \frac{\partial \bar{f}}{\partial v_\parallel}\right)}{(\omega - v_\parallel k_\parallel - \omega_D)} d^3 v$$

as

$$\varepsilon(\omega, \bar{f}) = 1 - P_k(\omega, \bar{f}) = 0$$

with

$$P_k(\omega, \bar{f}) = \frac{1}{n}\int \frac{\left(k_y \frac{c}{B}\nabla_r \bar{f} + \frac{q}{m}k_\parallel \frac{\partial \bar{f}}{\partial v_\parallel}\right)}{(\omega - v_\parallel k_\parallel - \omega_D)} d^3 v.$$

When we take $\bar{f}$ as a local Maxwellian, we recover the usual linear dispersion relation for the ITG. In contrast, when $\bar{f}$ is allowed to evolve, the dispersion relation also evolves.

For instance, we can compute the instantaneous flux of the distribution function (which in principle contains the information about fluxes of all its moments) as

$$\langle \tilde{v}_{Er}\tilde{f}\rangle = \mathrm{Re}\sum_k \tilde{v}^*_{Er,k}\tilde{f}_k$$
$$= \mathrm{Re}\sum_k i\frac{c}{B}k_y\frac{|\Phi_k|^2}{n}\frac{\left(k_y\frac{c}{B}\nabla_r\bar{f}+\frac{q}{m}k_\parallel\frac{\partial\bar{f}}{\partial v_\parallel}\right)}{(\omega_k - v_\parallel k_\parallel - \omega_D)},$$

and

$$\langle \tilde{E}_\parallel\tilde{f}\rangle = \mathrm{Re}\sum_k ik_\parallel\frac{|\Phi_k|^2}{n}\left[\frac{\left(k_y\frac{c}{B}\nabla_r\bar{f}+\frac{q}{m}k_\parallel\frac{\partial\bar{f}}{\partial v_\parallel}\right)}{(\omega_k - v_\parallel k_\parallel - \omega_D)}\right].$$

In other words, the generalized Fick's law takes the form

$$Q_f(v) = -D_{r,r}(v,k)\nabla_r f - D_{r,v}\frac{\partial}{\partial v_\parallel}\bar{f}, \tag{B1}$$

$$K_{\parallel f}(v) = -D_{v,r}(v,k)\nabla_r f - D_{v,v}\frac{\partial}{\partial v_\parallel}\bar{f}, \tag{B2}$$

where the quasi-linear phase space diffusion coefficients can be written as

$$D_{r,r}(v,k) = \mathrm{Re}\sum_k R_k n^{-1}|\tilde{v}_{Er,k}|^2,$$
$$D_{r,v}(v,k) = D_{v,r}(v,k) = \mathrm{Re}\sum_k R_k\frac{q}{m}n^{-1}\tilde{v}_{Er,k}\tilde{E}^*_{\parallel,k},$$
$$D_{v,v}(v,k) = \mathrm{Re}\sum_k R_k\left(\frac{q}{m}\right)^2 n^{-1}|\tilde{E}_{\parallel,k}|^2.$$

These "closures" allow us to write

$$\left(\frac{\partial}{\partial t}+v_\parallel\nabla_\parallel+\bar{\mathbf{v}}_E\cdot\nabla\right)\bar{f}+\nabla_r\cdot\left[D_{r,r}(v,k)\nabla_r f + D_{r,v}\frac{\partial}{\partial v_\parallel}\bar{f}\right]$$
$$+\frac{\partial}{\partial v_\parallel}\left[D_{v,r}(v,k)\nabla_r f + D_{v,v}\frac{\partial}{\partial v_\parallel}\bar{f}\right] = C(\bar{f}) + H(v,x), \tag{B3}$$

which is a renormalized evolution equation for $\bar{f}$ ala Dupree[50] as long as the response function R_k is properly renormalized.

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