Nonlinear Triad Interactions and the Mechanism of Spreading in Drift-Wave Turbulence

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(Received 28 February 2006; published 14 July 2006)

We present the results of a derivation of the fluctuation energy transport matrix for the two-field Hasegawa-Wakatani model of drift wave turbulence. The energy transport matrix is derived from a two-scale direct interaction approximation assuming weak turbulence. We examine different classes of triad interactions and show that radially extended eddies, as occurs in penetrative convection, are the most effective in turbulence spreading. We show that in the near-adiabatic limit internal energy spreads faster than the kinetic energy. Previous theories of spreading results are discussed in the context of weak turbulence theory.

DOI: 10.1103/PhysRevLett.97.024502 PACS numbers: 47.65.−d, 47.27.tb, 47.27.eb, 47.27.Jv

Turbulence spreading or radial transport of turbulence energy occurs because inhomogeneous turbulence generically tends to relax its intensity gradients and to entrain laminar or more weakly turbulent regions, which may be marginal or damped locally (i.e., regions which do not access external free energy sources directly). This causes spreading of turbulence from regions that are strongly driven (i.e., unstable regions) into regions which are locally damped or only weakly driven. Although the modeling of turbulence spreading in terms of intensity dependent "eddy diffusion" is an element of virtually every K-ϵ type model of fully developed turbulence, there is no corresponding model of turbulence spreading for "wave turbulence" in inhomogeneous, anisotropic media, such as magnetic fusion energy (MFE) plasmas. In MFE plasmas, which are relatively weakly turbulent, spreading occurs via linear or nonlinear mode couplings, i.e., toroidal coupling or wave interaction. Although the study of turbulence spreading in magnetically confined plasmas have started in the literature with a theoretical investigation [1], the interest in the subject is motivated by many observations of turbulent fluctuations in locally stable or damped regions of both simulations [2,3] and physical experiment [4].

Moreover, turbulence spreading is generally related to transport barrier back transitions and has recently been recognized [5–8] as an important issue in anomalous transport. It has also been linked to the breaking of gyro-Bohm scaling [9–11] in transport theory. Nevertheless, the study of nonlinear interactions in an inhomogenous medium in a general sense is not new. It is studied previously in the context of Langmuir waves [12], where the dynamics of mean flow and turbulence are represented by the equations of ideal hydrodynamics. More recently, a bivariate Burger's equation (also in qualitative agreement with ideal hydrodynamic hypothesis) was introduced for the nonperturbative dynamics of inhomogenous turbulence [13,14]. It is also noteworthy that a model of coherent three-wave coupling in an inhomogenous media was previously considered in order to explain the laser pellet irradiation problem [15].

Nonetheless, the current understanding of the fundamental dynamics of turbulence spreading as a result of nonlocal, nonlinear mode couplings is far from being satisfactory. In particular, the relation between the mechanisms of spreading and those of nonlinear wave interaction processes in drift wave turbulence is not well understood. Interactions involving zonal flows (i.e., $q_y = 0$, $q_\parallel \to 0$ modes) are especially perplexing in this context, and the subject of some controversy.

In this work, we elucidate the dynamics of turbulence spreading in weak, wave turbulence, and present a model of the process in terms of resonant three-wave interactions. We examine three major classes of triad interactions: "equilateral" triads of general orientation (couplings between three drift waves of comparable scale), isosceles triads when one of the legs is a long wavelength, low k convective cell (i.e., couplings between two drift waves and a hydrodynamic perturbation) and study two subcases of the latter, namely, the case where the short leg of the triad interaction is a zonal flow (i.e., $q_y = 0$, $q_\parallel \to 0$ $q_x \neq 0$), and the case when it is a streamer (i.e., $q_x = 0$, $q_\parallel \to 0$, $q_y \neq 0$). In view of the weak turbulence approximation, we consider only resonant interactions. We show that it is possible, within this framework, to systematically study the mechanism of spreading in weak wave turbulence and to quantitatively assess the relative importance of different classes of wave-wave interactions. This is accomplished by first examining the general structure of the intensity flux and then later constructing a transport model for the multicomponent fluctuation energy density.

Wave-dominated turbulence has certain essential features, which are fundamentally different from the conventional paradigm of fully developed Navier-Stokes turbulence. In fully developed fluid turbulence, it is common to model the overall effect of small scales on large scales as an effective "eddy" diffusivity—an effect which appears in all K-ϵ models. However, in the case of drift-wave turbulence, wave resonances play an essential role in regulating the diffusion induced by small scales, and neither homogeneity nor isotropy can be assumed. One way to deal with this problem is to compute the three-wave interaction induced *flux of turbulence energy*, using a Markovian two-scale direct interaction approximation (TSDIA) [16], assuming weak turbulence and weak mean flows. Weak turbulence, in this picture, implies taking only resonant contributions to the fluxes. This approach implicitly makes use of the random phase approximation (RPA) commonly made in weak turbulence studies. Use of weak turbulence here is mainly based on the observation that in most numerical simulations of plasma turbulent transport coefficients scale linearly with intensity (i.e., $D \sim |e\Phi/T_e|^2$ instead of $D \sim |e\Phi/T_e|$). The basic formulation of the model is based on a two-scale approach (i.e., $\Phi = \sum_{\mathbf{k}} \Phi_{\mathbf{k}}(X)e^{i\mathbf{k}\cdot\mathbf{x}}$) and requires a computation of the fluxes of nonlinear kinetic (i.e., $K = \langle|\nabla\Phi|^2\rangle$) and internal energy (i.e., $N = \langle n^2\rangle$) in the radial direction. The latter are roughly similar in various drift wave-turbulence models:

$$\Gamma_K = \frac{1}{2}\langle\Phi^2\hat{\mathbf{z}}\times\nabla(\nabla^2\Phi)\rangle_X = \mathrm{Re}\left[\frac{i}{6}\sum_{\mathbf{p}+\mathbf{q}+\mathbf{k}=0}(q_y q^2 + p_y p^2 + k_y k^2) \times\langle\Phi_{-\mathbf{k}}\Phi_{-\mathbf{q}}\Phi_{-\mathbf{p}}\rangle\right], \tag{1}$$

$$\Gamma_N = -\left\langle\frac{n^2}{2}\partial_Y\Phi\right\rangle = -\mathrm{Re}\left[\frac{i}{6}\sum_{\mathbf{p}+\mathbf{q}+\mathbf{k}=\mathbf{0}}(k_y\langle\Phi_{\mathbf{k}}n_{\mathbf{p}}n_{\mathbf{q}}\rangle + p_y\langle\Phi_{\mathbf{p}}n_{\mathbf{q}}n_{\mathbf{k}}\rangle + q_y\langle\Phi_{\mathbf{q}}n_{\mathbf{k}}n_{\mathbf{p}}\rangle)\right], \tag{2}$$

which are derived by substituting the Fourier expansions into the expressions for flux and averaging. Notice that the factor $1/6$ is the result of writing the permutations of wave numbers, explicitly. There are several observations that can be made directly upon examination of these expressions. First, close to the adiabatic limit (i.e., $\frac{k_\parallel^2 v_{\mathrm{the}}^2 L_n}{c_s} > 1$), where the dispersion relation for drift waves is $\omega_{\mathbf{k}} \approx k_y/(1 + k^2)$, the kinetic energy flux coefficient $\Lambda \equiv (q_y q^2 + p_y p^2 + k_y k^2)$ vanishes for wavelengths $k \ll 1$ (the most interesting limit for drift waves), whenever the three-wave resonance condition is satisfied (i.e., $\omega_{\mathbf{k}} + \omega_{\mathbf{p}} + \omega_{\mathbf{q}} = 0$, where $\mathbf{p} + \mathbf{q} + \mathbf{k} = \mathbf{0}$). Thus, somewhat surprisingly, the kinetic energy cannot "spread" itself in the adiabatic (i.e. Hasegawa-Mima) limit. Thus, three-wave interactions in drift wave turbulence can be said to possess an element of resiliency or self-binding. Second, if we pick one of the modes as a zonal flow (i.e., $q_y = 0$), the kinetic energy coefficient vanishes regardless of the collisionality limit, whenever there is resonance, since $\Lambda \sim k_y(k^2 - p^2) \sim \Delta\omega(1 + k^2)(1 + p^2)$. However, this is not true for "streamers", which are radially elongated structures (i.e., $q_x = 0$) that are effective in mixing the turbulence in the radial direction, since their flow is radial. This suggests that, not surprisingly, large scale radially extended cells are most effective at spreading turbulence and that zonal flows, which can shear apart such cells, must necessarily inhibit spreading, since they destroy the structures most likely to promote it.

We now proceed to discussion of a transport theory for fluctuation intensity. The aim here is to represent the flux of turbulence energy in drift wave turbulence (i.e., Γ_α, where different values of Greek indices correspond to N and K) as the product of a thermodynamic force vector $\partial_X N^\beta = \{\partial_X\langle\tilde{v}^2\rangle, \partial_X\langle\tilde{n}^2\rangle\}$ and a transport matrix $D^{\alpha\beta}$ so that Γ^α can be written in the form of a Fick's law:

$$\Gamma_\alpha \approx \sum_{\mathbf{p}} D_{\mathbf{p}}^{\alpha\beta}\partial_X N_{\mathbf{p}}^\beta \equiv D^{\alpha\beta}\partial_X N^\beta \tag{3}$$

resembling the flux-force relation form of collisional transport theory of gases. However, we note that here $D_{\alpha\beta}$ has off-diagonal components which are not positive definite and that $D^{KK} \neq D^{NN}$ in general. In (3), the elements of the transport matrix are

$$D_{\mathbf{p}}^{(KK)} \equiv \frac{1}{2}\int d^2\mathbf{k}\int d^2\mathbf{q}\frac{(k_y k^2 + p_y p^2 + q_y q^2)}{k^2 p^2 q^2} \times\delta(\mathbf{k}+\mathbf{p}+\mathbf{q})[\delta(\Delta\omega^+) + \delta(\Delta\omega^-)] \times\{[q_y(q^2 - p^2) + 2p_x\hat{\mathbf{z}}\times\mathbf{q}\cdot\mathbf{p}]K_{\mathbf{q}} + (k_y(k^2 - p^2) - 2p_x\hat{\mathbf{z}}\times\mathbf{q}\cdot\mathbf{p})K_{\mathbf{k}}\}, \tag{4}$$

$$D_{\mathbf{p}}^{(NN)} \equiv \frac{1}{2}\int d^2\mathbf{k}\int d^2\mathbf{q}\left(\frac{q_y^2}{q^2}K_{\mathbf{q}} + \frac{k_y^2}{k^2}K_{\mathbf{k}}\right) \times\delta(\mathbf{k}+\mathbf{p}+\mathbf{q})[\delta(\Delta\omega^+) + \delta(\Delta\omega^-)], \tag{5}$$

$$D_{\mathbf{p}}^{(NK)} \equiv -\frac{1}{2}\int d^2\mathbf{k}\int d^2\mathbf{q}\left(\frac{q_y p_y}{p^2}N_{\mathbf{q}} + \frac{k_y p_y}{p^2}N_{\mathbf{k}}\right) \times\delta(\mathbf{k}+\mathbf{p}+\mathbf{q})[\delta(\Delta\omega^+) + \delta(\Delta\omega^-)], \tag{6}$$

where $\Delta\omega^+$ is the mismatch between frequencies of three growing modes and $\Delta\omega^-$ is the mismatch when one of the modes is damped. The detailed derivation of Eqs. (4)–(6) can be found in [8]. The key steps in the derivation are the assumptions of two-scale evolution for the "beat mode" in the DIA, leading to the Fick's law form, and weak turbulence (also RPA), which results in the δ functions that impose the resonance conditions $\Delta\omega^\pm = 0$. Notice that in the case of the near-adiabatic limit, the damped mode is strongly damped [i.e., $\gamma^- \sim -c(1+k^2)/k^2$], so in this limit $\Delta\omega^-$ should be neglected. It is also useful to note here that we have neglected resonance broadening in the transport matrix, consistent with weak turbulence, and higher order nonlinear effects corresponding to nonlinear corrections to radial group velocity. Thus, the ''drift term'' ($\Gamma \sim V^{\alpha\beta}N^{\beta}$) consisting of these two higher order corrections is neglected. Here $D^{(KK)} \propto K$ is the nonlinear self-diffusion of kinetic energy, $D^{(NN)} \propto K$ is the nonlinear diffusion of internal energy by the drift motions and $D^{(NK)} \propto N$ is a radial stress acting on the local internal energy (and so is an off-diagonal term, and not a ''diffusion'' term).

Notice for the near-adiabatic case, the internal energy may spread, and since the kinetic energy is *linearly* coupled to the internal energy, it would still eventually follow N. In this scenario, N is expected to ''lead'' K at least by a few linear growth times. It is likely that this is a result of the dual cascade that results in N (which forward cascades) being mixed at smaller scales compared to K (which inverse cascades). If we consider the limit of ''passive scalar turbulence''. It is clear that n and n^2 (both being passive scalars) evolve the same way statistically. Since shear flows reduce the transport of n, they should also reduce the transport of n^2. Since the spreading of $N = n^2$ is the dominant cause in the overall spreading of energy, it becomes evident that radially sheared poloidal flows must reduce spreading of energy rather than induce it. Even though this statement is strictly true only for passive scalar evolution and the drift-wave equations are always linearly coupled, it is important to note that shear flow also reduces the linear coupling. These predictions are testable in direct numerical simulations.

We now discuss the predictions of the energy transport theory for three distinct classes of nonlinear interactions, which were previously mentioned. In all cases the energy transport matrix has been computed, explicitly.

When a spectrum consisting of a $q_x = 0$, $(q_\parallel, q_y) \neq 0$ drift wave, and two other drift waves from its resonance manifold (see Fig. 1) is considered in the near-adiabatic limit (further assuming $N_{\mathbf{q}} < N_{\mathbf{k}}$ and $K_{\mathbf{q}} < K_{\mathbf{k}}$, for simplicity), the transport coefficients are

$$D_{\mathbf{p}}^{(NN)} \sim \mathrm{Re}[\theta_{kpq}]\frac{k_y^2}{k_x^2+k_y^2}K_{\mathbf{k}},$$
$$D_{\mathbf{p}}^{(NK)} \sim \mathrm{Re}[\theta_{kpq}]\frac{k_y(q_y+k_y)}{k_x^2+(q_y+k_y)^2}N_{\mathbf{k}}. \tag{7}$$

On the other hand, a spectrum consisting of two drift waves and a zonal flow (or a $q_y = 0$ mode, see Fig. 2), which interact resonantly with each other, yields transport coefficients:

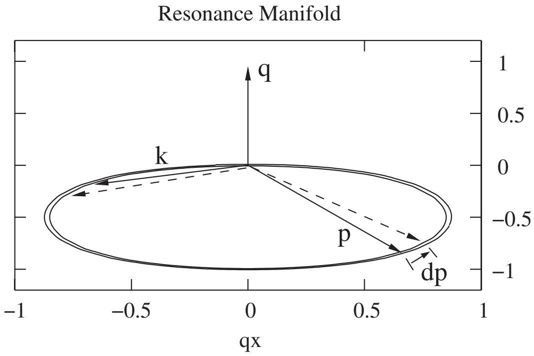

FIG. 1. The manifold of wave numbers that interact resonantly with a $p_x = 0$ mode. In a full spectrum one should integrate over the other two wave numbers of the triad to compute the diffusion coefficients [as in Eqs. (4)–(6)], only taking the resonant interactions into account. This can be achieved by computing the integrals along the resonance manifold as shown (up to a wave-number dependent coefficient).

$$D_{\mathbf{p}}^{(NN)} \sim \mathrm{Re}[\theta_{kpq}]\frac{k_y^2}{q_x^2/4+k_y^2}K_{\mathbf{k}},$$
$$D_{\mathbf{p}}^{(NK)} \sim \mathrm{Re}[\theta_{kpq}]\frac{k_y^2}{q_x^2/4+k_y^2}N_{\mathbf{k}}.$$

For this particular case, the coefficient of turbulence kinetic energy flux vanishes exactly, regardless of the collisionality limit. It is easy to see that in the limit of small $\bar{n}$, spreading is caused mainly by the beat term $\delta\bar{n}_{q,0}$ [i.e. $\langle\Phi nn\rangle \rightarrow \langle\Phi n\delta\bar{n}\rangle$]. Thus, only the internal energy spreads as a result of $\delta\bar{n}_{q,0}$ dynamics. It carries the kinetic energy via linear couplings. It is important to note that only $\delta\bar{n}_{q,0}$ plays a role in this ''three-wave interaction'', which results in spreading and that $\delta\bar{\Phi}_{q,0}$ has no effect.

One can consider a slight deviation from weak turbulence theory by considering the damping of the zonal flows by ion drag. When included, this damping effectively broadens the three-wave resonance in weak turbulence theory and results in a ''triad interaction time'' $\theta_{\mathbf{k},\mathbf{p},\mathbf{q}}$, such that $\mathrm{Im}[\theta_{\mathbf{k},\mathbf{p},\mathbf{q}}] \neq 0$. This allows the Fick's law for

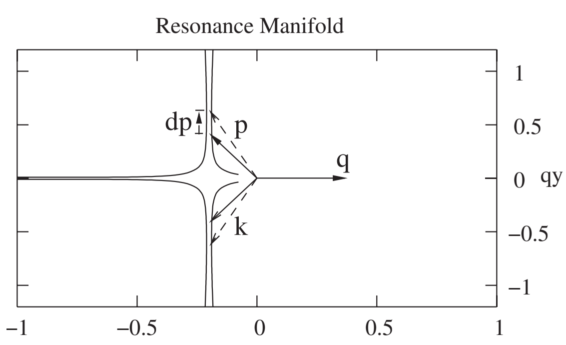

FIG. 2. The resonance manifold for the interaction involving a $q_y = 0$ mode.

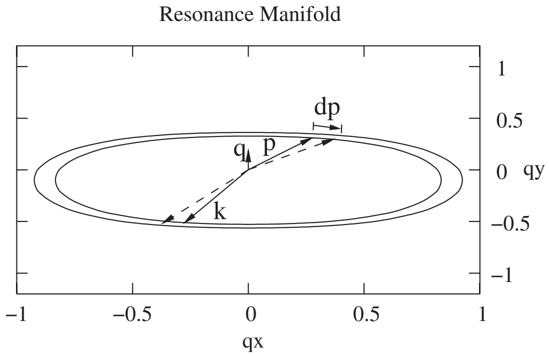

FIG. 3. The resonance manifold for the interaction involving a hydrodynamic streamer [i.e., $(k_x, k_\parallel) \rightarrow 0$].

the kinetic energy transport to be modified to $\Gamma_K \approx V^{KK}\tilde{K} + D^{KK}\partial_X\tilde{K}$, where V^{KK} is a radial propagation term in the form $V^{KK} \sim \nu_{\rm ZF}\bar{K}$. Even though this dependence of radial energy flux on zonal flow intensity seems to imply higher transport for higher zonal flow intensity, the term is also proportional to zonal flow damping, which is actually what determines the maximum value of zonal flow intensity. Thus we may say $V^{KK} \sim \nu_{\rm ZF}\bar{K}(\nu_{\rm ZF}) \sim f(\nu_{\rm ZF})$, where $f(\nu_{\rm ZF}) \rightarrow 0$ for both $\nu_{\rm ZF} \rightarrow 0$ or $\nu_{\rm ZF} > \gamma_l$ (since the ZFs are not excited).

Since the zonal flow is essentially a large scale hydrodynamic flow ($q_\parallel \rightarrow 0$), we should also compare to the hydrodynamic streamer ($q_y < 1$ and $q_\parallel \rightarrow 0$ (see Fig. 3). Equation (7) is valid also for streamers (but the resonance condition is different, which gives a larger $\tau_{ac} \equiv {\rm Re}[\theta_{kpq}]$). Kinetic energy also diffuses:

$$D_{\mathbf{p}}^{(KK)} \equiv {\rm Re}[\theta_{kpq}]\frac{(k_y k^2 + p_y p^2 + q_y^3)}{k^2 p^2 q_y^2} \times [k_y(k_y^2 - p_y^2) + 2k_x^2 q_y]K_{\mathbf{k}},$$

where we can set $k_x = p_x = 0$ and $p_y > 0$, $q_y > 0$, and $k_y < 0$, for example, to see that $D_{\mathbf{p}}^{(KK)} \sim \tau_{ac}3(q_y/p_y + 2)K_{\mathbf{k}}$, is positive (see Fig. 3, $k_y^2 > p_y^2$). Also, streamers remain correlated longer, since the change in their radial distribution as they spread, does not destroy their resonance. Thus one is inexorably lead to the conclusion that it is the large scale hydrodynamic streamers that most effectively transport energy (also streamers interact effectively with the most-unstable modes of the drift waves).

It has been suggested that zonal flows may "promote" spreading [17] in realistic toroidal geometry. However, simple physical intuition and direct gyrokinetic particle simulations indicate that the addition of external shear flows *reduces* turbulence spreading [18]. We argue that the reason the zonal flow might appear to play an important role in spreading is that fluctuation-fluctuation coupling is neglected in most of the models of turbulence-mean-flow interactions. As a result, the zonal flow is, by construction, the *only path* through which nonlinear energy transfer may occur. This in turn creates the illusion that turbulence spreading is due to wave-zonal flow interactions. We argue that for developed wave turbulence, the main cause of nonlinear spreading is the total contribution from direct interactions between fluctuations. Admittedly, our model, while allowing more nonlinear coupling processes, does not take realistic geometry effects into account, therefore a direct quantitative comparison with [17] is not meaningful.

In conclusion, we have constructed force-flux relations for the transport of turbulence intensity and calculated the transport matrix for fluctuation energy. The theory is cast in terms of wave interaction processes. We show that radially extended eddies are most effective for turbulence spreading while zonal flows inhibit spreading rather than promote it.

We thank X. Garbet, W. Wang, and the participants of *Festival de Théory at Aix en Provence* for many stimulating discussions. This research was supported by U.S. Department of Energy Grant No. FG02-04ER 54738 and U.S. DOE Contract No. DE-AC02-76-CHO-3073.

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