

UNIVERSITY OF CALIFORNIA SAN DIEGO

Dynamics of Staircase Formation and Persistence in Transport Systems

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by

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ABSTRACT OF THE DISSERTATION

Dynamics of Staircase Formation and Persistence in Transport Systems

by

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Doctor of Philosophy in Physics

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In this dissertation, we establish a unified theoretical foundation for staircase formation and persistence using a hierarchy of reduced physical models. The objective is to identify the minimal physical mechanisms responsible for staircase formation, determine the conditions that allow these structures to persist, and assess their relevance to transport regulation in magnetized plasmas.

In Chapter 1, we study staircase formation in passive scalar advection by stationary and fluctuating cellular flows. We show that layering emerges from the separation between rapid intra-cell mixing and slow inter-cell transport. Staircases remain robust to global transverse shear and weak vortex scattering when the diffusive timescale exceeds the turnover timescale

and profile curvature remains sufficiently large. Persistence is quantified using the cellular Peclet number and profile curvature.

In Chapter 2, we extend this model to active scalar transport by studying staircase formation in the magnetic potential of a two-dimensional magnetohydrodynamic system coupled to a fluctuating vortex array. Magnetic feedback suppresses transport across cell boundaries, increasing the disparity between intra-cell mixing and inter-cell transport. This produces a positive feedback loop in which weak magnetic fields reinforce barriers and promote sharper staircases. Magnetic staircases persist in both flux-expulsion and vortex-disruption regimes and can be sustained against resistive decay by stochastic forcing.

In Chapter 3, we combine the passive and active scalar results into a common framework for staircase formation and persistence. Staircase emergence is shown to arise from inhomogeneous transport generated by rapid local mixing, slow barrier-crossing transport, and feedback. We identify curvature and Peclet number as robust measures of resiliency and clarify how fluctuations, disorder, and magnetic feedback modify staircase morphology and lifetime.

In Chapter 4, we investigate staircase formation in reduced flux-driven models for the coupled evolution of temperature, turbulent kinetic energy, and mean flow. Unlike classical models based on a local layer-forming instability, persistent staircases arise through a metastable near-marginal regime with sustained transport contrast. Persistence is governed by the competition between transport redistribution, turbulent damping, and mean-flow shear. These results connect staircase dynamics to plasma transport self-organization and identify a distinct barrier-formation mechanism near marginal stability.

Introduction

In this dissertation, we study reduced physical models of staircase formation in magnetically confined plasmas. The primary goal is to isolate and understand the minimal physical mechanisms responsible for the creation, persistence, and regulation of layered transport structures in fusion plasmas. Such structures, commonly referred to as $E \times B$ staircases [1], consist of alternating regions of strong turbulent transport and localized transport barriers and have emerged as an important feature of near-marginal plasma turbulence [2]. Recent gyrokinetic simulations and experiments suggest that these structures regulate avalanche-like transport events, suppress radial transport spreading, and improve confinement through the formation of mesoscale transport barriers [3].

Despite their importance, the dynamics underlying staircase formation remain difficult to isolate in full-scale fusion simulations. Modern gyrokinetic and flux-driven simulations contain a wide range of coupled spatial and temporal scales, nonlinear interactions, and geometric effects, making it challenging to identify the essential physical ingredients responsible for transport self-organization. In addition, such simulations are computationally expensive and often difficult to interpret systematically. Experimental observations of staircase structures and avalanche regulation have also only recently become available [4], further motivating the development of simplified theoretical frameworks capable of isolating the dominant transport dynamics.

For this reason, reduced models remain an essential component of fusion theory. Throughout the history of plasma physics and fluid dynamics, simplified models have repeatedly provided the conceptual foundations needed to interpret complex nonlinear systems. Examples include Prandtl's mixing-length theory for turbulent transport [5], reduced magnetohydrodynamic de-

scriptions of plasma turbulence [6], and minimal models of transport near marginal stability [2]. Such approaches sacrifice detailed realism in favor of isolating dominant mechanisms, identifying scaling behavior, and constructing physically interpretable macroscopic measures.

The present work follows this philosophy by studying a sequence of progressively richer reduced models of staircase formation. Beginning from passive scalar transport in cellular flows and extending toward active transport systems with self-consistent turbulent feedback, the objective is to identify which mechanisms are fundamental to layered transport formation and which are secondary consequences of more complicated plasma dynamics. In this sense, the reduced models studied here are not intended to reproduce the full complexity of magnetic confinement devices, but rather to capture the essential transport physics associated with radial redistribution, inhomogeneous mixing, and barrier formation.

Understanding such dynamics is important not only from the perspective of nonequilibrium statistical physics, but also because turbulent transport strongly constrains the confinement performance of fusion plasmas. Before discussing these reduced models and the associated staircase dynamics, it is first necessary to review the basic principles of magnetic confinement fusion, the transport processes that limit confinement, and the physical mechanisms that motivate the reduced descriptions developed throughout this dissertation.

0.1 Global energy demand

Global energy demand has continued to increase in recent decades, driven by industrial growth, electrification, and the rapid expansion of emerging technologies. At present, global energy production remains dominated by fossil fuels, whose continued use contributes substantially to greenhouse gas emissions and climate change [7]. These concerns, together with long-term issues of energy sustainability and security, have motivated the search for reliable carbon-free energy sources.

Among proposed alternatives, nuclear fusion is particularly attractive due to its large

energy density, abundant fuel supply, and lack of direct carbon emissions. The most accessible fusion reaction for energy production is the deuterium–tritium (D–T) reaction,



in which a deuterium nucleus fuses with a tritium nucleus to produce helium and a neutron while releasing a large amount of energy through mass conversion. Fusion fuel is effectively abundant, since deuterium occurs naturally in seawater and tritium may be bred from lithium [8]. In addition, fusion avoids the long-lived radioactive waste associated with conventional nuclear fission [9].

Since the first laboratory fusion experiments in the 1930s [10], substantial progress has been made toward controlled thermonuclear fusion. However, a commercially viable fusion reactor has not yet been achieved. The principal challenge is the realization of the extreme conditions necessary for fusion reactions to occur. Reacting nuclei must overcome Coulomb repulsion and therefore require kinetic energies corresponding to temperatures of order $\sim 10\text{keV}$ for D–T fusion. In such conditions, matter becomes fully ionized and forms a plasma consisting of ions and electrons.

0.2 Confinement problem

The objective of controlled fusion is to achieve self-sustained burning of fusion fuel, namely, to maintain fusion reactions for a sufficient duration to produce net useful energy. This requires confining a plasma at sufficiently high density and temperature while minimizing energy losses over long timescales. These requirements are summarized by the Lawson criterion [11], which for deuterium–tritium (D–T) fusion is commonly written as

$$n_e T \tau_E \geq 3 \times 10^{21} \text{ keVs/m}^3. \quad (2)$$

Here, n_e denotes the electron density, T the plasma temperature, and τ_E the energy confinement time, defined as the characteristic timescale over which energy is retained within the plasma. Satisfying this condition is necessary to achieve break-even or net energy gain. Consequently, maintaining high plasma pressure while suppressing energy losses remains a central challenge in fusion research.

Plasma confinement relies on the strong interaction between charged particles and magnetic fields. In the presence of a magnetic field, particles execute cyclotron motion perpendicular to the field while moving relatively freely along magnetic field lines. As a result, transport across the magnetic field is substantially reduced, suggesting that appropriately designed magnetic configurations can confine a hot plasma for extended periods. This principle forms the basis of magnetic confinement fusion.

Among magnetic confinement concepts, the tokamak is the most extensively studied and experimentally successful configuration. The tokamak (“toroidal chamber with magnetic coils”), proposed by Andrei Sakharov and Igor Tamm in 1951, consists of a toroidal geometry in which the magnetic field is composed of a strong toroidal component generated by external coils and a poloidal component produced by a toroidal plasma current. Together, these fields generate helical magnetic field lines that wind around the torus and improve particle confinement.

In a purely toroidal magnetic field, magnetic field inhomogeneity and curvature produce particle drifts that lead to enhanced radial transport and degraded confinement. The tokamak mitigates these effects through magnetic field line helicity, which averages radial drifts over particle trajectories and thereby improves confinement performance.

Tokamaks currently achieve the highest confinement performance among magnetic fusion devices. For example, the Joint European Torus (JET) achieved a triple product of approximately 4.7×10^{20} keV s/m³ [12], approaching within roughly an order of magnitude of the Lawson criterion. Therefore, tokamaks remain the leading candidate for realizing sustained fusion energy production.

Despite this progress, major challenges remain. In particular, steep temperature and

density gradients lead to microinstabilities that generate turbulence and enhanced transport. These turbulent transport processes substantially exceed classical predictions and are therefore commonly referred to as anomalous transport. Since anomalous transport strongly degrades confinement, understanding the origin and regulation of plasma turbulence remains essential for the development of magnetic fusion energy. In this context, drift-wave turbulence and anomalous transport provide a fundamental paradigm for describing transport in magnetized plasmas and will be discussed in the following section.

0.3 Anomalous transport

Collisional processes make plasma confinement in tokamaks inherently imperfect, resulting in a finite energy confinement time τ_E . Since confinement performance strongly impacts the feasibility of fusion reactors, improving τ_E remains a central objective in magnetic fusion research [13]. If radial transport is approximated as diffusive, the confinement time scales inversely with the effective diffusivity,

$$\tau_E \sim \frac{a^2}{D}, \quad (3)$$

where a is the tokamak minor radius.

In a collisional plasma, transport is predicted to occur through Coulomb collisions, leading to the classical diffusivity

$$D_c \sim \rho_e^2 \nu_{ei}, \quad (4)$$

where ρ_e is the electron gyroradius and ν_{ei} is the electron–ion collision frequency. Toroidal geometry and trapped-particle effects further enhance transport, producing neoclassical transport,

$$D_{\text{neo}} \sim (1 + q^2) D_c, \quad (5)$$

where q is the safety factor [14]. However, experiments consistently observe transport levels

substantially larger than both classical and neoclassical predictions. This excess transport is commonly referred to as anomalous transport.

Anomalous transport is widely attributed to plasma microturbulence [15]. Within a mixing-length framework, the turbulent diffusivity may be estimated as

$$D_T \sim \tilde{v} l_{\text{mix}}, \quad (6)$$

where $\tilde{v}$ is a characteristic fluctuation velocity and l_{mix} is the mixing length [16]. In magnetized plasmas, this estimate leads naturally to diffusivities between the Bohm and gyro-Bohm limits,

$$D_B \sim c_s \rho_s, \quad (7)$$

$$D_{gB} \sim c_s \rho_s \frac{\rho_s}{L_n}, \quad (8)$$

where c_s is the ion sound speed, ρ_s is the ion sound gyroradius, and L_n is the density gradient scale length. Experimental transport levels are frequently observed between these two scalings.

Despite anomalous transport, tokamaks remain the leading magnetic confinement concept due to their comparatively strong confinement performance. In the 1970s, ion temperatures approaching 6.5 keV were achieved using neutral beam injection [17]. However, confinement was observed to degrade with increasing heating power [18]. This trend changed with the discovery of the high-confinement (H-mode) regime in 1982 [19], which demonstrated substantially improved confinement properties. Understanding the origin of such confinement improvement requires examining the dynamics of drift-wave turbulence and zonal-flow regulation, which are discussed in the following section.

0.4 Drift-wave turbulence

Since heat and particle sources are localized near the plasma core, tokamak plasmas naturally develop radial gradients in density and temperature, and therefore pressure. In the pres-

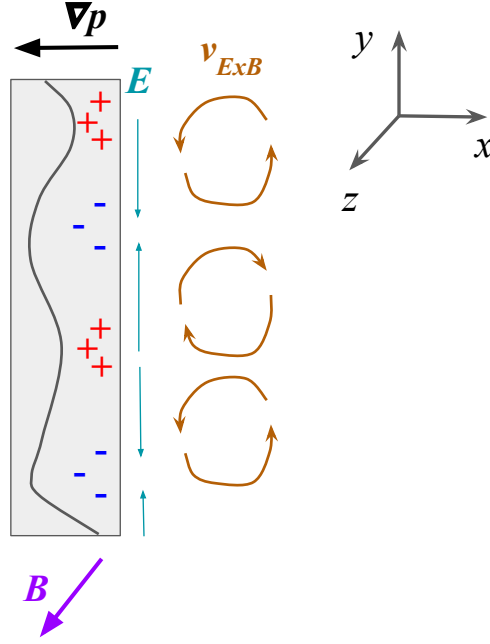


Figure 1. Drift-wave mechanism showing $\mathbf{E} \times \mathbf{B}$ convection in a nonuniform magnetized plasma.

ence of a magnetic field, these gradients drive species-dependent diamagnetic drifts. Fluctuating electric fields also generate the $\mathbf{E} \times \mathbf{B}$ drift,

$$\mathbf{v}_E = -\frac{c}{B^2} \nabla \tilde{\phi} \times \mathbf{B}, \quad (9)$$

where $\tilde{\phi}$ is the fluctuating electrostatic potential. Unlike diamagnetic drifts, the $\mathbf{E} \times \mathbf{B}$ drift is independent of particle charge and mass and therefore convects both ions and electrons together across the magnetic field.

Drift waves arise from the interaction between pressure-gradient-driven diamagnetic motion and $\mathbf{E} \times \mathbf{B}$ advection. In the adiabatic limit, electrons rapidly equilibrate along magnetic field lines, producing an approximate Boltzmann response,

$$\frac{\tilde{n}_e}{n_0} \simeq \frac{e\tilde{\phi}}{T_e}, \quad (10)$$

which couples density and electrostatic potential fluctuations [20]. In realistic plasmas, colli-

sions and finite parallel electron response introduce a phase shift between these fluctuations, destabilizing the drift wave and leading to exponential growth of perturbations. Nonlinear mode interactions subsequently drive the plasma into a turbulent state.

Drift-wave turbulence is widely regarded as a primary source of anomalous transport in magnetized plasmas. In addition to enhancing radial transport, drift-wave turbulence can also generate zonal flows through nonlinear mode coupling and Reynolds stress. Since zonal flows play a central role in turbulence regulation and confinement transitions, they are discussed in the following section.

0.4.1 Zonal flows and transport barriers

A complete description of drift-wave turbulence must include zonal flows, which are axisymmetric, low-frequency shear flows with zero poloidal and toroidal mode numbers. Zonal flows arise through nonlinear interactions among unstable drift-wave modes and play a central role in regulating turbulent transport. Reduced models such as the Hasegawa–Mima and Hasegawa–Wakatani equations provide a minimal theoretical framework for describing drift-wave turbulence and zonal-flow generation in magnetized plasmas [21, 22]. In these systems, drift-wave turbulence generates Reynolds stress,

$$\overline{\tilde{v}_x \tilde{v}_y}, \quad (11)$$

which transfers energy from turbulent fluctuations into zonal flows.

Since zonal flows are predominantly poloidally and toroidally symmetric, they do not directly contribute to radial transport through $\mathbf{E} \times \mathbf{B}$ advection. Instead, their primary effect is to shear and decorrelate turbulent eddies, thereby reducing turbulent transport (see Fig. 2). The shearing action of zonal flows modifies the radial wavenumber of turbulent fluctuations

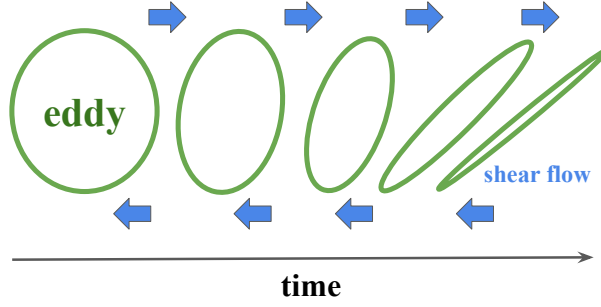


Figure 2. Example of “eddy shearing” due to zonal flows. Zonal flows regulate turbulence by “eddy shearing”, in which a turbulent eddy will gradually be distorted and stretched until it’s destroyed.

according to

$$k_x = k_{x,0} - k_y V_y' t, \quad (12)$$

where V_y' is the zonal-flow shear. This process transfers turbulent energy to smaller spatial scales, where dissipation is enhanced. Consequently, zonal flows act as regulators of turbulence and are widely believed to play a central role in confinement improvement and the transition from low- to high-confinement regimes.

The interaction between drift-wave turbulence and zonal flows is commonly described within a predator–prey framework, in which drift waves act as the energy source while zonal flows suppress turbulence through shear-induced decorrelation [23]. These dynamics may be represented by the reduced system [24],

$$\frac{\partial W}{\partial t} = \alpha W - \beta W Z - \gamma W^2, \quad (13)$$

$$\frac{\partial Z}{\partial t} = \beta W Z - \delta Z - \varepsilon Z^2, \quad (14)$$

where W and Z denote the drift-wave turbulence intensity and zonal-flow energy, respectively. Here, α represents linear instability drive, β the nonlinear transfer of energy from turbulence to zonal flows, δ zonal-flow damping, and γ and ε nonlinear dissipation. This reduced system

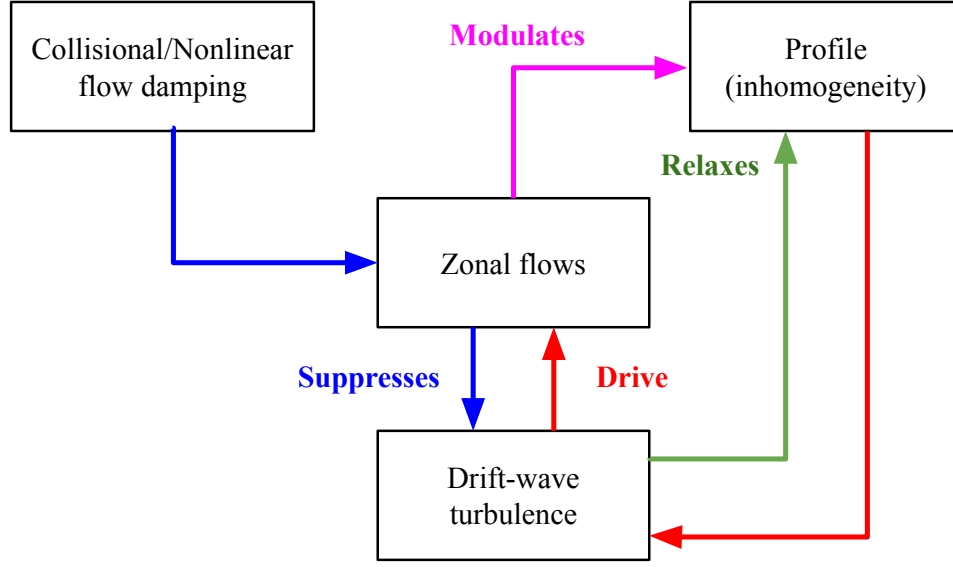


Figure 3. Illustration of basic feedback loop in the drift-wave/zonal flow system (predator-prey dynamics).

captures the essential feedback loop governing turbulence self-regulation in magnetized plasmas (see Fig. 3).

Turbulence-generated zonal flows can develop in both the plasma core and edge, where they contribute to the formation of transport barriers and improved confinement. In particular, sufficiently strong shear can locally suppress turbulent transport, leading to the formation of internal and edge transport barriers associated with enhanced confinement states [19]. Numerical and experimental studies further suggest that zonal flows may self-organize into multiple quasiperiodic shear layers with characteristic mesoscopic spacing. Such structures indicate that transport regulation in magnetized plasmas is fundamentally connected to the interplay between turbulence spreading, shear suppression, and nonlocal transport dynamics, which motivates the discussion that follows.

0.5 Turbulence spreading and avalanches

Thus far, drift-wave turbulence has been discussed primarily as a consequence of local linear instability. In this picture, turbulence develops when the local temperature or density gradient exceeds a critical threshold [2]. However, experiments and simulations have demonstrated that significant turbulence may persist even in regions that are locally stable. This behavior is attributed to turbulence spreading, in which turbulence generated in unstable regions propagates nonlinearly into neighboring stable regions through mode coupling and radial scattering processes.

Turbulence spreading implies that transport is fundamentally nonlocal. The local gradient alone is insufficient to determine the turbulence level, since fluctuations generated elsewhere may invade and sustain transport in nominally stable regions. In reduced descriptions, this process is commonly represented by a nonlinear diffusion equation for the turbulence intensity,

$$\partial_t I = \partial_x (D_I \partial_x I) + \dots, \quad (15)$$

where I denotes turbulence intensity and the effective diffusivity D_I depends on the turbulence level itself. Consequently, turbulence behaves as a self-propagating field capable of spreading across the plasma [25].

The discussion of turbulence spreading naturally leads to avalanche dynamics and self-organized criticality (SOC). In SOC systems, slow external driving produces states that evolve toward marginal stability, where transport occurs through intermittent, bursty relaxation events. The classical example is the sandpile model [26], in which local threshold exceedance triggers cascades spanning multiple spatial scales (see Fig. 4). Similar behavior has long been proposed in magnetically confined plasmas, particularly in regimes operating near gradient marginality (see Table 1).

In plasma systems, avalanches correspond to rapid, spatially extended transport events

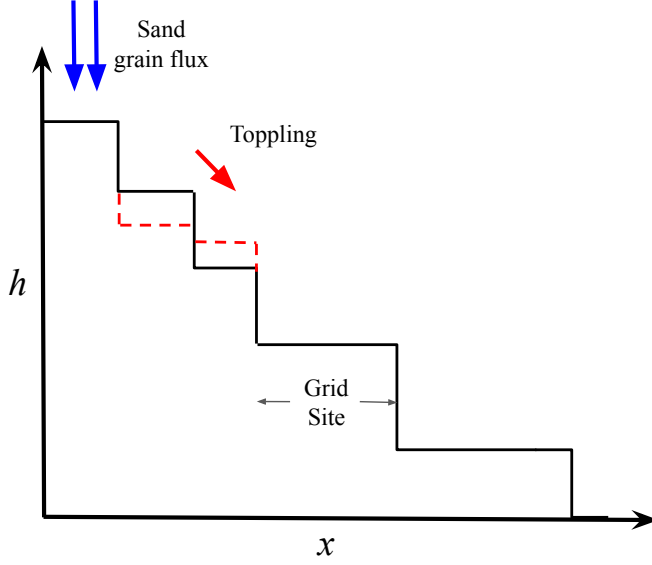


Figure 4. The one-dimensional “sandpile” model. There is a wall at the left and sand can exit only through the right. A comparison of the sandpile model to MFE is provided in Table 1.

that relax unstable gradients through collective dynamics. Rather than purely local diffusion, transport proceeds through intermittent bursts capable of propagating across substantial radial distances. Numerical simulations and experiments indicate that such events can contribute significantly to heat and particle transport [27, 28]. Measurements of fluctuation statistics and long-range temporal correlations further support the presence of nonlocal transport dynamics in tokamaks [29].

Avalanches and zonal flows are strongly coupled. Zonal-flow shear can suppress avalanche propagation by reducing the radial extent of turbulent structures, while avalanche-driven transport can generate zonal flows through enhanced Reynolds stress [3]. The resulting competition between turbulent spreading and shear suppression produces self-organization of transport and flow structures across mesoscopic scales.

These nonlocal transport dynamics are also closely connected to confinement transitions and transport barrier formation in magnetized plasmas, particularly in regimes operating near

Sandpile	MFE
Pile	Confinement
Grid site/cell	Local fluctuation/eddy
Toppling	Local instability/turbulence mechanism
N (number of grains toppling)	Eddy mixing rule
Random rain of grains	External heating (random heat input to the system $\rightarrow$ SOC)
Sand grain flux	Heat/ particle flux
Average slope	Mean profile
Avalanche	Avalanche
Local rigidity	Profile stiffness

Table 1. Similarities between the sandpile model and turbulence in MFE.

marginal stability.

0.6 H-mode and confinement improvement

Interest in the viability of tokamaks was significantly revitalized in 1982 with the discovery of the high-confinement (H-mode) regime [30]. Experiments on the ASDEX tokamak demonstrated that when the auxiliary heating power exceeds a threshold, the energy confinement time increases by approximately a factor of two. This transition, known as the L–H transition, corresponds to a spontaneous shift from low-confinement (L-mode) to high-confinement (H-mode) operation and is now widely associated with the emergence of strong sheared $\mathbf{E} \times \mathbf{B}$ flows at the plasma edge.

The L–H transition is generally understood as a consequence of turbulence suppression by shear flows [31, 32]. In drift-wave turbulence, transport is mediated by convective eddies. Sufficiently strong shear tilts and decorrelates these eddies, reducing turbulent transport and leading to the formation of an edge transport barrier (ETB). Similar mechanisms can also produce internal transport barriers (ITBs) within the plasma core, resulting in improved confinement and steeper pressure gradients. However, steep gradients may also lead to operational challenges,

including impurity accumulation and edge-localized modes (ELMs), which can produce large transient heat loads on plasma-facing components. Consequently, reactor-relevant operation requires balancing confinement improvement against impurity control, power exhaust, and plasma–wall interaction constraints.

Since the discovery of H-mode, numerous advanced confinement regimes associated with transport barrier formation have been identified [33, 34]. Experimental and numerical studies further suggest that transport barriers need not occur as isolated structures. Near marginal stability, quasiperiodic barrier layers may emerge within the plasma interior, producing staircase-like transport states composed of multiple shear layers and localized barriers. These layered transport states motivate the discussion of the $E \times B$ staircase in the following section.

0.7 $E \times B$ staircase

Recent numerical and experimental studies indicate that near-marginal magnetized plasmas naturally evolve toward states consisting of avalanche-dominated transport regions separated by localized shear layers acting as mesoscale transport barriers [35]. These self-organized structures, commonly referred to as $E \times B$ staircases, have emerged as an important mechanism for turbulent transport regulation in magnetically confined plasmas.

The $E \times B$ staircase is the plasma analogue of the potential-vorticity staircase observed in geophysical fluid dynamics. It consists of a quasiperiodic radial structure composed of relatively well-mixed regions (“steps”) separated by narrow shear layers (“jumps”) that locally suppress transport. Avalanche-like transport dominates within the step regions, while zonal-flow shear and profile corrugations concentrate near the interfaces. The staircase therefore represents a spatial segregation between turbulent mixing and transport suppression.

Staircase formation results from the interaction between avalanche-driven transport, turbulence spreading, and zonal-flow shear near marginal stability. Drift-wave turbulence drives zonal flows through Reynolds stress, while zonal-flow shear suppresses radial transport through

eddy decorrelation. At the same time, avalanches redistribute gradients and turbulent intensity across neighboring regions. Near marginal stability, these competing processes can self-organize into a quasiperiodic arrangement of transport barriers and mixing regions.

The dynamics of staircase formation are closely related to the Kubo number, which characterizes the degree of stochasticity and transport decorrelation in turbulent systems [33]. In plasma transport theory, the Kubo number may be written as

$$\text{Ku} = \frac{\delta_r}{\Delta_r} \sim \frac{\tau_{ac}}{\tau_{NL}}, \quad (16)$$

where δ_r is the radial autocorrelation length, Δ_r is the radial decorrelation length, τ_{ac} is the autocorrelation time, and τ_{NL} is the nonlinear turnover time. Physically, the Kubo number measures the degree to which transport trajectories remain correlated over time.

When $\text{Ku} \ll 1$, decorrelation is rapid and transport is predominantly diffusive, consistent with quasi-linear transport theory. In contrast, for $\text{Ku} \gg 1$, transport becomes increasingly coherent and nonlocal, and may acquire a percolative character in which particles propagate along long-lived connected pathways rather than through stochastic diffusion. Magnetized plasmas frequently operate near the intermediate regime $\text{Ku} \sim O(1)$, where coherent structures and stochastic transport coexist.

This intermediate regime is particularly relevant to staircase formation. Near $\text{Ku} \sim O(1)$, turbulent structures remain sufficiently coherent to sustain mesoscale transport barriers and profile corrugations while still permitting turbulence spreading and avalanche propagation between adjacent regions. As a consequence, transport becomes spatially inhomogeneous, producing localized barriers that regulate radial transport and partially suppress turbulent spreading. The present work primarily considers the regime $\text{Ku} \lesssim 1$, where transport remains predominantly diffusive, though coherent mesoscale structures may persist for long times.

The formation of such layered transport states motivates the reduced-model approach developed throughout this dissertation. In particular, the focus is on understanding how localized

barriers, inhomogeneous mixing, and transport redistribution emerge from comparatively simple transport dynamics.

0.8 Layered transport and analogies to fluids

Layered transport structures arise in a wide range of nonequilibrium systems, including magnetized plasmas, rotating geophysical fluids, stratified turbulence, and doubly diffusive convection (DDC) [36]. Although the underlying physics differs across these systems, they frequently exhibit spontaneous segregation into relatively well-mixed regions separated by sharp interfaces or transport barriers. These similarities suggest that staircase formation reflects a broader class of self-organized transport dynamics associated with turbulent mixing and nonlinear transport regulation.

In geophysical fluids, layered transport is closely connected to potential-vorticity (PV) homogenization and jet formation [37]. Turbulent mixing tends to homogenize PV within localized regions while sharpening gradients at their boundaries, producing quasiperiodic PV staircases separated by zonal jets that inhibit cross-interface transport. Similar ideas also appear in stratified fluids through the Phillips effect, in which homogeneous stirring spontaneously generates density layering and sharp interfaces [38]. These systems provide important conceptual analogies for understanding how turbulent transport can self-organize into mesoscale barrier structures.

A common theme in many staircase models is the interaction between imposed transport scales and emergent dynamical scales. One important example is the Rhines scale in drift–Rossby turbulence [39, 40], which marks the transition between turbulent decorrelation and wave-dominated dynamics. Related ideas also appear in stratified turbulence through the Ozmidov scale, which characterizes the scale at which turbulent overturning competes with buoyancy effects [38]. Such emergent scales frequently regulate transport anisotropy, mixing efficiency, and mesoscale pattern formation.

These concepts appear in several reduced models of layered transport and staircase formation. Reduced turbulence models [41, 34, 42], wave-kinetic descriptions [43], and mean-field transport theories [44] all predict the emergence of self-organized mesoscale structure through nonlinear feedback between transport and profile evolution. A particularly influential example is the model of Balmforth, Llewellyn-Smith, and Young (BLY) [38], originally developed to describe density staircases in stratified fluids. In the BLY model, transport depends on the interaction between an imposed stirring scale and an emergent Ozmidov scale, producing alternating regions of strong mixing and sharp interfaces.

The approach adopted in this dissertation is most closely related to homogenization dynamics and transport self-organization in weakly overlapping transport structures. In particular, the focus is on staircase formation in arrays of interacting transport cells with disparate mixing and transport timescales. This perspective provides a minimal and physically transparent model for understanding layered transport formation in both passive and active scalar systems.

0.9 Cellular transport model

Since magnetized plasmas frequently operate near $Ku \sim O(1)$, it is useful to study intermediate regimes between purely diffusive transport and strongly coherent or percolative transport. Systems such as Taylor dispersion [45] and cellular flows [46, 47, 48] provide useful analogues. In these systems, motion is largely deterministic along streamlines, while weak diffusion enables transport between adjacent structures, producing nontrivial global transport dynamics.

Consider an array of weakly overlapping convective cells of characteristic size d subject to a global scalar gradient. Motion within each cell is dominated by convection with characteristic velocity v_0 , while transport between neighboring cells occurs diffusively across thin boundary

layers of width $\delta \ll d$. The corresponding Peclet number is

$$\text{Pe} = \frac{v_0 d}{D_0} \gg 1, \quad (17)$$

indicating that convection dominates transport within individual cells.

The effective diffusivity on scales much larger than the cell size may be estimated using a random-walk argument. The transport step size is $\Delta x \sim d$, the characteristic step time is $\Delta t \sim d/v_0$, and only a fraction

$$f_{\text{active}} \sim \frac{\delta}{d} \quad (18)$$

of the domain participates in inter-cell transport. The effective diffusivity is therefore

$$D_{\text{eff}} \sim f_{\text{active}} \frac{(\Delta x)^2}{\Delta t} \sim \delta v_0. \quad (19)$$

Balancing convection and diffusion within the boundary layer gives

$$\delta^2 \sim D_0 \frac{d}{v_0}, \quad (20)$$

which yields

$$D_{\text{eff}} \sim \sqrt{D_0 v_0 d} = \sqrt{D_0 D_{\text{cell}}}, \quad (21)$$

where

$$D_{\text{cell}} = v_0 d \quad (22)$$

is the characteristic convective transport coefficient associated with cellular motion. Thus, the effective diffusivity is the geometric mean of diffusive and convective transport processes.

This system naturally generates staircase-like scalar profiles consisting of relatively well-mixed cell interiors separated by narrow gradient layers. The cellular array therefore provides a minimal model for studying inhomogeneous mixing, layered transport, and the formation of multiple transport barriers. The remainder of this dissertation focuses on the dynamics of such cellular transport systems and their relevance to staircase formation in passive and active scalar transport.

0.10 Organization of remaining chapters

In this dissertation, we investigate staircase formation and persistence using a hierarchy of reduced physical models. Beginning with passive scalar transport in cellular flows, we progressively introduce fluctuations and feedback to identify the minimal mechanisms responsible for layered transport structures and their persistence. This progression culminates in reduced flux-driven models of turbulent plasma transport, where staircase states emerge through near-marginal self-organization rather than a local layer-forming instability.

In Chapter 1, we study staircase formation in passive scalar advection by stationary and fluctuating cellular flows. We show that layering emerges from the separation between rapid intra-cell mixing and slow inter-cell transport. Staircases remain robust to global transverse shear and weak vortex scattering when the diffusive timescale exceeds the turnover timescale and profile curvature remains sufficiently large. Persistence is quantified using the cellular Peclet number and profile curvature.

In Chapter 2, we extend this framework to active scalar transport by studying staircase formation in the magnetic potential of a two-dimensional magnetohydrodynamic system coupled to a fluctuating vortex array. Magnetic feedback suppresses transport across cell boundaries, increasing the disparity between intra-cell mixing and inter-cell transport. This positive feedback strengthens transport barriers and promotes sharper, longer-lived staircase structures. The effects of magnetic Reynolds number, stochastic magnetic forcing, and vortex disruption are also

examined.

In Chapter 3, we combine the passive and active scalar results into a common framework for staircase formation and persistence. Staircase emergence is shown to arise from inhomogeneous transport generated by rapid local mixing, slow barrier-crossing transport, and feedback. We identify curvature and Peclet number as robust measures of staircase resiliency and clarify how fluctuations, disorder, and magnetic feedback modify staircase morphology and lifetime.

In Chapter 4, we investigate staircase formation in reduced flux-driven models for the coupled evolution of temperature, turbulent kinetic energy, and mean flow. Unlike classical staircase theories based on a local layer-forming instability, persistent staircases arise through a metastable near-marginal regime with sustained transport contrast. Persistence is governed by the competition between transport redistribution, turbulent damping, and mean-flow shear. These results connect staircase dynamics to plasma transport self-organization and identify a distinct barrier-formation mechanism near marginal stability.

In Chapter 5, we summarize the principal results of the preceding chapters and discuss directions for future research, including extensions to higher-dimensional transport models, self-consistent fluctuation dynamics, and direct comparisons with experimental measurements and gyrokinetic simulations.

Chapter 1

Staircase resiliency in a fluctuating cellular array

1.1 Introduction

1.1.1 Background

Staircases [37] – structures of long-range layered order [49] – have been predicted and observed in applications such as doubly diffusive convection (e.g., stellar interiors, oceans, and planetary atmospheres) [36, 50], rotating geophysical fluids (e.g., potential vorticity (PV) jumps associated with sharp jets (zonal flows)) [37], fluids with stratified density [51], and magnetized plasmas [52, 1, 53, 54]. In all of these, sharp gradient layers act as barriers to transport. On account that layering is observed in many different types of systems, it is beneficial to understand how staircases form – i.e., how ‘inhomogeneous mixing’ is realized? Such mixing results in the local sharpening of a scalar’s (e.g., PV, temperature, etc.) gradient. When the gradient steepens, a pattern of “jumps” and “steps” develop in the scalar profile, hence the staircase.

In magnetically confined plasma, layering is manifested by the formation of an $\mathbf{E} \times \mathbf{B}$ staircase – a quasi-periodic pattern of mean field and fluctuation intensity. A signature of this structure is the containment of avalanche activity, which can be detrimental to confinement [3]. Both simulation and experimental data have shown a pressure profile that resembles a staircase. At “jumps” (strong localized temperature gradients), turbulence drives an enduring layer of

localized shear that acts as a permeable transport barrier. In regions of “steps” (mixing region), there is strong, avalanche-like transport. Here, the interspersed regions of localized shear reduce the spread of avalanche-like transport.

There are two common approaches when explaining the phenomenon of layering in magnetic confinement. One approach proposes that the formation process is due to the interplay of mixing on two length scales [38]. Here, bistability, which is key to staircase formation, enters via a gradient dependent mixing length. The mixing length is determined by an excitation scale and the Rhines scale [42]. The emergent scale for drift-wave turbulence is the Rhines scale [39, 40], which is defined by the cross-over between the wave interaction mismatch frequency and the decorrelation rate characteristic of the inverse cascade. Note that both the local profile scale length and the turbulence intensity enter the Rhines scale – hence its “emergent” character. The interplay of processes on two length scales triggers self-sharpening of modulations, leading to a staircase. For an extensive study on the two scale mixing process, see [41, 34, 43, 44]. The other approach is the jams model [55, 56, 57]. In this alternative picture, the staircase forms due to a time delay between temperature modulations and local heat flux. For a more detailed discussion of these two approaches, see Appendix 6.1.

So far, nearly all suggestions involve a feedback process between zonal shears, turbulence, and their mutual interaction by modulation processes. However, perhaps the *simplest* explanation of inhomogeneous mixing is to observe that it occurs when the turbulence field is organized in an array of marginally overlapping convective cells (See Fig. 1.1). Such a configuration is expected when instabilities are weakly excited but not strong enough to produce large transport (i.e., when turbulence is relatively weak, cells overlap only slightly, and profiles are “near-marginal”). Such states are characteristic of “stiff profiles”, which adopt roughly the same shape regardless of the applied heating and fueling profiles [58].

Marginally overlapping cells capture two different transport mechanisms. The two transport methods can be understood using the setup of overlapping and nearly-overlapping cells (See Fig. 1.2). In the former, particles can scatter directly from cell to cell. A classic example of

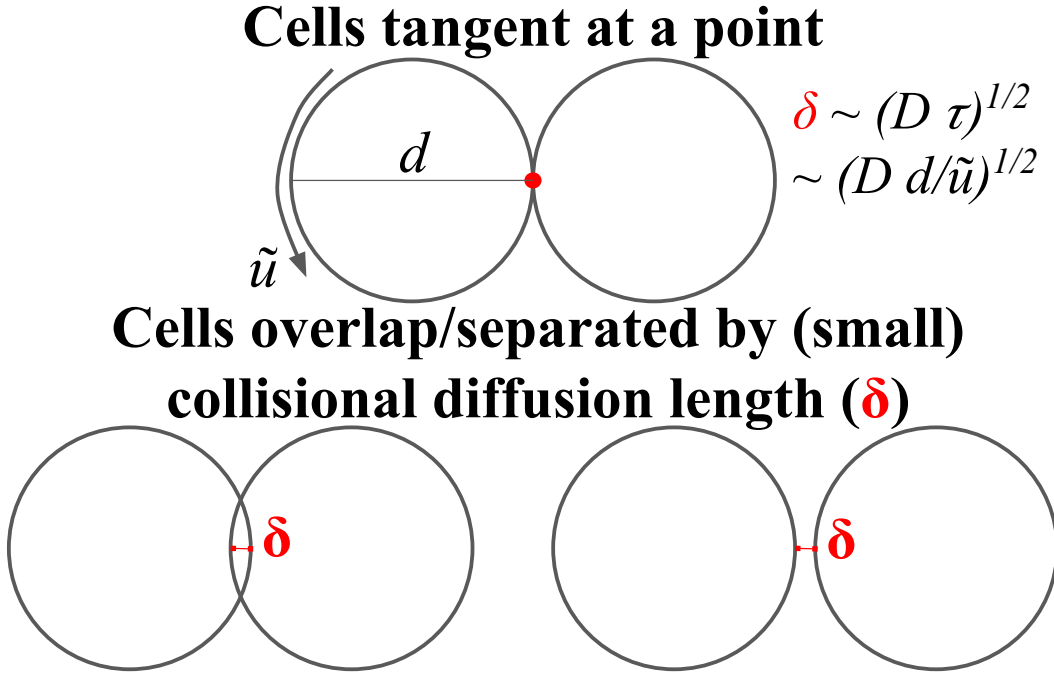


Figure 1.1. Illustration of marginally overlapping cells. Marginally overlapping cells are cells that are tangent at a point, or which intersect over, or are separated by, a distance of no more than a collisional diffusion length.

the latter is the problem of passive-scalar advection by an array of stationary convective cells [46, 48, 47]. A feature of this model is that there is a coexistence and interplay of fast (cell turn-over) and slow (diffusion) transport time scales. In this setup, many overturns occur in one diffusion time, thus resulting in inhomogeneous mixing. As depicted in Fig. 1.4, the system has a global density gradient in the x direction. As the cells rotate, they capture and trap particles. Due to a global diffusion D , the particles can be kicked off a streamline at a narrow gap (δ) between two convective cells. Note that we assume that the characteristic scale of the cells is d , hence $d \gg \delta$. Here, the cell boundary acts as a transport barrier; therefore, the *simplest* case of staircase formation occurs in a stationary array of eddies and passive scalar dynamics (See Fig. 1.3). It is important to note that in this setup, ‘feedback’ is not necessary to form a staircase structure.

A remarkable result which is characteristic of this model is the effective diffusivity (D^*) caused by the interplay of fast and slow transport. To derive the effective diffusivity, we consider

a random walk with a diffusion coefficient calculated by $(\Delta x)^2/\Delta t$, where Δx and Δt represent the cell length along (d) the global density gradient and the boundary layer scattering time $[(d/\delta)\tau_H]$, respectively. Here, τ_H represents the cell circulation time. Therefore, we have the following,

$$D^* \approx \frac{(\Delta x)^2}{\Delta t}. \quad (1.1)$$

Since particles undergo a pure random walk at the boundary layer, we can define δ as $\sqrt{D\tau_H}$. To formulate an expression for τ_H , we introduce the cellular streamfunction

$$\psi_0 = \sin \pi(x/d) \sin \pi\beta(y/d), \quad (1.2)$$

and flow velocity

$$\mathbf{u} = (\tilde{u}d/\pi)\hat{z} \times \nabla\psi, \quad (1.3)$$

where $\tilde{u}$ is the maximum flow velocity. We can define the circulation time around the roll as

$$\tau_H = \frac{d}{\tilde{u}\beta}.$$

Here, the term β is the ratio of cell lengths along ($d = d_x$) and perpendicular (d_y) to the global density gradient. Plugging in these values into Eq. (1.1) results in

$$D^* \approx \sqrt{DD_{\text{cell}}} = D\sqrt{\text{Pe}}, \quad (1.4)$$

where $D_{\text{cell}} = \tilde{u}d\beta$. Here Pe represents the Peclet number,

$$\text{Pe} = \frac{\tau_D}{\tau_H}, \quad (1.5)$$

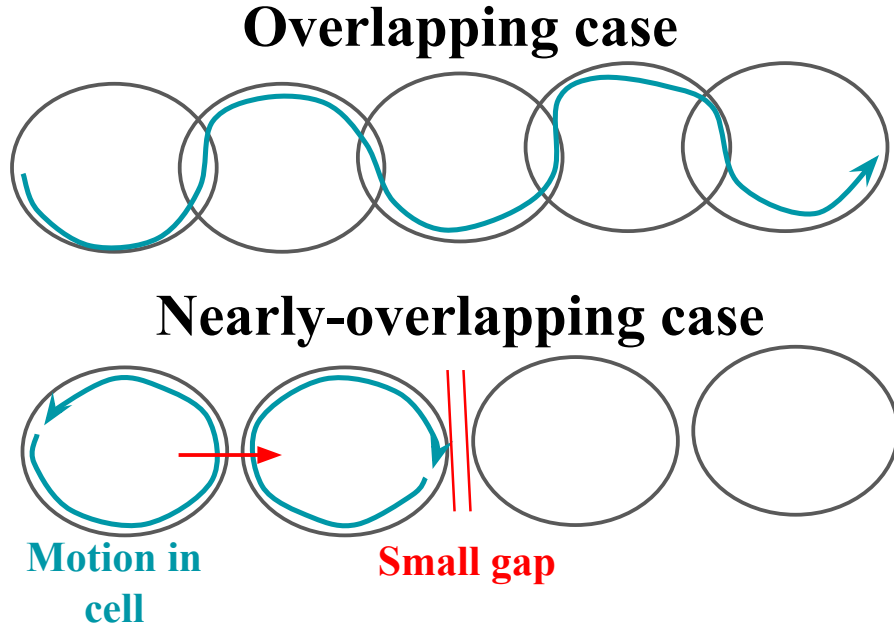


Figure 1.2. Particle transport in overlapping and nearly-overlapping cells. In the overlapping case, the particle can scatter directly from cell to cell. For the nearly-overlapping case, there is a coexistence of: Fast transport (mixing in cell) and slow transport (kicks between cells).

which is the ratio of the time for diffusion of a particle through a roll ($\tau_D = d^2/D$) to the time of circulation around the roll. Since layering in a cellular array requires that the cell circulation time be much less than the diffusion time, the result in Eq. (1.4) is only valid in the regime of $Pe \gg 1$. It is worth emphasizing that the effective diffusivity is not determined by simple addition as in $D^* = D + D_{\text{cell}}$, but by the geometric mean $\sqrt{DD_{\text{cell}}}$.

Many but not all properties of the nearly-overlapping cells are present in magnetic confinement devices. For example, convection cells take the form of the $\mathbf{E} \times \mathbf{B}$ convection, zonal flows represent the boundary regions between cells, diffusion represents ambient collisional diffusion, and scalar concentration represents the spatial propagation of turbulent intensity. The propagation of turbulent intensity enables turbulent heat transport and thus carries a heat flux. Due to these similarities, it's convenient to review the simplest incarnation of the staircase, which is the cellular array with two disparate time scales. This model is a minimal approach to layering, which complements other ideas such as the $\mathbf{E} \times \mathbf{B}$ “predator-prey” model, where drift-wave

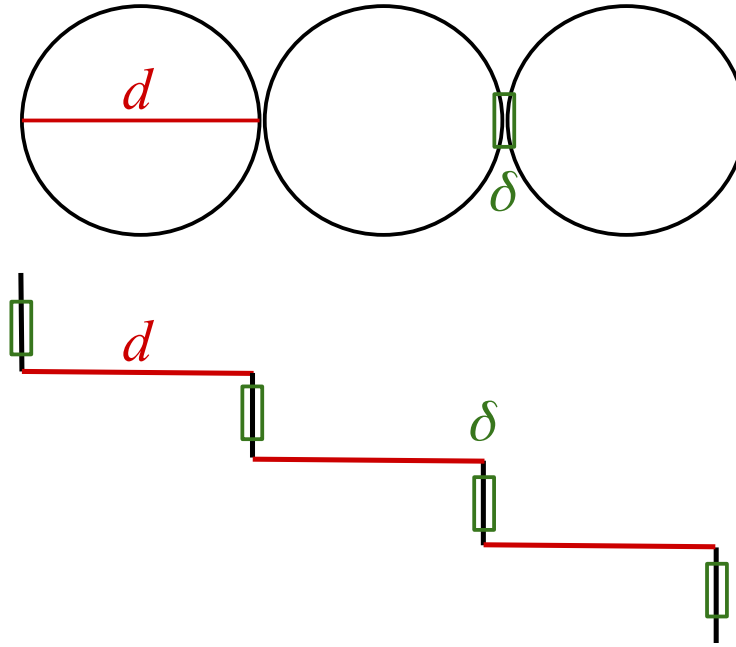


Figure 1.3. Illustration of a staircase profile. Domains of cellular mixing (“steps”) and regions of cell boundaries (“jumps”) are denoted by d (red) and δ (green), respectively.

turbulence and zonal flow populations interact via a dynamical feedback loop. Note that although this model incorporates features relevant to magnetically confined plasmas, it should not be construed as an exact representation thereof. For a brief discussion on staircases in magnetic confinement, see Appendix 6.1.

In this paper, we explore structural fluctuations and consider the effects of an additional shear streamfunction on the fixed cellular array. Here, we answer the following:

1. Will the staircase structure form in the presence of global transverse shear and turbulence-induced mixing?
2. How does global transverse shear affect the mixing of scalar concentration? If we increase the strength of the shear, what will happen to the staircase?
3. In a fluctuating vortex array, how does the staircase structure change? What are the criteria for a staircase to be considered resilient?

4. As scalar concentration flows in the fluctuating vortex array, will the staircase step or barrier develop first?
5. As vortices are scattered, how will the effective velocity of scalar concentration front change?
6. In the fluctuating vortex array, how does effective diffusivity deviate, if at all, from the $D\sqrt{\text{Pe}}$ scaling?

In the paper, we address these six questions and interpret the results in the context of magnetically confined plasmas.

1.1.2 Fixed Cellular Array Model

The governing equation for passive-scalar transport is

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) n = D \nabla^2 n, \quad (1.6)$$

where n is the scalar concentration, $\mathbf{u}$ the fluid velocity [See Eq. (1.3)], and D the background diffusion coefficient. For our simulations, we set the values of $\tilde{u}$ to 0.5. The dimensionless streamfunction produces the flow structure

$$\psi = \psi_0 + \alpha \psi_{\text{shear}}. \quad (1.7)$$

In Section 1.1.2 and Section 1.2, the value of β is set to one. In Section 1.3, we calculate the average value of β by measuring the length of the cells along and perpendicular to the mean gradient. For $\alpha = 0$, the streamfunction creates an array of convection cells [46, 48, 47]. In section 1.2, we explore the case of $\alpha \neq 0$. We note that Eqs. (1.3) and (1.7) follow the same normalization as the streamfunction and fluid velocity given in [46].

The passive-scalar equation is numerically solved on a two-dimensional grid where x and y range from $[0, L]$. The value of L and d are set to 2π and $\pi/3$ and are kept fixed for all

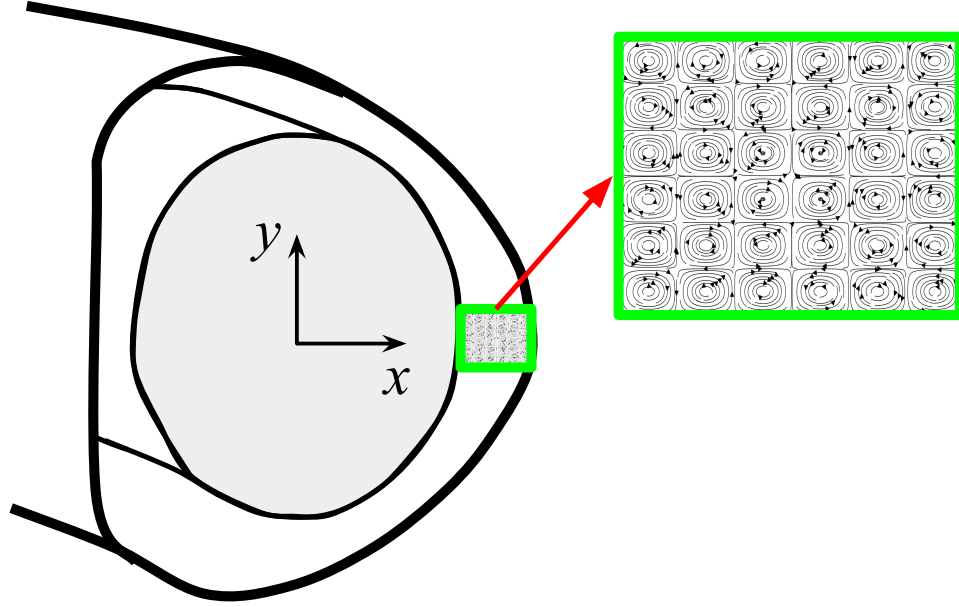


Figure 1.4. Illustration of the model solved in this paper. The passive-scalar equation utilizes both Neumann and periodic boundary conditions to model the physics of core to edge in fusion devices. In our model, x and y can represent the radial and poloidal direction, respectively.

numerical simulations and calculations performed throughout the paper. For our simulation domain, the value of d results in an array of 6×6 stationary convective cells.

We impose Neumann boundary conditions in x and periodic boundary conditions in y on the passive-scalar equation. The direction of the mean gradient is set by the Neumann boundary conditions and is similar in design to those in [59]. The Neumann boundary conditions are of the form

$$\Gamma|_{x=0} = -1 + \eta \tilde{\Gamma}, \quad (1.8)$$

$$\Gamma|_{x=L} = -1. \quad (1.9)$$

Here, $\eta \tilde{\Gamma}$ corresponds to a random scalar concentration flux, thus producing a noisy flux-driven system. The term $\eta \tilde{\Gamma}$ is also used for diagnostic purposes, such as mapping the path of the scalar concentration via a pulse train. The use of these particular boundary conditions help approximate

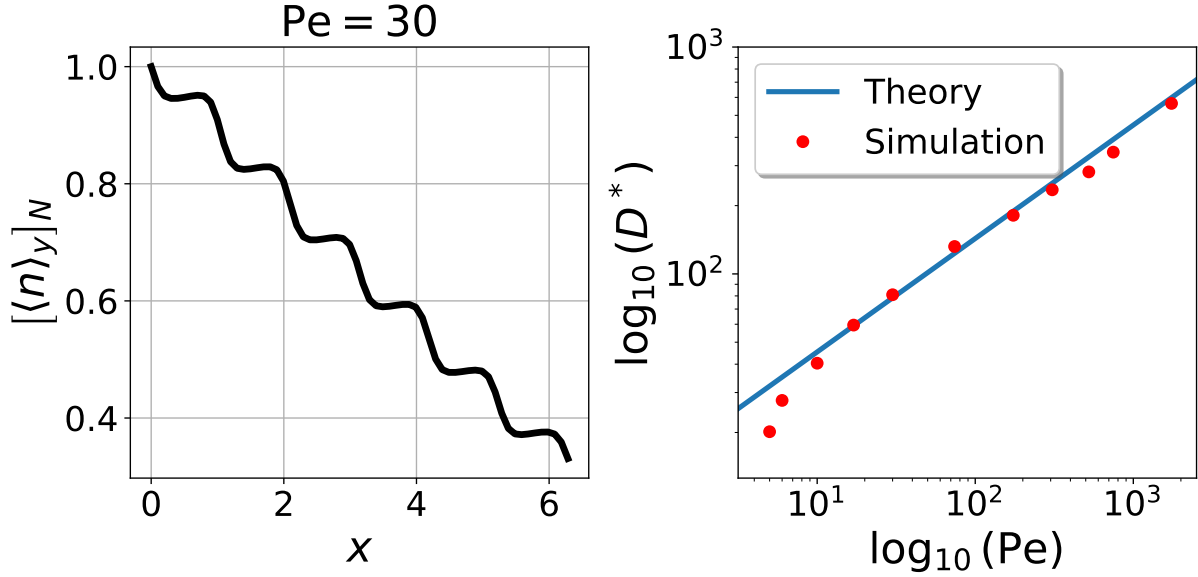


Figure 1.5. Staircase profile as function of x for $Pe = 30$ and Log-Log plot of effective diffusivity D^* as a function of Pe . The validated scaling [Eq. (1.4)] and computational results are represented by the solid blue line and red points, respectively. Computational results follow a $1/2$ slope for large Pe number. Note that $\langle n \rangle_y$ is normalized by its max value.

the fixed flux condition and periodic structure found in real physical systems (e.g., magnetic confinement). An example of a physical system that demonstrates these boundary conditions is shown in Fig. 1.4.

In order to solve the passive-scalar transport equation, we use an explicit finite difference numerical scheme. The time derivative uses forward Euler and the spatial derivative uses a centered difference approach. The numerical results are accurate up to first-order in time and second-order in space.

We test the accuracy of the numerical code on the array of 6×6 stationary convective cells. The accuracy of the code is verified by producing a staircase structure and comparing the computed effective diffusivity to the validated effective diffusivity shown in Eq. (1.4). To determine the effective diffusivity, we drive a flux of scalar concentration at one side of the array until the staircase profile forms. We generate the staircase profile by taking an average of n along

the direction perpendicular to the mean gradient

$$\langle n(x) \rangle_y = \frac{1}{L} \int_0^L n(x, y) dy. \quad (1.10)$$

Thereafter, we use Fick's law to calculate the effective diffusivity (D^*),

$$D^* = -\frac{(x_2 - x_1)^2}{\Delta t} \frac{n_{\text{tot}, t_f}(x = [x_1, x_2]) - n_{\text{tot}, t_i}(x = [x_1, x_2])}{\langle n_{t_f}(x_2) \rangle_y - \langle n_{t_f}(x_1) \rangle_y}. \quad (1.11)$$

Here x_1 and x_2 represent $(1/4)L$ and $(3/4)L$, respectively. Note that the difference between t_f and t_i is 8.3 diffusive cycles (i.e., $\Delta t = 8.3\tau_D$). For the region $[x_1, x_2]$, we define the parameter n_{tot} at a given time as

$$n_{\text{tot}} = \frac{2}{L^2} \int_{x_1}^{x_2} \int_0^L n(x, y) dy dx.$$

For a detailed explanation of how we obtain D^* , see Appendix 6.3. We find that for a wide range of D , the staircase profile forms as long as $\text{Pe} \gg 1$. Additionally, we determine that the numerical code is sufficiently accurate and agrees with Eq. (1.4). A plot of $\log_{10} D^*$ vs $\log_{10} \text{Pe}$ is shown in Fig. 1.5 comparing both simulation and theory. Here, we demonstrate an agreement between simulation and theory for $\text{Pe} \gg 1$.

The outcome of studies regarding both transport and staircase profile resiliency to the effects of global shear and turbulence-induced variability in the flow structure are determined and discussed. The remainder of the paper is outlined here. In Section 1.2, we study staircase resiliency by probing the fixed-cellular array using a global shear streamfunction. In Section 1.3, the cellular array is relaxed by switching to a fluctuating vortex array. Here the vortex array is allowed to fluctuate by slowly increasing the Reynolds number in the Navier-Stokes equation. Finally in Section 1.3.1, 1.3.2, and 1.3.3, we discuss the staircase in the fluctuating vortex array, the criteria for staircase resiliency, and scalar concentration transport in the fluctuating vortex array.

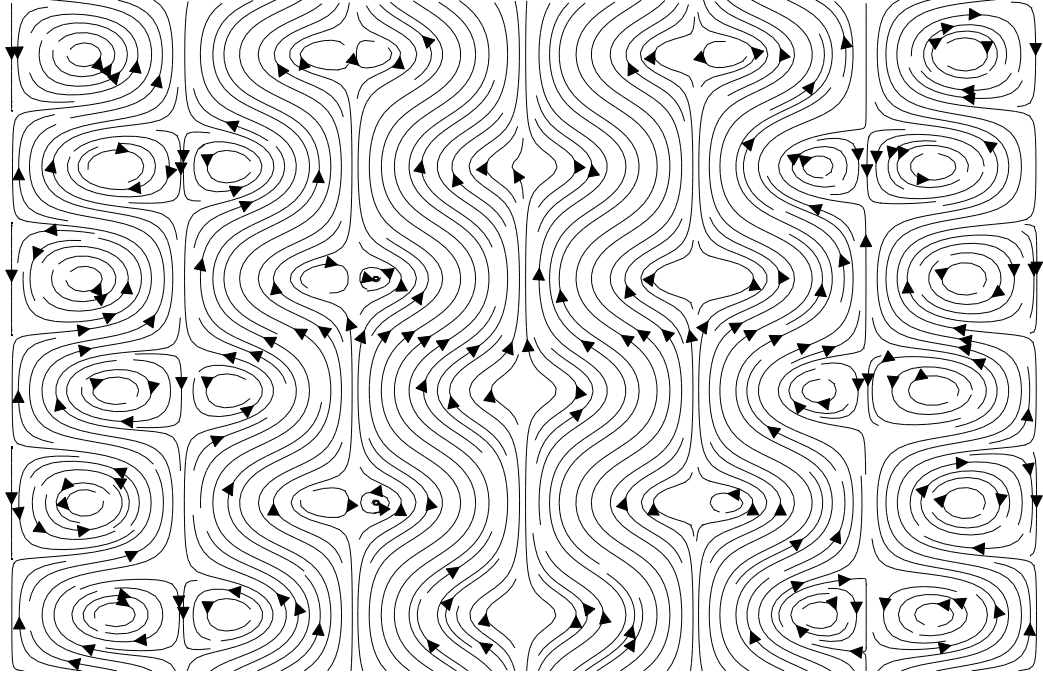


Figure 1.6. Array of convective cells with parabolic shear ($m = 1$) in the y direction for the case of $\alpha = 6$. The strength of the shear in the flow is controlled by α . The increase in α , destroys the array cell structure.

1.2 Global Shear on Fixed Cellular Array

The effects of global velocity shear on the staircase structure are explored by including an additional shear flow

$$\psi_{\text{shear}} = -\cos \frac{mx}{2}, \quad (1.12)$$

into the streamfunction [Eq. (1.7)]. As clarification, the term “shear” refers to advection in the y direction. The addition of the global shear streamfunction creates a new time scale, which will call the shear dispersion time scale [$\tau_{\text{adv}} = 12\pi/(\tilde{u}\alpha m)$]. It is important to make the distinction that τ_{adv} represents the time for advection by ψ_{shear} across the box in y (i.e., $\tau_{\text{adv}} = L/u_S$). Note that this is different from the familiar shearing time [$\tau_S = 1/(du_s/dx)$], which is equal to

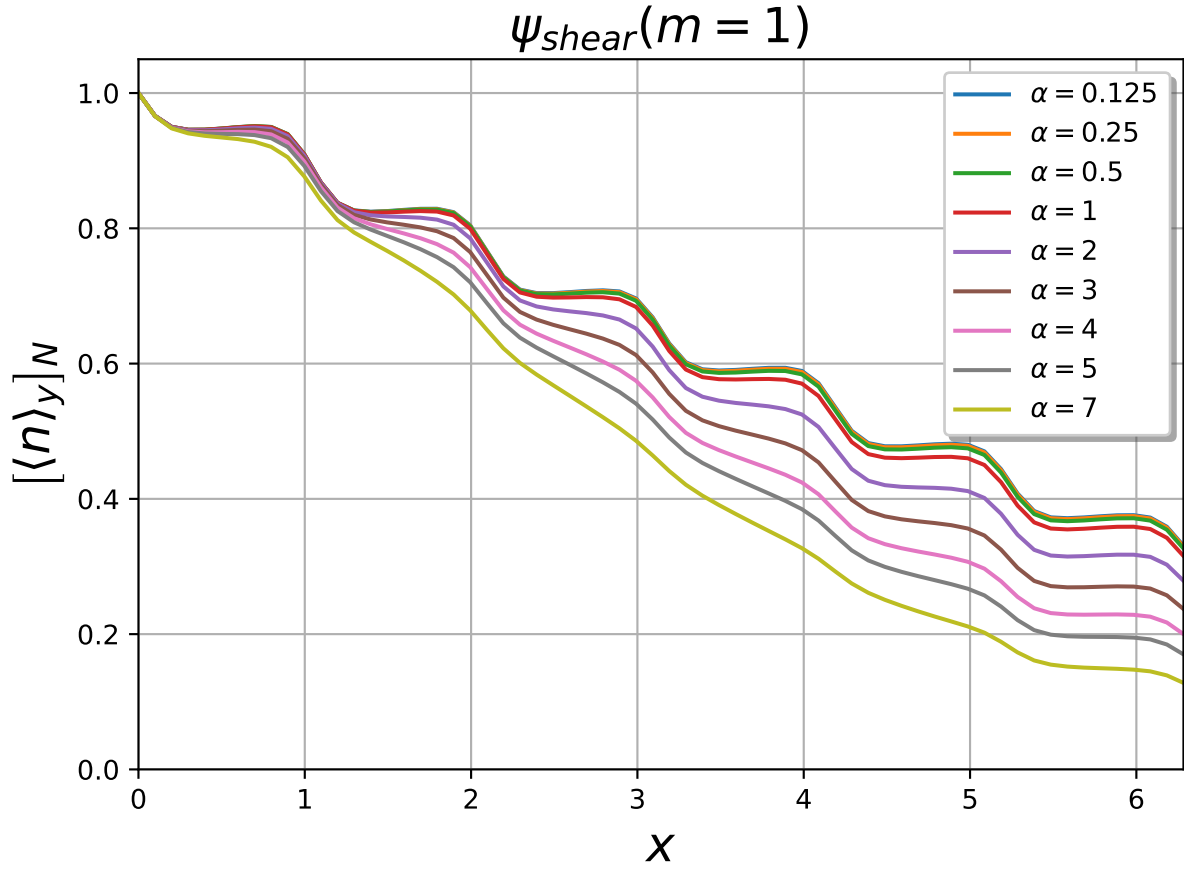


Figure 1.7. Scan of normalized averaged scalar concentration in y as a function of x for $m = 1$ and different α values. As the shear strength increases, the staircase profile breaks down. Global shear flow dissolves staircase steps.

$$(1/3md)\tau_{adv}.$$

We conduct a scan of m (i.e., the number of waves per distance L) at different integer values and different strengths, α . As a preliminary analysis, we study the physical effects that the global shear streamfunction has on the original cellular array flow structure. In Fig. 1.6, we demonstrate that increasing the amplitude of a quarter sinusoidal wave ($m = 1$) disrupts the cellular mixing structure. As stated in the introduction, cellular mixing structures are essential for the confinement of scalar concentration. Thus, it is reasonable to expect that the destruction of the cells will affect the scalar staircase profile and scalar transport.

In order to understand global transverse shear effects, we compare the three different

time scales τ_H , τ_D , and τ_{adv} . The ratio of the diffusion time scale to the shear dispersion time scale results in

$$\frac{\tau_D}{\tau_{\text{adv}}} = \frac{\alpha m \tilde{u} d^2}{12\pi D}. \quad (1.13)$$

Here, we find that when the product of α and m is greater than $12\pi D/(\tilde{u}d^2)$, the global transverse shear transports the passive-scalar at rate faster than diffusion. Thus, when $\alpha m > 12\pi D/(\tilde{u}d^2)$, the shear dispersion time becomes an intermediate time, between τ_D and τ_H . To find the upper limit such that τ_{adv} becomes shorter than τ_H , we take the ratio of the circulation time scale to the shear dispersion time scale

$$\frac{\tau_H}{\tau_{\text{adv}}} = \frac{\alpha m d}{12\pi\beta}. \quad (1.14)$$

Keep in mind that the value of β is set to one for this section. Here, we find that when the product of α and m is greater than $12\pi\beta/d$, the shear dispersion gives an effective mixing rate faster than inner-cell mixing. We summarize the order of the time scales in Table 1.1.

The simulations for this section span the space of $0 < \alpha m < 45$. We cover this parameter space to capture the effects of global transverse shear on the scalar concentration staircase structure. By running simulations for different values of α and m , we can also determine the consequences that the order of time scales has on the transport of the scalar concentration.

We begin by analyzing the case of $m = 1$. For $m = 1$, we increase the value of α and study the change in $\langle n(x) \rangle_y$. We find that as the shear strength increases, the staircase profile breaks down. The staircase profile breaks down in the sense that staircase steps dissolve (See Fig. 1.7). As sheared flow transports the scalar concentration in the y direction, the gradient $d\langle n \rangle_y/dx$ increases in the region where global shear is dominant. Due to the nature of the shear streamfunction, shear is weak at the boundaries, hence the flat steps persist there ($u_S(x=0) = u_S(x=L) \sim \sin(mx/2) = 0$).

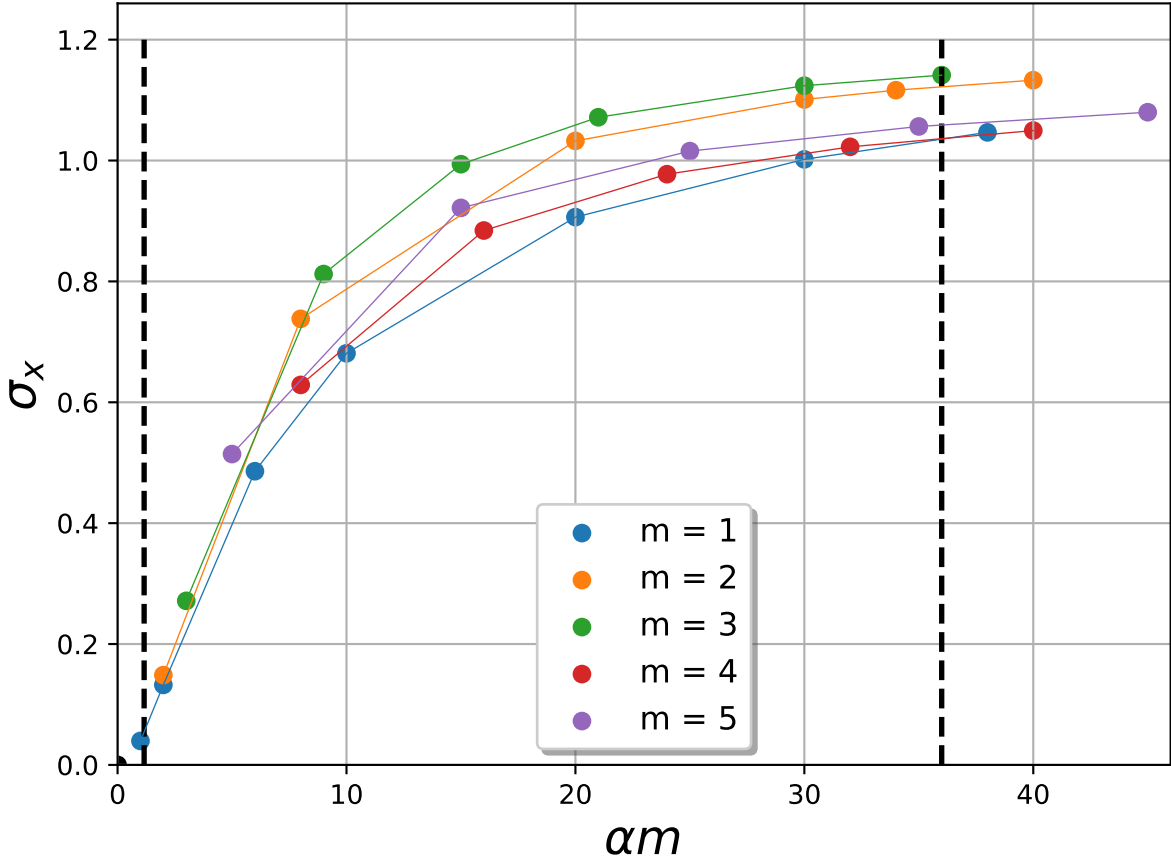


Figure 1.8. Variance in x as a function of αm . For different mode numbers m , the variance grows logarithmically. We conclude that for $\alpha m > 12\pi\beta/d = 36$, the average scalar concentration profile will no longer change significantly and will be of similar form for different m .

As stated in Section 2.1, the staircase structure has interspersed barriers and steps, which limit transport and trap the scalar concentration. The fast mixing in the cells trap scalar concentration, and inter-cell barriers limit transport due to a slow diffusion time scale. By destroying the steps, the interspersed trapping of scalar concentration (due to inner-cellular mixing) no longer occurs. In order to understand how different values of m breakdown the staircase profile, it's convenient to assess the deviation of the $\langle n \rangle_y$ profile from the original staircase profile ($\alpha = 0$). By doing so, we can also measure the total scalar concentration in the flow as a function of staircase profile deviation.

To quantify, for different m , the deviation or breakdown of the staircase profile, the

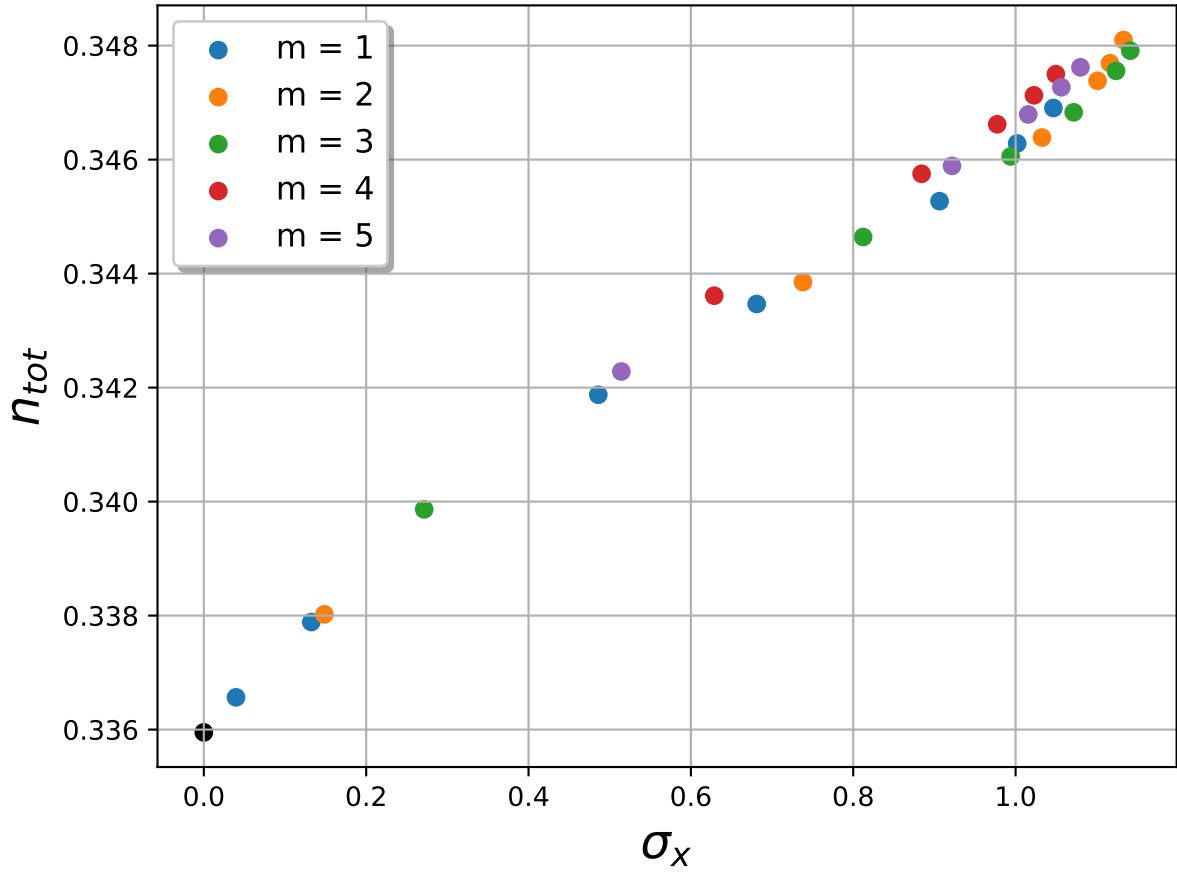


Figure 1.9. Total scalar concentration as a function of the variance in x . The total scalar concentration increases as the scalar concentration profile deviates from the staircase ($\alpha = 0$) profile. Global shear weakens mixing in the x direction. The value of $d\langle n \rangle_y / dx$ increases with shear, thus improving confinement of scalar concentration.

variance with respect to x is calculated for different α values,

$$\sigma_x^2 = \int_0^L (\langle n \rangle_y - \langle n_0 \rangle_y)^2 dx. \quad (1.15)$$

Here, we define the baseline staircase profile ($\alpha = 0$) as $\langle n_0 \rangle_y$. The variance measures the deviation between the $\alpha \neq 0$ profiles and the baseline staircase. In Fig. 1.8, we plot the square-root of Eq. (1.15) as a function of αm . In the plot, we mark the transition points where the different order in time scales occur ($\alpha m = 12\pi D / (\tilde{u} d^2)$ and $\alpha m = 12\pi\beta / d$) as black dashed vertical lines. The graph shows a similar trend in the variance for distinct mode numbers (m).

Cases	Time scales
$\alpha = 0$	$\tau_H < \tau_D$
$0 < \alpha m < 12\pi D/(\tilde{u}d^2)$	$\tau_H < \tau_D < \tau_{\text{adv}}$
$12\pi D/(\tilde{u}d^2) < \alpha m < 12\pi\beta/d$	$\tau_H < \tau_{\text{adv}} < \tau_D$
$12\pi\beta/d < \alpha m$	$\tau_{\text{adv}} < \tau_H < \tau_D$

Table 1.1. Global shear time scale scenarios.

For each mode number, the increase in α results in a significant increase in the deviation up to values of $\alpha m \sim 12\pi\beta/d$. For values of $\alpha m > 12\pi\beta/d$, the variance plateaus at $\sigma_x \sim 1.19$ for all m . The plot of the variance shows that up to a critical shear of $\alpha m = 12\pi\beta/d$, the $\langle n \rangle_y$ profile for different mode numbers will be similar in form and no longer notably change throughout the regime of $\tau_{\text{adv}} < \tau_H < \tau_D$. To summarize, we show that the $\langle n \rangle_y$ profile deviates significantly in the regime of $\tau_H < \tau_{\text{adv}} < \tau_D$, but goes unchanged once $\alpha m > 12\pi\beta/d$.

Next, we want to explore how the order in time scales affects the dynamics of the scalar concentration transport. In Fig. 1.7, we observe that as α increases, the staircase steps gradually dissolve, resulting in an increase in $d\langle n \rangle_y/dx$. Thus, the scalar concentration is no longer trapped by the cells, as advection in the y -direction becomes the dominant form of transport. With increasing $d\langle n \rangle_y/dx$, the transport (in x) due to cell circulation is reduced. To assess whether shear diminishes mixing along x and its relation to the deviation of the staircase profile, we calculate the total scalar concentration for various values of m and α

$$n_{\text{tot}} = \frac{1}{L^2} \int_0^L \int_0^L n(x, y) dy dx. \quad (1.16)$$

In Fig. 1.9, we show a line with positive slope between the total scalar concentration and σ_x . We find that as the pattern of staircase corrugation decays, the total scalar concentration increases. Hence, the increase in global shear weakens/strengthens the mixing of scalar concentration along/perpendicular to the mean gradient. We note that as σ_x approaches 1.19, the total scalar

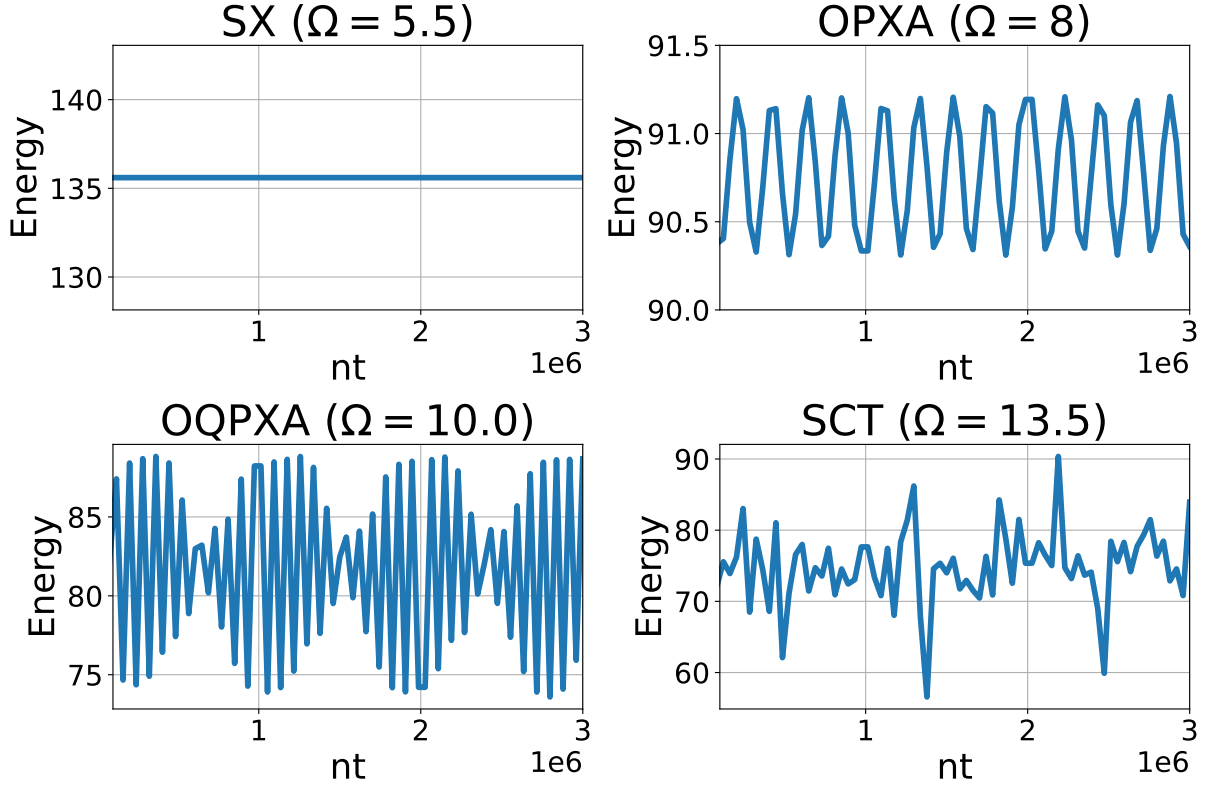


Figure 1.10. Energy time-trace for the different stages of the fluctuating vortex array. Here nt represents the time-steps in the DNS.

concentration does not change significantly. Thus, transport of scalar concentration can no longer be suppressed once it surpasses a value of $\alpha m = 12\pi\beta/d$. Nevertheless, results show that an additional global sinusoidal shear flow *weakens* the staircase structure and reduces the transport of scalar concentration in x for values of $\alpha m > 12\pi D/(\tilde{u}d^2)$.

The question of whether Eq. (1.4) can help explain the suppression of scalar concentration may arise. The answer to this question is no. With increasing shear strength (α), the global shear disrupts cellular mixing structures, causing scalar concentration to advect in the y -direction. In this context, explaining the suppression of scalar concentration cannot be achieved via Eq. (1.4). This is due to the condition that requires cell boundaries to be intact.

In the next section, the cellular array will be allowed to relax and fluctuate. There, we will discuss the effects that the relaxed vortex array has on the staircase robustness and scalar

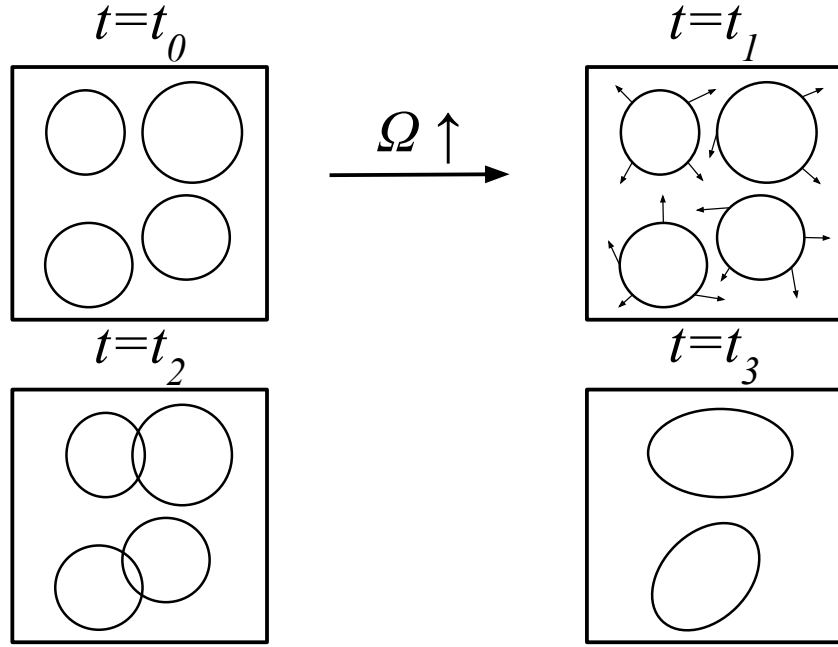


Figure 1.11. An increase in Ω , such that $\Omega > \Omega_s$, perturbs the location of the cell. Cells then run into each other and merge. Here, Ω_s represents the onset of cellular fluctuations.

transport.

1.3 Relaxation in a Fluctuating Vortex Array

Up till now, the research presented in this paper has involved only a slight modification to the problem of the fixed cellular array studied in [46]. We argue that the fixed cellular array setup is contrived and overly constrained; therefore, it is an unrealistic representation of physical systems observed in nature. Such physical systems will manifest variability, fluctuations, and intrinsic jitter. Thus, the cell pattern is more like a *melting vortex array* than a frozen array. Hence, we study the dynamics of a less constrained cell array (i.e., vortex array with fluctuations) and ask how resilient the staircase profile is in the presence of fluctuations. In addition to that, we give answers to the following:

1. What happens to interspersed regions of strong scalar concentration mixing as cells relax?

What about other cellular interactions?

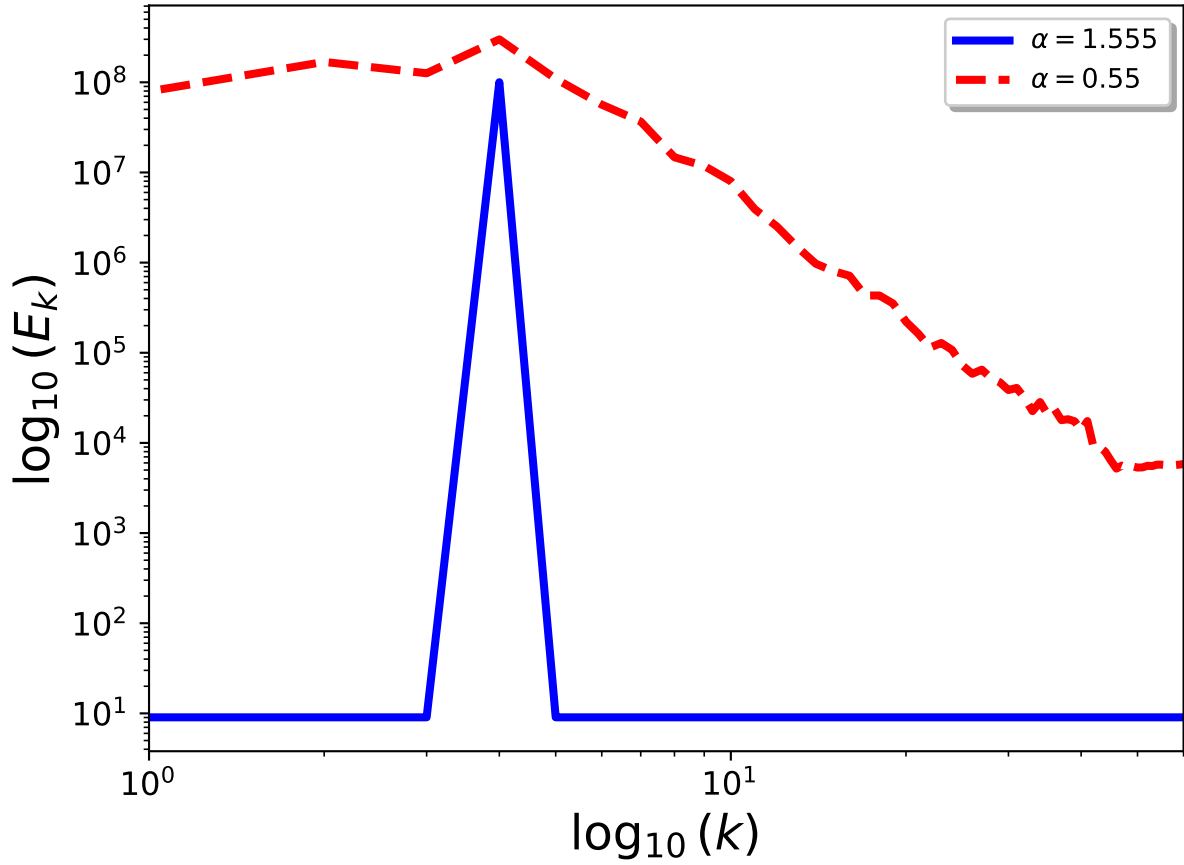


Figure 1.12. Log-log plot of the energy as a function of the wavenumber k for different α . The decrease in the α parameter results in large scale flows. The solid blue line denotes the zero temperature cellular flow ($\Omega \sim 5.7$), where all energy is mainly focused on $k = 4$. As α decreases, the energy grows for smaller k , as shown in the dashed red line.

2. What is the behavior of the scalar concentration trajectory through the vortex array? As we increase fluctuations in the vortex array, how will the speed and trajectory of the scalar concentration front change (See Fig. 1.24)?
3. How does the increase in scattering in the vortex array affect the transport of scalar concentration?

To answer these questions, we use the model of a “Melting Vortex Crystal”, advanced in [60, 61]. In this setup, a spatially periodic force creates the vortex array pattern. The vortex array flow is driven and damped by both viscosity and drag. The fluctuations of the vortex array can be

Transitions	Ω Range
SX	$\Omega < 6$
SXA	$6 < \Omega < 7$
OPXA/OQPXA	$7 < \Omega < 13$
SCT	$13 < \Omega$

Table 1.2. Different stages of the fluctuating vortex array. In this paper we consider small and large Ω any value within SX and SCT range, respectively.

explored systematically by scans of modest Reynolds numbers. Here, the Reynolds number is set by the forcing strength. Such a variable vortex array flow is akin to a melting crystal, where “melting” is related to the turbulent mixing field. By systematically varying the vortex array state (i.e., the degree of melting), one can explore the connection between cell size perturbations and staircase-variability.

The governing equation for the vortex array is the 2D Navier-Stokes (NS) equation, which can be written in nondimensional form as

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_{fv} \cdot \nabla \right) \omega = \frac{1}{\Omega} \nabla^2 \omega + F_\omega - \alpha \omega, \quad (1.17)$$

$$\nabla^2 \psi_{fv} = \omega. \quad (1.18)$$

We write the 2D incompressible NS equation in terms of the streamfunction ψ_{fv} and vorticity $\omega = \nabla \times \mathbf{u}_{fv}$, where $\mathbf{u}_{fv} = \hat{\mathbf{z}} \times \nabla \psi_{fv}$ is the fluid velocity at position $\mathbf{x}$ and time t . Note that $\mathbf{u}_{fv}$ is not the same flow velocity [Eq. (1.3)] we insert into the passive-scalar transport equation [Eq. (1.6)]. For a derivation of Eq. (1.17) and control parameters, see Appendix 6.2. The spatially periodic force F_ω in the NS equation is defined as

$$F_\omega \equiv -n^3 [\cos nx + \cos ny] / \Omega. \quad (1.19)$$

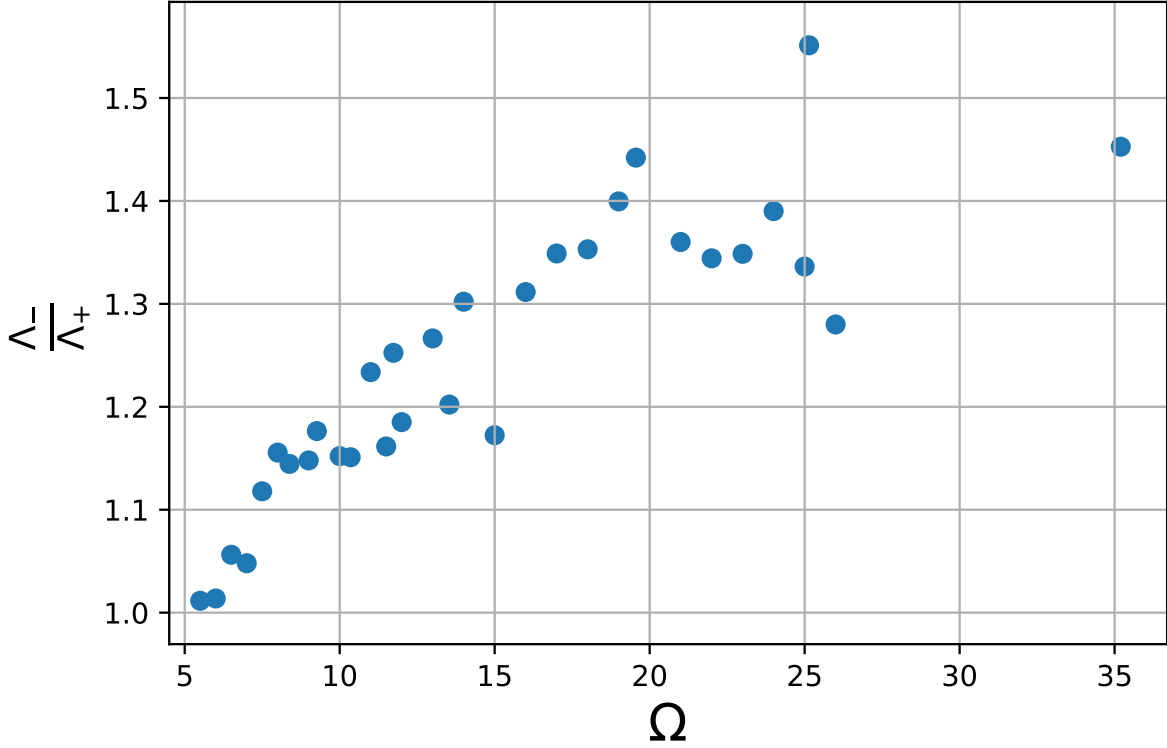


Figure 1.13. Plot of saddle area (Λ_-) to center area (Λ_+) as a function of Ω . In the regime of $\tau_v > \tau_S$, fluctuations lead to large scale flows, thus the area of saddles increase compared to area of centers.

Here n represents the injection wave vector and is set to $n = 4$ for all simulations. The boundary conditions for Eq. (1.17) are periodic in both x and y direction. In Eq. (1.17), there are two control parameters: the nondimensional Ekman-friction α , and Ω . Here, Ω is equivalent to the familiar and intuitive Reynolds number. Note that the velocity is set by the forcing. We can define Ω as the ratio of the viscous time scale ($\tau_v = n^2/k^2\nu$) to the cell circulation time scale ($\tau_H = nk\nu/F_{\text{amp}}$). The alpha parameter is defined as the ratio of cell circulation time scale to the mean shearing time scale ($\tau_S = 1/\alpha'$). Thus, the two control parameters are

$$\Omega = \frac{\tau_v}{\tau_H}, \quad (1.20)$$

$$\alpha = \frac{\tau_H}{\tau_S}. \quad (1.21)$$

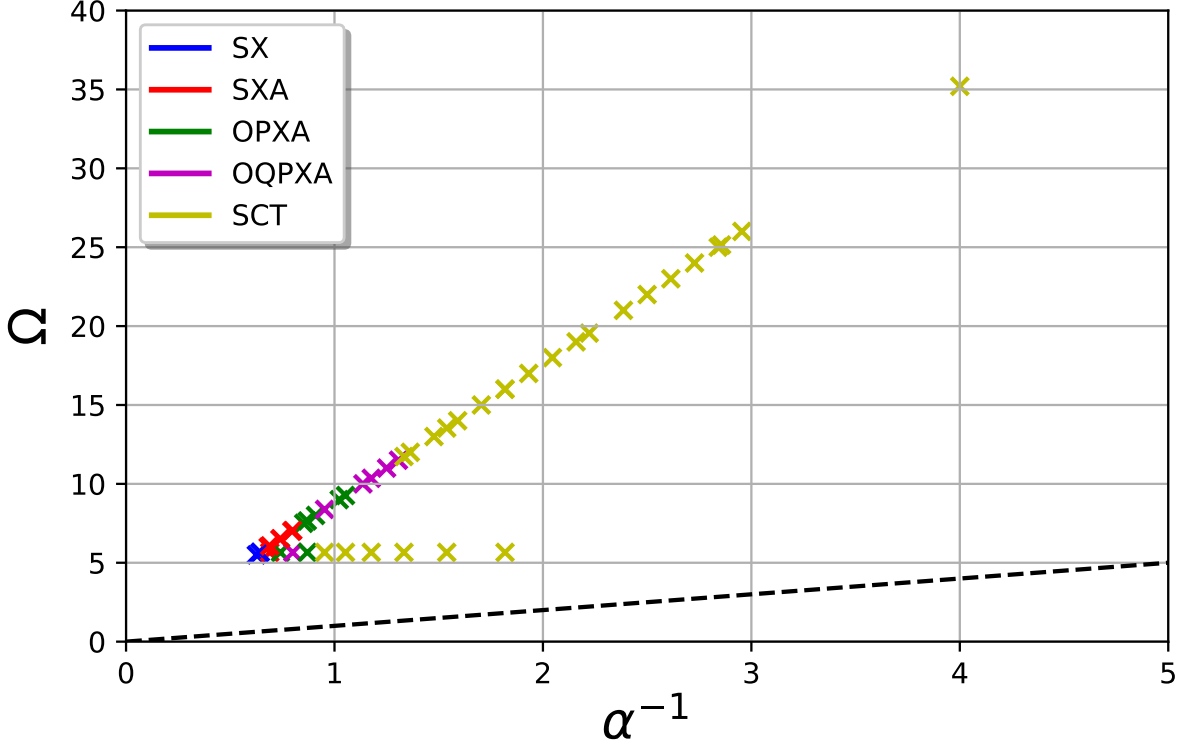


Figure 1.14. Plot of simulations in Reynolds (Ω) and $1/\alpha$ number space. Here Ω and $1/\alpha$ control turbulent mixing and large scale flows. The different stages of the fluctuating vortex array are highlighted in different colors. All simulations ran in this study fall in the regime of $\tau_v > \tau_s$.

Here F_{amp} is the forcing amplitude, k inverse length scale, ν the kinematic viscosity, and α' the dimensional Ekman-friction. The value of α affects large scale flow and Ω affects small scale flow and also the strength of the forcing function, F_ω . In this setup, the spatially periodic force F_ω yields, at $\Omega < 5.5$, a zero fluctuating vortex array.

The NS equation is solved numerically by using a pseudo-spectral method with 2/3 dealiasing cut-off and the 4th-order Runge-Kutta method for the time steps. The time step used here is $\delta t = 0.01$ with a spacial grid set to $N = 128$. To make sure that the temporal evolution of the system is obtained accurately, typical runs consist of 3×10^6 time steps. We remind the reader that this numerical scheme differs from that used to solve Eq. (1.6). Here, we employ the same numerical methods as Refs. [60, 61], aiming to replicate the results obtained in those works. By employing this numerical scheme, we can precisely generate, classify, and validate

various stages in the evolution of the fluctuating vortex array.

Before discussing the destabilization of the zero-temperature vortex array state, the initial vorticity in the DNS has the form,

$$\omega = \omega_s + 10^{-4} \sum_{m_1=1, m_2=1}^{2,2} [\sin(m_1x + m_2y) + \cos(m_1x + m_2y)] m_2^2 / \sqrt{m_1^2 + m_2^2},$$

where $\omega_s = -n[\cos(nx) + \cos(ny)]$. The system is evolved by the application of the force F_ω . Here, exploration of the different fluctuating vortex array stages is done by slowly increasing the value of Ω . Note that this study focuses on low-modest Reynolds numbers ($\Omega = 4 - 40$), mainly exploring a regime of spatio-temporal chaos. The fluctuating vortex array model allows us to realize similar physics to that which occurs in actual physical systems (e.g., drift-wave turbulence). For all simulations, the following are output (a) the total kinetic energy $E(t) \equiv \bar{\mathbf{u}}_{fv}^2$ and (b) the streamfunction ψ_{fv} .

We use the value of Ω to categorize the different stages in the evolution of the fluctuating vortex array. We denote the onset of cellular fluctuations as Ω_s . Here Ω_s is the effective Reynolds number where cellular fluctuations occur. When $\Omega < \Omega_s = 6$, the vortex array is referred to as a zero-temperature vortex array. The vortex array is allowed to fluctuate by increasing Ω from the stable state Ω_s , in increments of 0.5. This allows us to clearly mark transitions between states of the flow in the vortex array. The transitions are marked as follows: SX, the zero-temperature vortex array imposed by the force; followed by SXA, steady distorted array; leading to a distorted array that oscillates in time, either periodically (OPXA) or quasiperiodically (OQPXA); lastly, the system becomes disordered and exhibits spatio-temporal chaos and turbulence (SCT). Here spatio-temporal chaos means that the temporal evolution of the system becomes disorder and the spatial organization of the vortex array is distorted (i.e., flow patterns appear and disappear in irregular fashion). A table of these transitions is provided in Table 2.2. In Fig. 1.10, we show the energy time-trace for each of these transitions. These stages are categorized based on the behavior of the energy time-trace and its autocorrelation. In the autocorrelation analysis

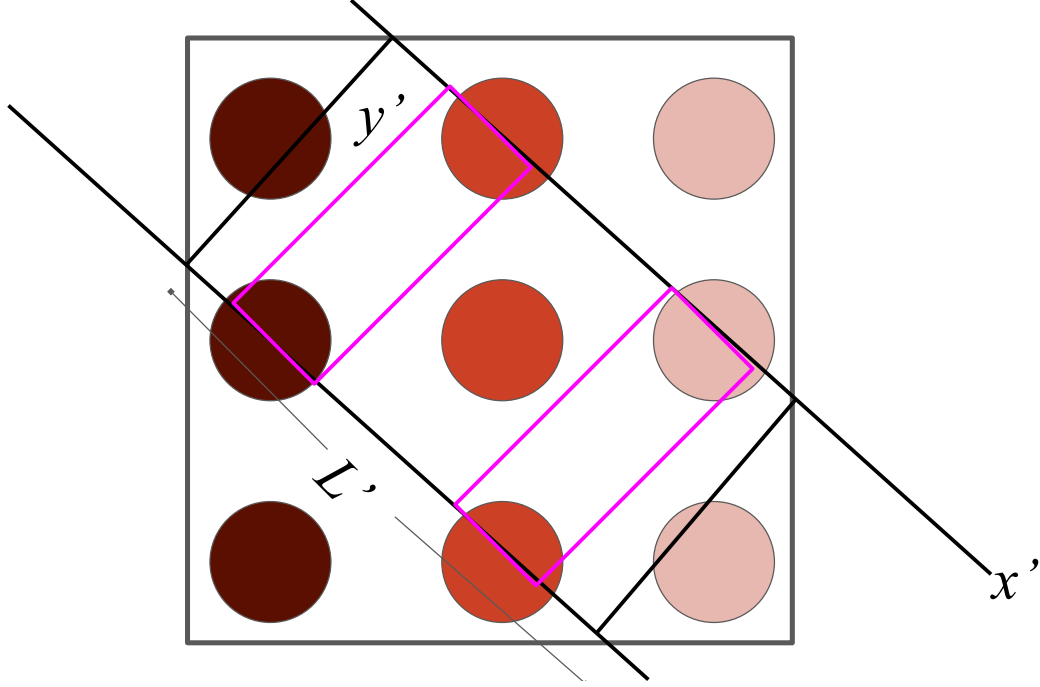


Figure 1.15. Illustration of the incline scalar concentration averaging. We note that averaging along y' as a function of x' leaves out cells due to y' and x' cutoffs. The cutoffs will make it appear as if $\langle n \rangle$ leaves steps.

of the energy, we measure the relationship of the observations between the different points in time, and thus seek a pattern or trend in the time series. For states of SX and SXA ($\Omega < 7$), the energy autocorrelation is 1 for all lags, while for OPXA and OQPXA ($7 < \Omega < 13$), the energy autocorrelation oscillates with the frequency of the periodic fluctuation for lag greater than zero. In the case of the chaotic/turbulent state ($\Omega > 13$), the energy autocorrelation value is low (~ 0) for all lags.

Before injecting the passive scalar into the flow, we address the question of “cellular interactions” as the cells are allowed to relax. This discussion will be of use when later discussing the staircase structure and transport of scalar concentration in the vortex array. It’s important to note that the flow will evolve differently depending on the values chosen for k , ν , and α' . For our study, we assign the following values to each parameter: $k = 10$, $\nu = 10^{-2}$, and $\alpha' = 0.55$. To understand the consequence of this, we have to look at the time scale definition of the control parameters $\Omega = \tau_v/\tau_H$ and $\alpha = \tau_H/\tau_S$. At this point, the alert reader will notice that

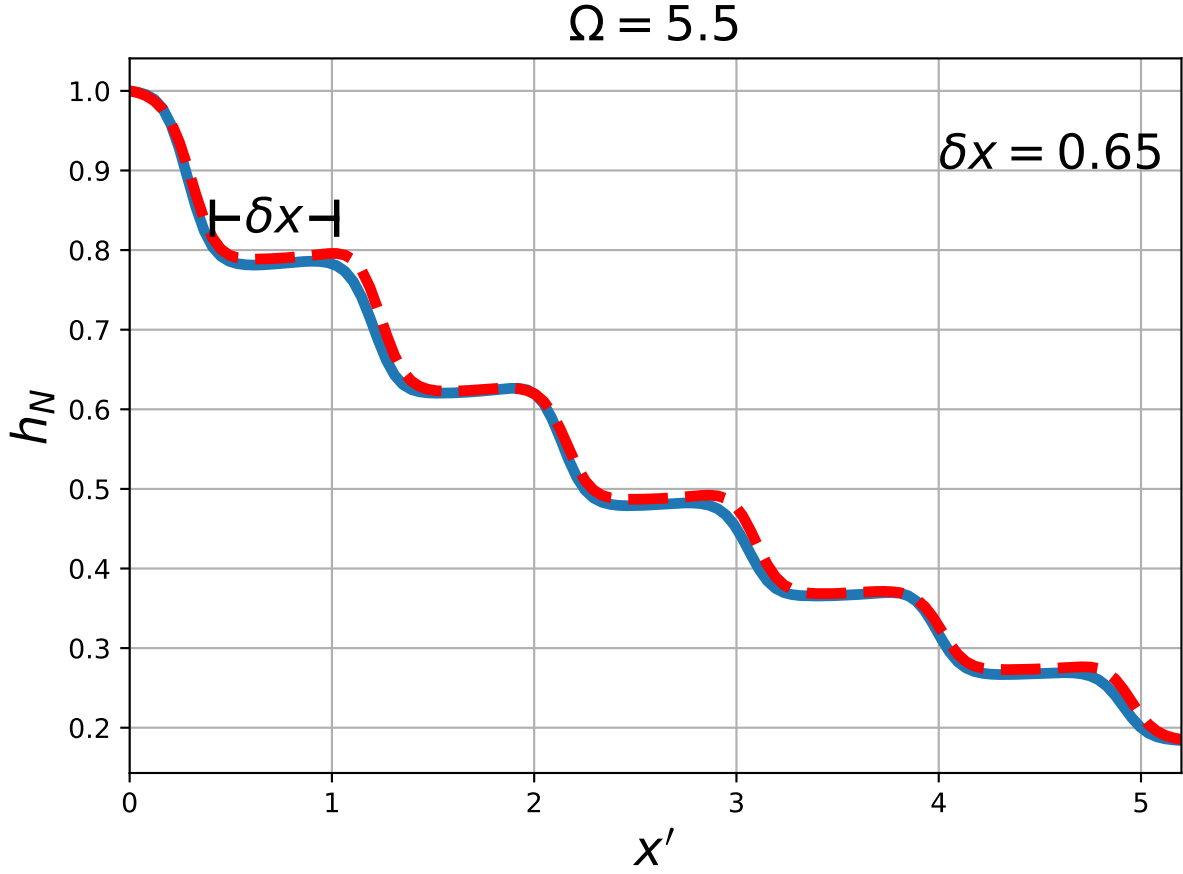


Figure 1.16. Illustration of the zero temperature cellular flow structure. The solid blue and dashed red plot lines correspond to averaging in the $\theta = -\pi/4$ and $\pi/4$ direction. In the zero temperature vortex array, the staircase profile for the blue (solid) and red (dashed) normalized $h(x')$ plot lines are the same.

the amplitude of the forcing (F_{amp}) couples α and Ω . Therefore, the two values are inversely proportional to each other. It is clear from Fig. 1.14 that $\Omega\alpha > 1$ for all simulations. In time scale notation, this means that $\tau_S < \tau_v$, which implies that large scale flows develop at a rate faster than fluid momentum diffuses over a scale length (n/k). Here, the diffusion of the fluid momentum is associated with the excursion of a cell during cell array fluctuations.

As a consequence of the $\tau_S < \tau_v$ condition, as Ω increases, cells merge. An increase in Ω , such that $\Omega > \Omega_s$, perturbs the location of the cells. This perturbation causes cells to run into each other and merge (See Fig. 1.11). The merging of flow lines results in vortices merging and forming larger vortex structures, which over time, develop into large scale flows. We confirm

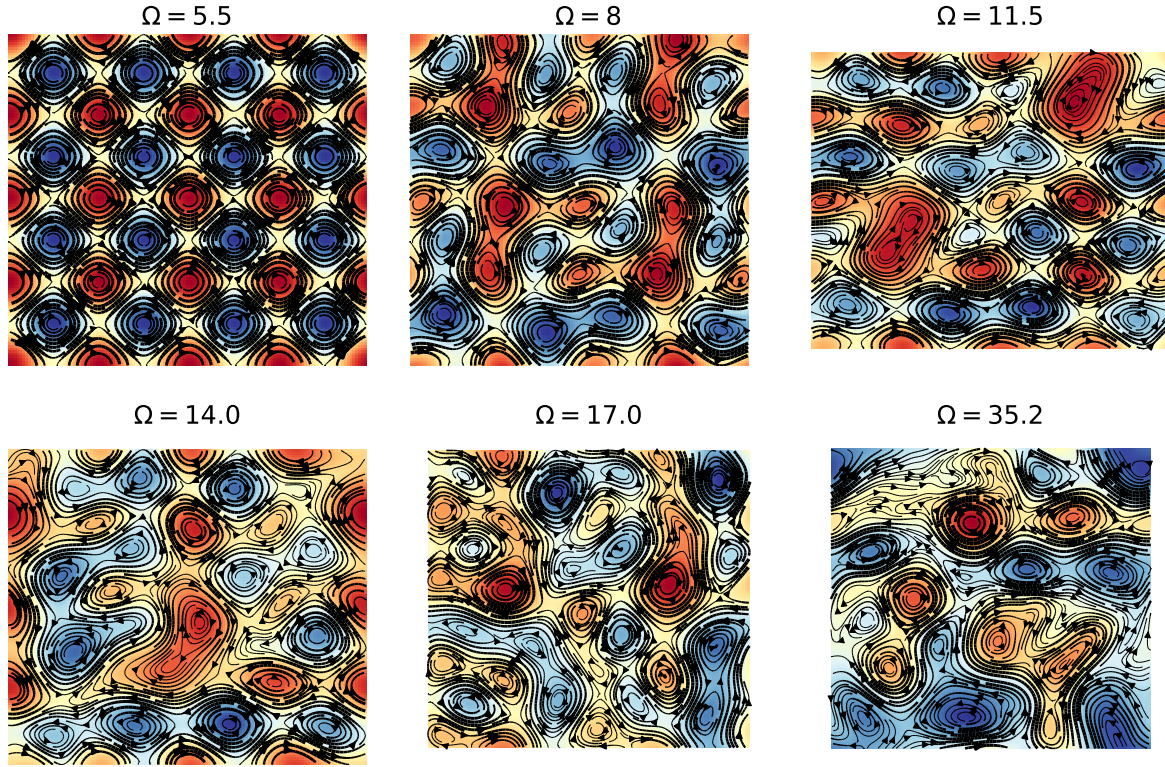


Figure 1.17. Contour plots, illustrating different stages of the fluctuating vortex array. As Ω increases, there is a merger of vortices along with distortions of the cellular flow array. Growth in Ω leads to scattered vortices and large scale flow structures.

this by qualitatively studying the contour plots of the streamfunction at different stages of the fluctuating vortex array (See Fig. 1.17). In addition to this, we observe that only vortices of the same sign merge.

To quantitatively analyze the formation of large scale flows, we investigate the influence of the nondimensional Ekman-friction parameter (α) on the fluctuating vortex array. In Fig. 1.12, we present an investigation of the α parameter. The solid blue line in the graph depicts the energy of the zero-temperature vortex array in wavenumber (k) space. When α is sufficiently large (~ 1.6), the parameter dampens both large and small scale flows. For large α , the flow follows the frozen vortex array pattern dictated by the external force F_ω . Here, energy is concentrated at the wave injection number $k = 4$. While maintaining a constant Ω ($\Omega \sim 5.7$), we decrease the value of α . The graph's dashed red line illustrates the evolution of the energy as α transitions to

a lower value. At a lower value of α , the energy increases significantly for values of k less than 4. This demonstrates that small values of α imply *large* scale flow structures (where the value of α serves as a control for large scale flows).

As an extension to the previous assessment, we use the Okubo-Weiss field to measure the evolution of large scale flows relative to vortices as the value of Ω increases. The Okubo-Weiss field (Λ) is defined as

$$\Lambda = \frac{\partial u_x}{\partial x} \frac{\partial u_y}{\partial y} - \frac{\partial u_x}{\partial y} \frac{\partial u_y}{\partial x}. \quad (1.22)$$

The Okubo-Weiss field gives the difference between square vorticity and square shear strain [62, 63]. The quantity Λ characterizes the local topology (i.e., Gaussian curvature of streamlines) of an incompressible two-dimensional fluid flow. Here, Λ is a distinctive measure of fluid properties, particularly in distinguishing and describing regions dominated by deformation and rotation. In Fig. 1.13, we plot the ratio of $\Lambda < 0$ to $\Lambda > 0$, which corresponds to the ratio of the area of saddles to the area occupied by centers, respectively. The increase in Ω results in an increase in the population of saddles relative to centers, which coincides with our previous assessment based on the evolution of energy in wavenumber space.

For different stages of the fluctuating vortex array, the streamfunction (ψ_{fv}) is output from the vorticity equation [Eq. (1.17)]. It is important to note that $(\nabla \psi_{fv})_{fv}$ is dimensionless, thus when inserting into Eq. (1.3) we make the following transformation, $\nabla \psi = (\pi/d) (\nabla \psi_{fv})_{fv}$. We insert this into the passive-scalar equation [Eq. (1.6)] through the velocity term [Eq. (1.3)],

$$\mathbf{u} = (\tilde{u}d/\pi) \hat{\mathbf{z}} \times \nabla \psi = \tilde{u} \left(\hat{\mathbf{z}} \times (\nabla \psi)_{fv} \right). \quad (1.23)$$

By doing this normalization, we can compare passive scalar transport in the fluctuating vortex array to the fixed cellular array. The scalar concentration is evolved in this flow using the same boundary conditions discussed in Sec. 1.1.2. In the following subsections, we present the effects

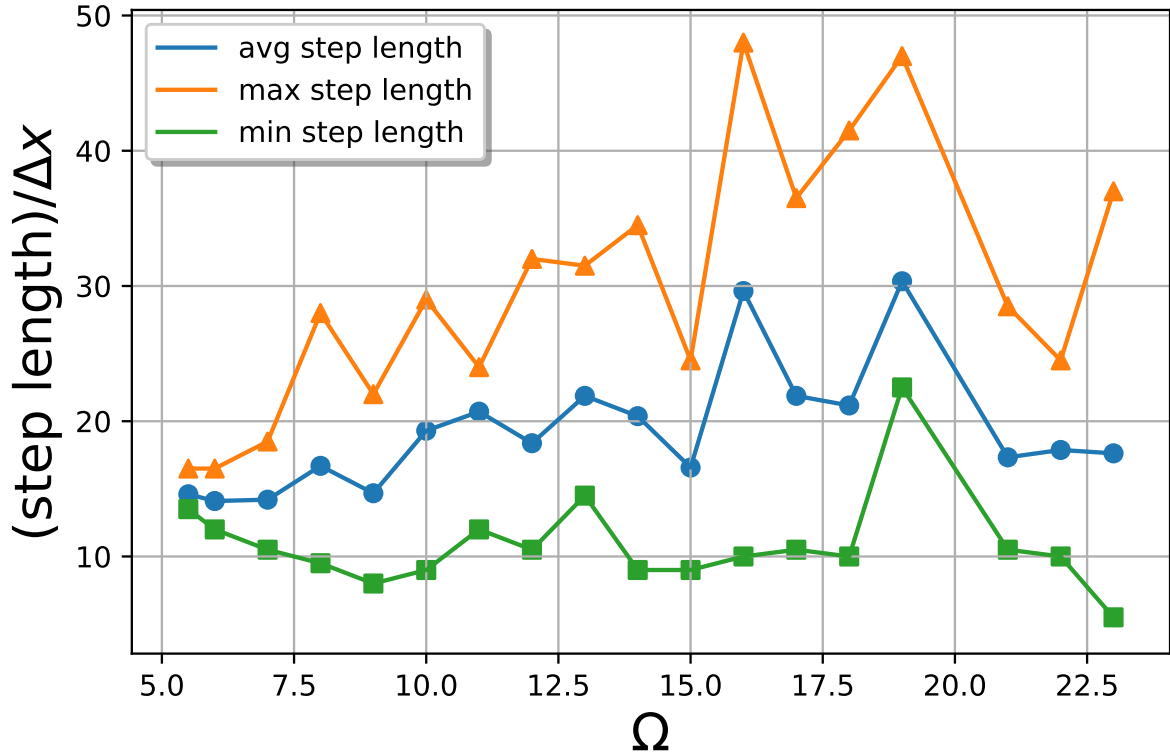


Figure 1.18. Plot of step size divided by the spatial grid spacing ($\Delta x = L/N$) as a function of Ω . As the vortex array fluctuates, cells begin to merge, leading to overall average step size and max step size increase.

large scale flows and vortex fluctuations (in the regime of $\tau_v > \tau_s$) have on staircase structure resiliency and passive-scalar transport.

1.3.1 Staircase Profile in the Vortex Array

Now that we have a general understanding of how Ω affects cellular interactions and flow dynamics, we drive a flux of scalar concentration (similar to Sec. 1.1.2) in the flow created by Eq. (1.23). We stress that simply using Eq. (1.10) will not result in a staircase profile. Here, the average of the scalar concentration must be calculated at different orientations in order to recover the staircase profile. By doing so, we can establish a connection with the fixed cellular array (i.e., baseline case) and show pattern formation occurs. In addition, this also demonstrates that oblique staircases exist, which *can* influence transport in x .

To produce the staircase profile, we rotate the scalar concentration field in two different directions, $\theta = \pm\pi/4$, with respect to x . In Fig. 1.15, we illustrate the averaging along one of the orientations. Therefore, we now define the average of the scalar concentration as

$$h(x') = \langle n(x') \rangle_{y'} = \frac{1}{L'} \int_0^{L'} n(x', y') dy'. \quad (1.24)$$

Here, the terms x' and y' represent the x -axis and y -axis in the two different orientations. In terms of L , the value of L' is $0.82L$. To setup a baseline case, we consider the zero-temperature vortex array. In this stage, the average scalar concentration profile [Fig. 1.16] takes the form of the staircase presented in [46]. We note that for the case of $\Omega = 0$ (i.e., no fluctuations), the result is the same since the vortices become locked in the cellular array pattern.

We repeat the same process for different stages of the fluctuating vortex array. As cells are scattered from the zero-temperature state, the cells begin to merge, resulting in larger cell structures. The formation of large cell structures is a consequence of the $\tau_S < \tau_V$ condition, which we discuss in detail in Sec. 1.3. In Fig. 1.22, we examine two different cases, namely $\Omega = 8$ and $\Omega = 11.5$. Here, the two contour plots of the scalar concentration demonstrate that cell streamlines connect for several cell structures. The connection of cell streamlines reflect as mergers in staircase steps in the average scalar concentration profile. By looking at the profile of h , one can mistakenly interpret that there are no steps or that steps are missing. To explain this, we emphasize that steps become elongated due to vortex mergers and that the missing steps correspond to the cutoffs of the incline scalar concentration averaging, as shown in Fig. 1.15. The rest of the fluctuating vortex array stages are summarized through analysis of the step-size [Fig. 1.18] and curvature of the staircase profile [Fig. 1.19] as a function of Ω .

To assess the mergers in steps, we measure the step length at different values of Ω (See Figs. 1.18, 1.22, 1.23, and 1.30). Here, the average and maximum step length in the staircase structure grows with Ω . This analysis verifies that cell mergers result in elongated staircase steps (as discussed in Fig. 1.22). The plot of the step-size also confirms that as Ω increases, the cells

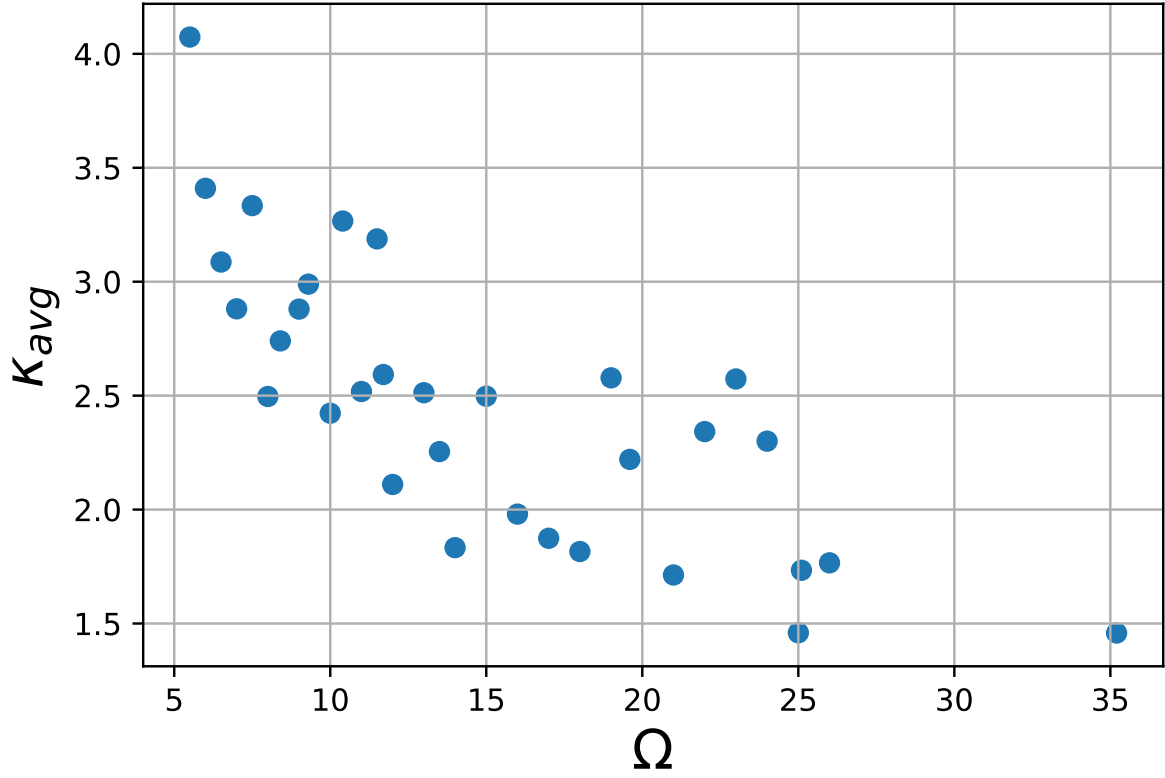


Figure 1.19. The plot of the curvature as a function of Ω . As cells scatter, the merger of steps decrease interspersed regions of corrugation, hence a decline in $h(x')$ curvature.

in the flow grow in size.

In addition to step measurements, we calculate the curvature of the scalar concentration profile. Based on our study of Figs. 1.18 and 1.22, as the vortex array relaxes, the number of steps decrease due to mergers. The result of step mergers means there are fewer interspersed regions of corrugations. In order to measure the interspersed regions of corrugations, we calculate the curvature of the scalar concentration profile

$$\kappa = \int_0^{L'} \frac{h''(x')}{(1 + h'(x')^2)^{3/2}} dx', \quad (1.25)$$

where $h(x')$ represents Eq. (1.24). We discern in Fig. 1.19 that the increase in Ω results in the decrease of κ . It is important to note that the value of κ does not vanish, but instead

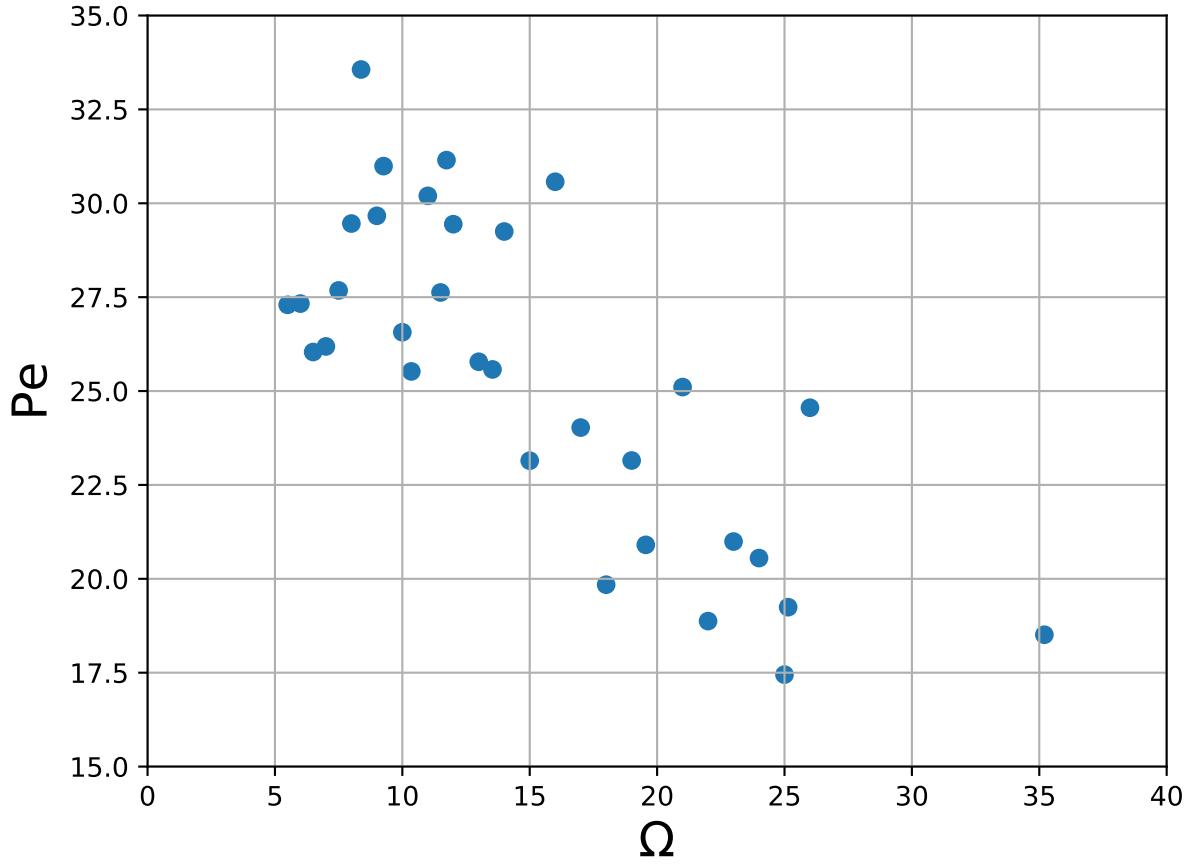


Figure 1.20. Peclet number as a function of Ω . As the vortex array is allowed to fluctuate, the two disparate time scales change. Fluctuations of the cells, weakens the cell mixing time scale.

remains above ~ 1.5 for large values of Ω . Here, κ quantifies and describes the step length scale in the staircase. Large and small values of curvature represent small steps and big steps, respectively. Similarly, the value of κ can also be interpreted as a measure of the number of steps or interspersed regions of corrugations in the staircase profile. Hence, the profile curvature is a signature of layering.

1.3.2 Staircase Resiliency

In this subsection, we provide an in-depth discussion on staircase resiliency. So far, we have only provided evidence that staircase structures persist up to modest values of Reynolds numbers. In order to give a precise meaning to the statement of “resiliency”, a set of criteria

Staircase Resiliency Criterion

Peclet Number	$Pe \gg 1$
Curvature	$\kappa \geq 1.5$

Table 1.3. Criterion of staircase resiliency.

must be established.

First and foremost, $Pe \gg 1$ is a necessary condition for the formation of transport barriers in the process of scalar mixing, as mentioned in Section 2.1. Maintaining $Pe \gg 1$ over a relevant range of Reynolds numbers is thus one natural criterion for “resilience”. It is of use then to calculate Pe for different stages of the fluctuating vortex array. Since we control the max flow velocity and background diffusivity ($D = 1.7 \times 10^{-2}$), the Pe number only depends on the dimensions d_x and d_y of the cells. As discussed in Section 1.1.2, d represents the length of the cell in the direction of the mean gradient (d_x) and β represents the ratio of d_x to the length of the cell perpendicular to the mean gradient (d_y). By using the streamfunction (ψ_{fv}), we can determine the average size of the cells in the flow in order to calculate the Pe number. In Fig. 1.20, we show for values of Ω greater than 25 that Pe decreases up to a value of ~ 17.5 . Hence, the $Pe \gg 1$ criterion is satisfied for the range of $0 < \Omega < 40$.

On the topic of Pe , it is useful to measure the ratio of Pe to Reynolds numbers used here. We can define the Schmidt number, which measures the ratio of momentum diffusivity to scalar diffusivity,

$$Sc = \frac{\tau_D}{\tau_v}. \quad (1.26)$$

For our simulations, we operate in a regime of $Sc = [0.5, 7.5]$. It is unclear whether maintaining $Sc \sim \mathcal{O}(1)$ is a strict or necessary requirement in order to sustain the staircase structure. In future work, we plan to study the Sc number in more detail.

$$\Omega = 18.0$$

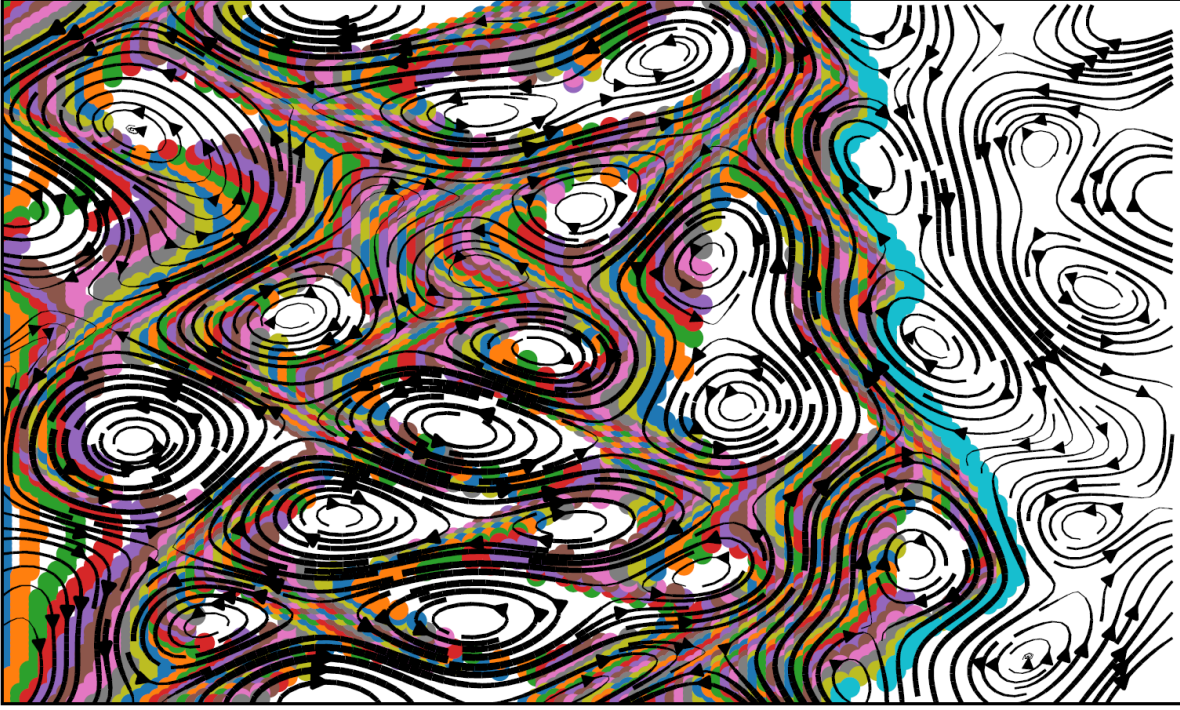


Figure 1.21. In the process of forming a staircase structure, the scalar concentration first forms a “web”. The scalar concentration flows along areas of strong shear. The holes correspond to vortex structures. Here, the barriers are formed first because scalar concentration travels along regions of strong shear. Over time, scalar concentration slowly enters the vortices and becomes homogenized.

A second criterion is that a staircase should maintain a sufficiently high curvature (equivalent to sustaining a sufficient number of steps) over the relevant range of Ω values. The precise critical value of curvature (κ) is not determined, though our studies suggest $\kappa \gtrsim 1.5$ is an adequate value for a staircase. In Fig. 1.19, the plot of κ vs Ω shows that κ is greater than 1.5 for most values of Reynolds numbers. Even for the largest Reynolds number studied, κ does not vanish, though it does decrease as the staircase coarsens due to mergers. To substantiate this assertion, we present Figs. 1.23 and 1.30, which illustrate the presence of staircase structures at the highest values of Ω . Hence, we identify the condition $\kappa \geq 1.5$ as a second criterion.

Therefore, resilience of a staircase over a given range of Ω requires:

1. $Pe \gg 1$, so barriers are formed and maintained.

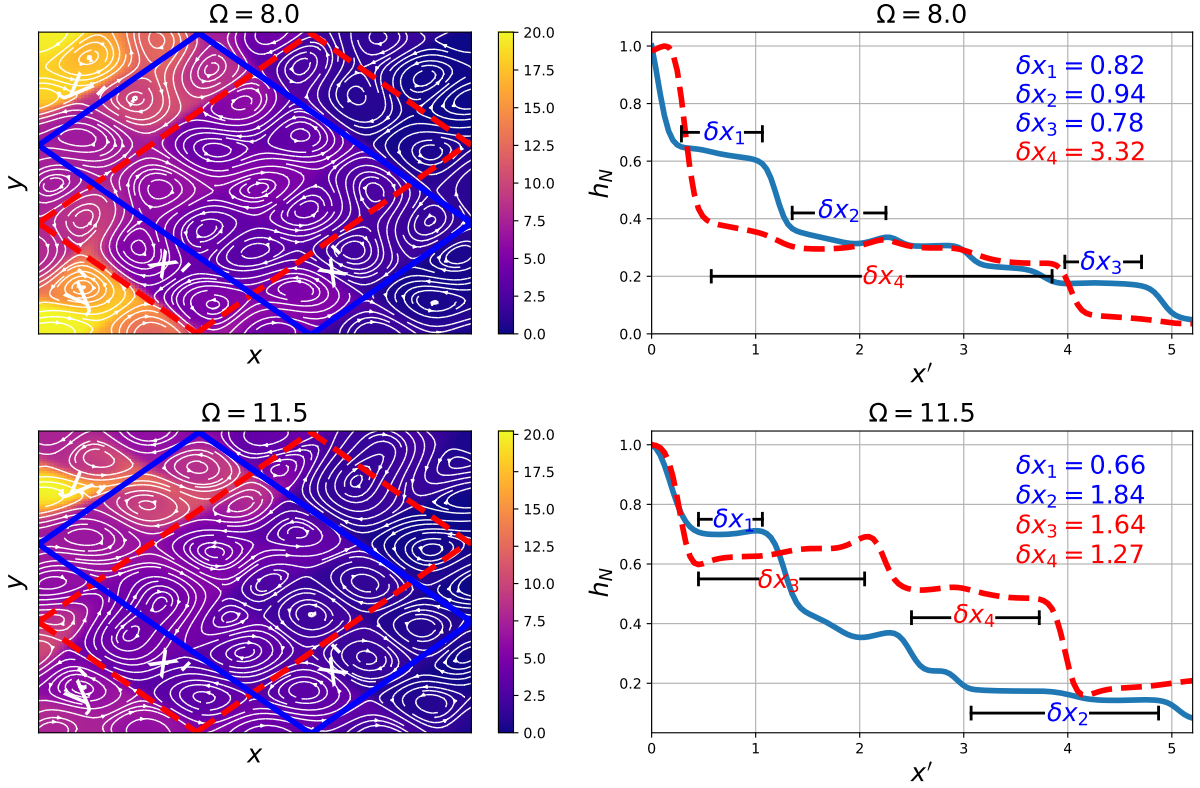


Figure 1.22. Snapshot of passive-scalar concentration for the case of $\Omega = 8.0$ and $\Omega = 11.5$. On the left figures, the streamlines display mergers/connections of vortices. These mergers are reflected on the normalized $h(x')$ profiles on the right, by the elongated staircase step on the solid blue and dashed red line. The blue (solid) and red (dashed) plot lines correspond to averaging in the $\theta = -\pi/4$ and $\theta = \pi/4$ directions with respect to x . As the cells fluctuate from the zero temperature state, the staircase structure remains robust. We indicate steps as regions where $dh/dx' \sim 0$. These are clearly separated by distinct jumps (i.e., regions of corrugations).

2. $\kappa \geq 1.5$, so a requisite number of steps is sustained.

Clearly, criterion (1) is based on first principles, while criterion (2) is empirical, and emerges from our studies. Finally, we speculate that the curvature criterion (2) may be ultimately related to the $\text{Pe} \gg 1$ criterion (1). However, we are unable to report anything further at this time. We summarize the criterion for resiliency discussed here in Table 1.3. In the following subsection, we will direct our attention to the transport of the scalar concentration in the vortex array.

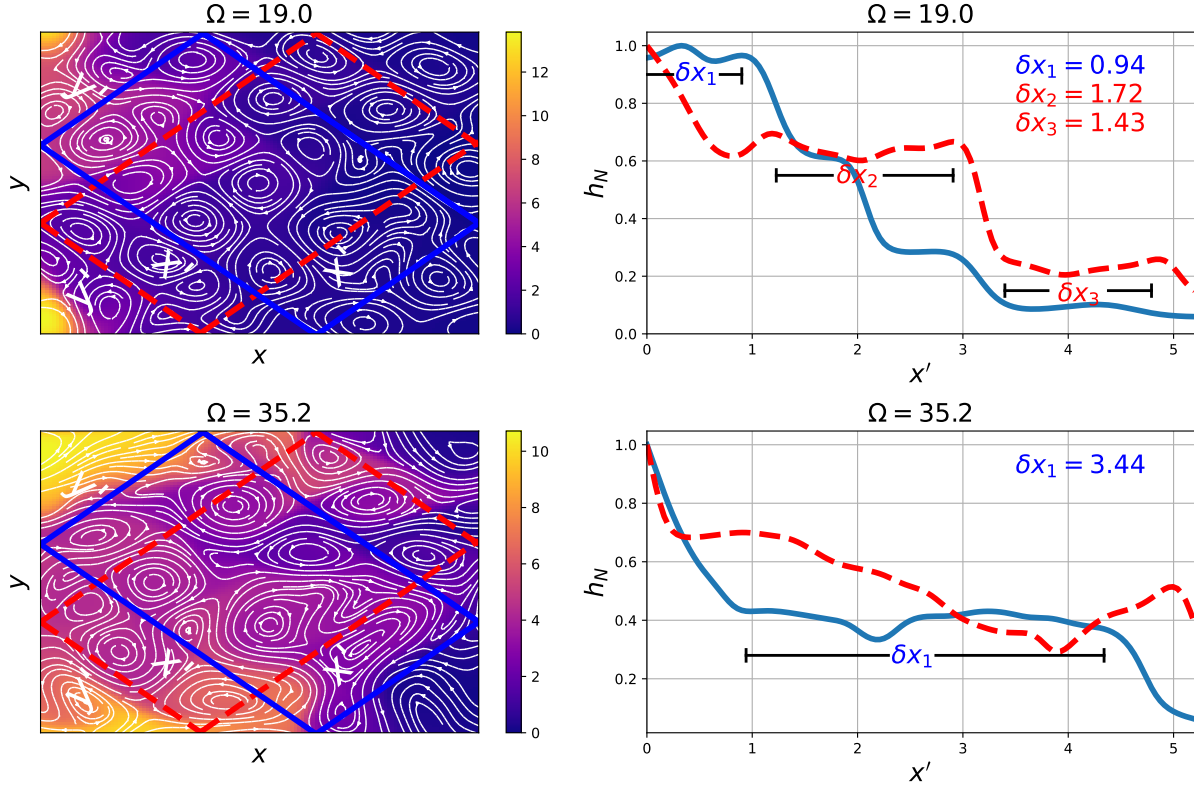


Figure 1.23. Snapshot of passive-scalar concentration for the case of $\Omega = 19.0$ and $\Omega = 35.2$. Similar to Fig. 1.22, cell mergers are reflected on the normalized $h(x')$ profiles on the right. Here, we demonstrate evidence that the staircase structure persists up to the value of $\Omega = 35.2$

1.3.3 Transport in the Vortex Array

In this subsection, we explore the trajectory and transport of scalar concentration in the vortex array. Note that the scalar concentration trajectory is the path that an element of scalar concentration (i.e., scalar particle) takes to reach the right boundary (See Fig. 1.24). In the process of scalar concentration reaching the right boundary it's of use to understand how interspersed regions of corrugations form (i.e., how a staircase structure forms).

We find that in the process of forming a staircase, the scalar concentration forms a “web” structure. The web structure represents the flow of scalar concentration along regions of strong shear and around vortices. In Fig. 1.21, we see that barriers in the staircase form before steps because scalar concentration travels *along* cell boundaries first. Based on previous

examination of the Okubo-Weiss field (See Fig. 1.13), the increase in Ω will result in a wider scalar concentration path (i.e., the web will form thicker fibers). To understand how the widening of the scalar concentration path affects the scalar trajectory, we apply a pulse train of scalar concentration to the left boundary [Eq. (1.11)].

When scalar concentration is injected into the flow via a pulse train method, a flamelet network pattern forms (See Figs. 1.25 and 1.26). As stated in Ref. [64], flamelet fronts are visible in the regime of Peclet and Damköhler (Da) numbers much greater than one. The nondimensional number Da is defined as the ratio of cell circulation to reaction time. Here, the reaction time corresponds to the speed of the scalar concentration front propagation, which is faster than the cell rotation speed. We define “front” as the scalar concentration’s furthestmost point in x . The regime of $Pe \gg 1$ and $Da \gg 1$ is also known as the weak advection regime [65, 66, 67, 68, 69]. The weak-stirring of scalar concentration process ensures that steep gradients persist at the boundaries of the cells, as shown in Fig. 1.21. Yet, as the flame front propagates, the vortex entrains the flame front over time by a sort of “homogenization” process (i.e., by the synergy of differential rotation and diffusion) [70]. This explains why we still see the formation of the staircase in Fig 1.22.

From earlier analysis, we know that scalar concentration travels along contours of strong shear; therefore, we can use the Okubo-Weiss field to map the trajectory of the scalar concentration. To map the path of the scalar concentration in the Okubo-Weiss field, we connect regions of strong shear to their nearest strong shear neighbor, as shown in Fig. 1.26. Proof that the scalar travels along areas of strong shear is firmly established by comparing the paths found in the Okubo-Weiss field to those in the scalar field. Here, it is significant to note that the scalar concentration travels along strong shear but not across it.

By mapping the scalar concentration trajectory, we learn how the vortex array affects the speed of the scalar concentration front propagation. Here, we use the map of the trajectory to determine the travel distance. The travel time is determined by the number of time steps required for the scalar concentration’s furthestmost point (in x) to reach the right boundary. While pulsing

scalar concentration, we mark points (in green) on the scalar concentration contours (Fig. 1.25) when n attains a critical value ($n_{\text{crit}} = 0.1$). This enables us to quantify the travel time of the scalar concentration. We show the average scalar speed as a function of Ω in Fig. 1.27. The plot shows that the scattering of vortices leads to an overall decrease in scalar concentration velocity. An idea that is relevant here is the *least time criterion*. As the cellular array scatters, the path of least time increases in length (similar to scattered path of light in atmosphere). We also find that the average scalar concentration front speed follows a similar trend to that of the staircase curvature plot as we increase Ω . We speculate that curvature may ultimately be related to the scalar concentration front speed,

$$\begin{aligned}\bar{u} &\sim \frac{1}{\Omega^a}, \\ \kappa &\sim \frac{1}{\Omega^b}.\end{aligned}$$

Here, a and b represent different constants. Therefore, $\bar{u} \sim \kappa^{a/b}$.

Finally, we investigate how the effective diffusivity for the relaxed vortex array differs from the fixed cellular array. Again, applying Fick's law, we find that as the cells fluctuate, the effective diffusivity also fluctuates but remains close to the validated effective diffusivity [Eq. (1.4)]. We show this in Fig. 1.28a. Note that we calculate D^* for the two orientations at points $x_1 = (1/4)L'$ and $x_2 = (3/4)L'$ using the method described in Appendix 6.3. Although cell structures fluctuate, we note that as long as the cell boundaries are not destroyed, the transport across cell boundaries will not change. The reason for deviation in the effective diffusivity is due to geometrical properties of the cell structure and the cell turn-over velocity.

As mentioned in Section 1.3.1, all simulations ran in this experiment use a fixed background diffusivity; therefore, only cell dimensions and cell turn-over velocity ($0 < u_{\text{cell}} < \bar{u}$) will determine the effective diffusivity. Based on the definition of Eq. (1.4), the only significant

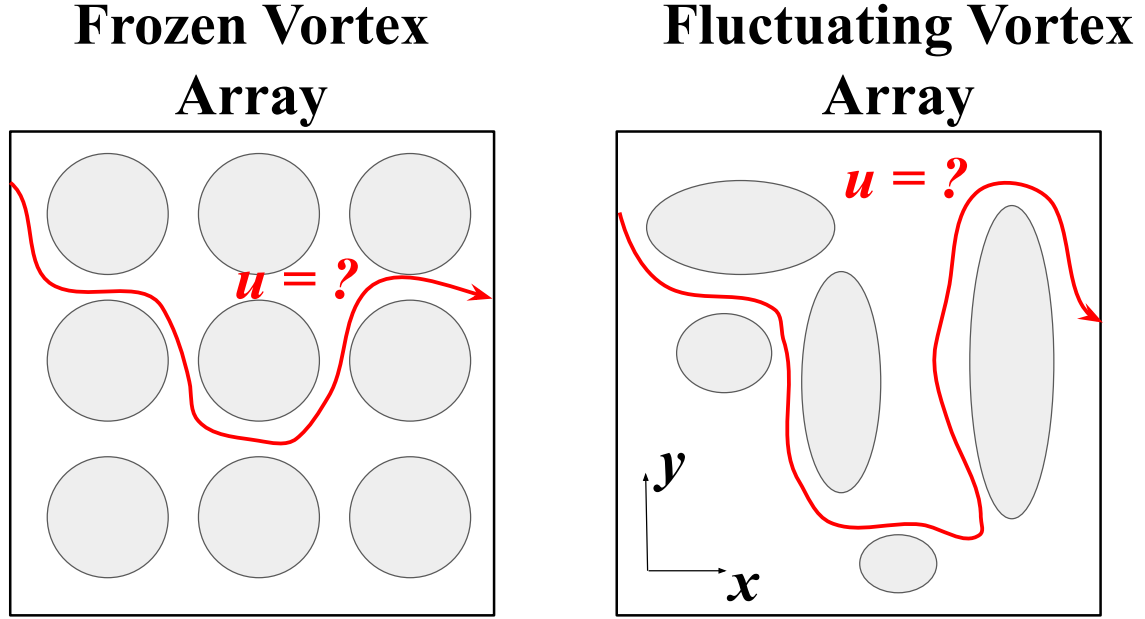


Figure 1.24. The scalar concentration trajectory at various stages of the fluctuating vortex array. It's important to note that the scalar concentration trajectory is the path that an element of scalar concentration (i.e., scalar particle) takes to reach the right boundary. The solid curved (red) line represents the path of the scalar particle.

values in D_{cell} are d_x , d_y , and u_{cell} . Thus, we can define a scaling for the effective diffusion as

$$D^* \propto \sqrt{u_{\text{cell}} d_x \beta}, \quad (1.27)$$

where β is defined as d_x/d_y . As depicted in Fig. 1.17, the cells exhibit variations (both d_x and d_y change) with the incremental rise in the value of Ω . In this paper, we will examine the impact of d_x and d_y , as our emphasis is on the impact of cell geometry on pattern formation.

A plot of β as a function of Ω is shown in Fig. 1.28b, which demonstrates similar characteristics to the effective diffusion plot. The β shown in the graph is an average of all cell sizes for each Ω ; nevertheless, the mean of the aspect ratio shows the direct effects the dimensions of the cells have on transport. If d_x is greater than d_y , D^* will be greater than the effective diffusivity for the fixed cellular array, which we refer to as D_{RB} . If d_x is less than d_y , the effective diffusivity is less than D_{RB} . We confirm this theory by analyzing the cellular flow at

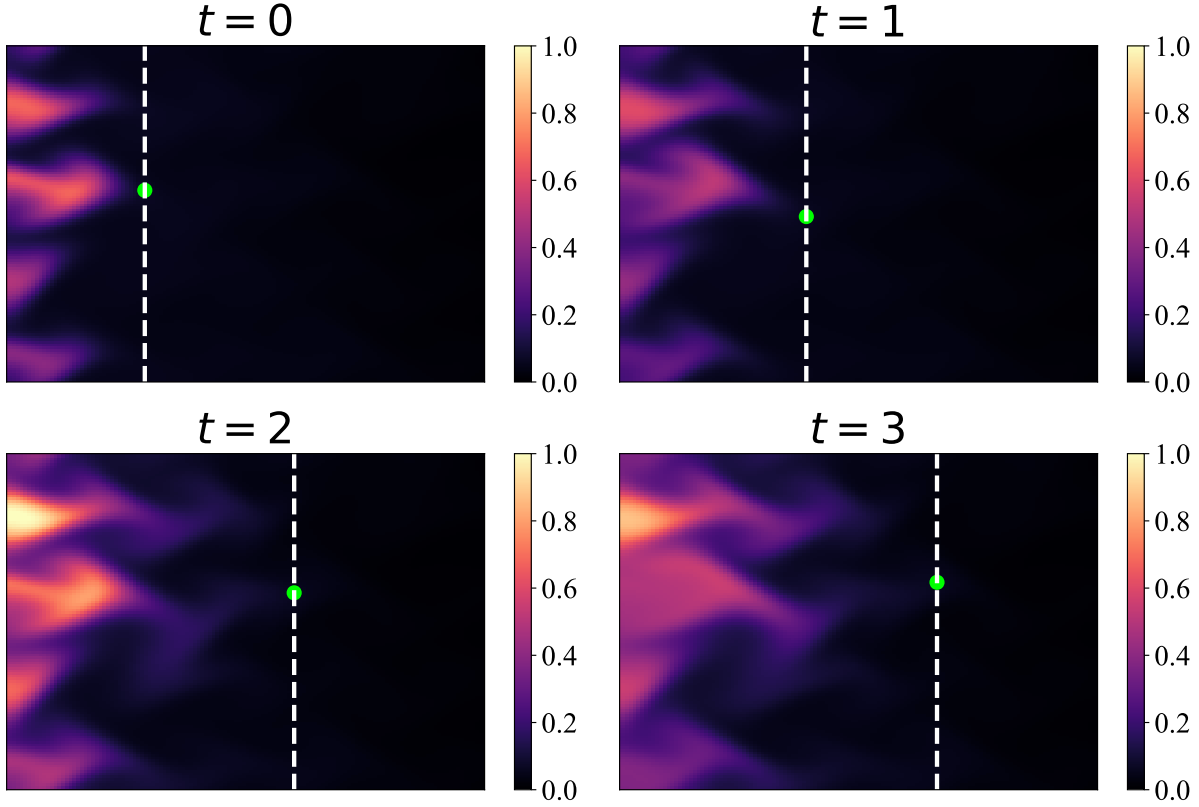


Figure 1.25. Pulse train of scalar concentration in $\Omega = 11.5$ flow. Flame front propagates, over time vortex entrains flame front. At different times, we mark the position of $n_{\text{crit}} = 0.1$ in green (light gray).

different orientations. We use $\Omega = 10.35$ to illustrate this point (See Fig. 1.29). As shown in Fig. 1.29, $\beta \sim 1$ is the average for the cells, but there are outliers (due to cell mergers) that make the average go above or below $\beta = 1$. This is a reason why D^* deviates somewhat from D_{RB} .

To assess the relationship between cell dimensions and D^* , we compute the correlation coefficient between Fig. 1.28a and its corresponding d_x^2/d_y value, which results in a value of 0.47. The computed value suggests a moderately positive correlation between cell dimensions and D^* . It is important to note that another factor to consider is the circulation speed of the cells in the region where D^* is measured. Given the variability in cell lengths, the cell turn-over velocity will also exhibit fluctuations, influencing the effective diffusivity. Nonetheless, our findings indicate that with knowledge of cell dimensions d_x and d_y , one can qualitatively approximate the trend of the effective diffusivity in a fluctuating vortex array.

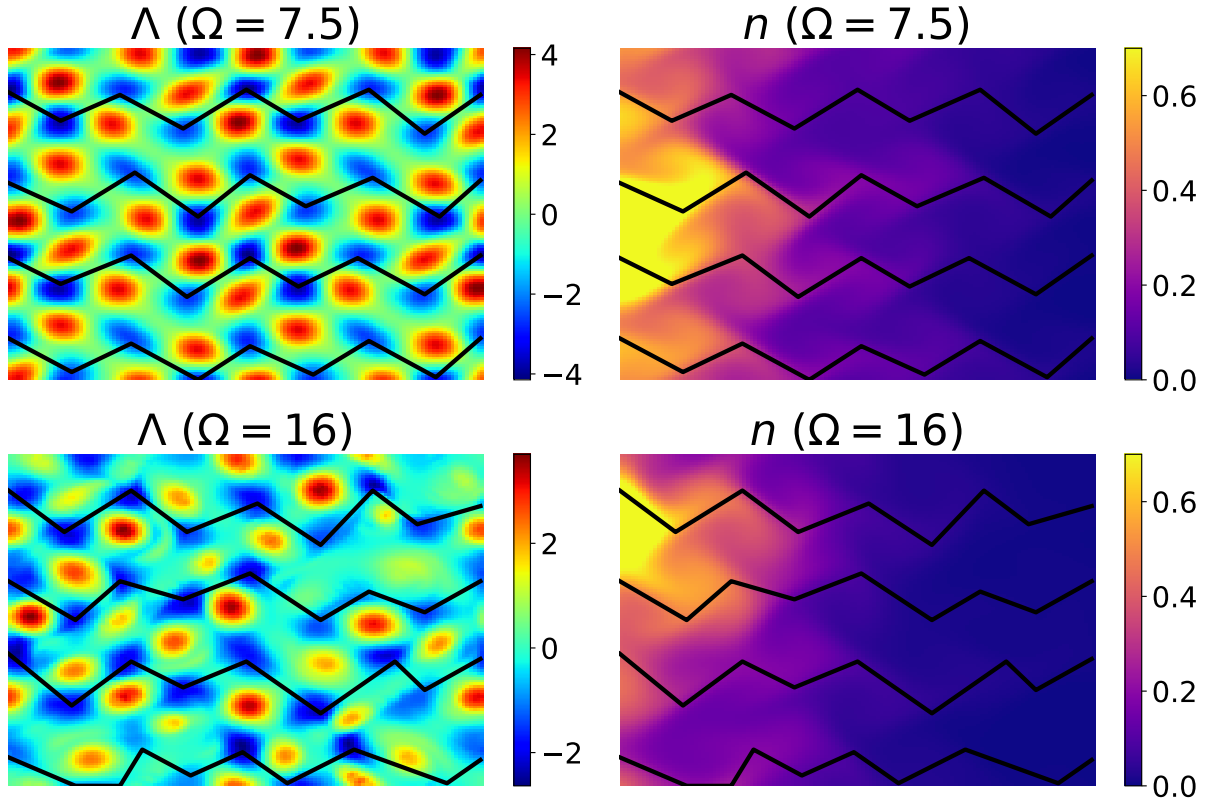


Figure 1.26. In Okubo-Weiss field areas of strong shear ($\Lambda < 0$) are connected to their nearest strong shear neighbor. The connections in the Okubo-Weiss field are mapped to the scalar concentration contour to show that scalar concentration travels along areas of strong shear.

1.4 Conclusion

In this paper, we examine a minimal approach to layering to explore staircase resiliency and scalar concentration transport in a very simple system. The constrained cellular array, which produces layering as a consequence of two disparate time scales, is probed by introducing global transverse shear and turbulence-induced mixing in the cellular flow. It's important to note that there is no dynamic feedback necessary. Here, we summarize the principal results of this paper:

1. In an array of cells, the staircase structure forms in the presence of global transverse shear and turbulence-induced mixing. Each, of course, affects the staircase structure differently.
2. Global transverse shear introduces a shear dispersion time scale, which *enhances* the

mixing of scalar concentration along y . This shear suppresses transport along x , resulting in an improvement in the confinement of scalar concentration.

3. Staircases form in the fluctuating vortex array over a broad range of modest Reynolds numbers ($0 < \Omega < 40$). N.B., regions of corrugations lessen, due to cell mergers. The staircases in the fluctuating vortex array satisfy the two criteria for resiliency, $Pe \gg 1$ and $\kappa \geq 1.5$.
4. Scalar concentration travels along regions of strong shear, creating a “web” structure. Here, staircase barriers form first, and scalar concentration “homogenizes” in vortices at a later time.
5. The scattering of vortices leads to an overall decrease in scalar concentration front velocity. The paths are those of the least time.
6. If background diffusion is kept fixed, cell geometric properties can qualitatively approximate the trend of the effective diffusivity. The effective diffusivity of the fluctuating vortex array does not deviate significantly from $D\sqrt{Pe}$.

The results of these simulations and their implications for a physical system, such as drift-wave turbulence near marginal stability in magnetic confinement devices, are reviewed and elucidated below.

The addition of a global transverse shear [Eq. (1.12)] to the fixed cellular array introduces a shear dispersion time scale, τ_{adv} . We find that for a critical shear strength [$\alpha m = 12\pi D/(\tilde{u}d^2)$], the global transverse shear dissolves the staircase structure and increases $d\langle n \rangle_y/dx$. Therefore, transverse shear reduces the mixing of scalar concentration along the mean gradient (See Figs. 1.7 and 1.9). For shear strength values such that $\tau_S < \tau_H < \tau_D$, the mixing in y of scalar concentration can no longer be enhanced significantly regardless of the value of α and m . We summarize the different time scale orders and their corresponding range of αm values in Table 1.1.

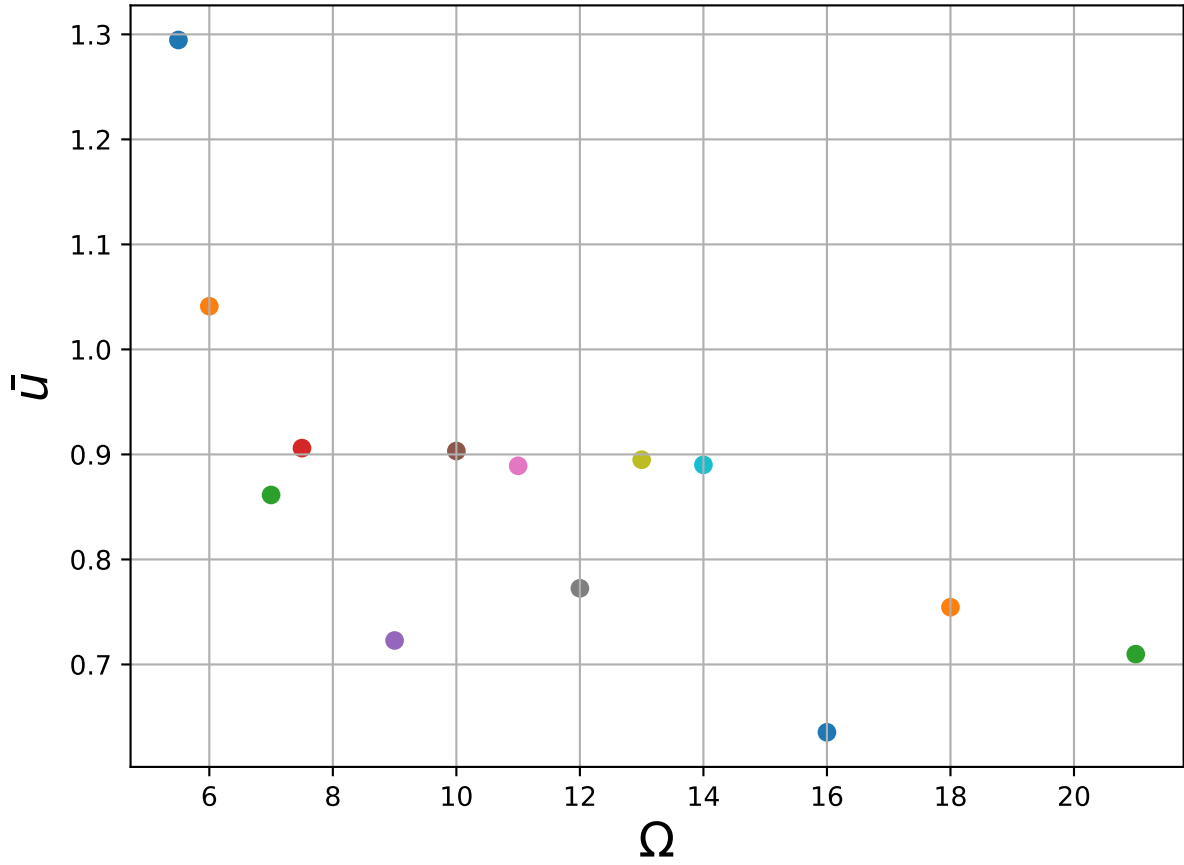
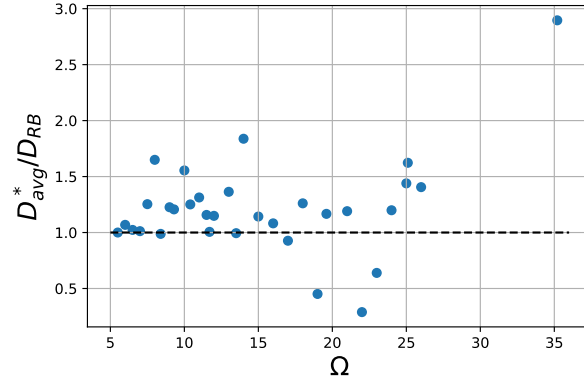


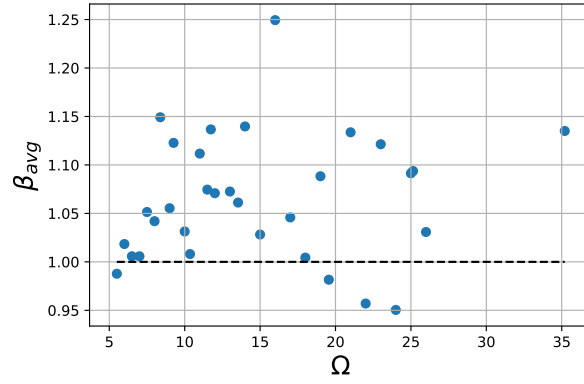
Figure 1.27. Plot of the scalar front propagation speed as a function of Ω . The scattering of vortices leads to a decrease in front propagation speed. As the cellular array scatters, the path of least time increases in length.

The problem presented in Ref. [46] and in the global shear flow section involves fixed ordered arrays. As mentioned in Section 1.3, an actual physical system will manifest variability. Therefore, we introduced variability in the cellular array via forcing and drag in the NS equation [Eq. (1.17)] to study a less constrained cellular array. Here, Reynolds (Ω) and Ekman-friction (α) numbers control turbulent mixing and large scale flows. By injecting scalar concentration into the relaxed cellular array, we find that the staircase is resilient to cell fluctuations. Our results indicate that staircase structures persist, though staircase steps become less regular due to cell mergers.

We established a set of criteria for the statement of “resiliency”. Two appropriate criteria



(a)

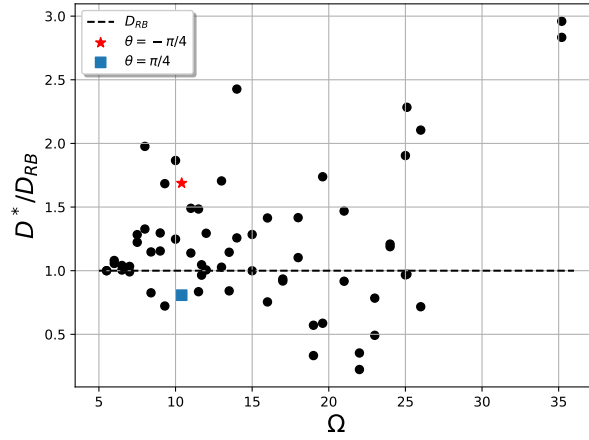


(b)

Figure 1.28. Plot of effective diffusion (D^*) over the zero temperature effective diffusion (D_{RB}) as a function of Ω . The effective diffusion is shown to hug the line of 1, hence it does not deviate significantly from D_{RB} . The effective diffusivity is defined as $\propto D\sqrt{\text{Pe}}$. Keeping all quantities fixed, the only parameter that affects D^* is β , which corresponds to the ratio of the cell's length d_x/d_y . We see through (b) the direct effects that the geometrical properties of the cells has on the value of D^* .

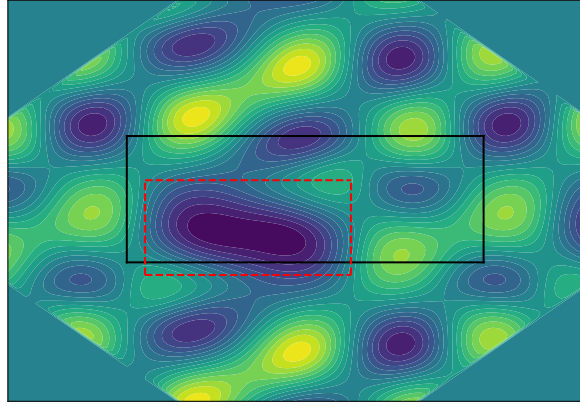
for a staircase structure to be resilient are (1) $\text{Pe} \gg 1$ and (2) $\kappa \geq 1.5$. The first criterion is a fundamental requirement so that barriers are formed and maintained, and the second criterion establishes that a staircase should sustain a sufficient number of steps. For the range of $0 < \Omega < 40$, criterion (1) is satisfied. The second criterion is empirical and emerges from our studies. Over the relevant scope of Ω values, criterion (2) is maintained.

By closely analyzing the global structure of the scalar field, the scalar concentration first travels along regions of strong shear. This implies that staircase barriers form before “homoge-



(a)

$\theta = -45$



(b)

Figure 1.29. The effective diffusion at different orientations as a function of Ω and the contour image of ψ at $\Omega = 10.35$ rotated at $\theta = -\pi/4$. The orientation of the dashed red boxed cell significantly influences the placement of D^* either below or above the dotted line.

nization” takes place inside the vortices. The analysis is confirmed through the visualization of flame fronts in our simulations. Here, flame fronts occur in the regime of weak cell advection (Fig. 1.25 and 1.26).

In addition, by tracking the scalar concentration front in the flow, we determine the path and time it takes for an element of scalar concentration to travel along the mean gradient. We discover that as cells fluctuate, the scalar concentration path of least time *increases*.

Finally, we find that the effective diffusivity for the relaxed cellular array does not deviate significantly from that of the constrained cellular array (See Fig. 1.28a). We find that as long as

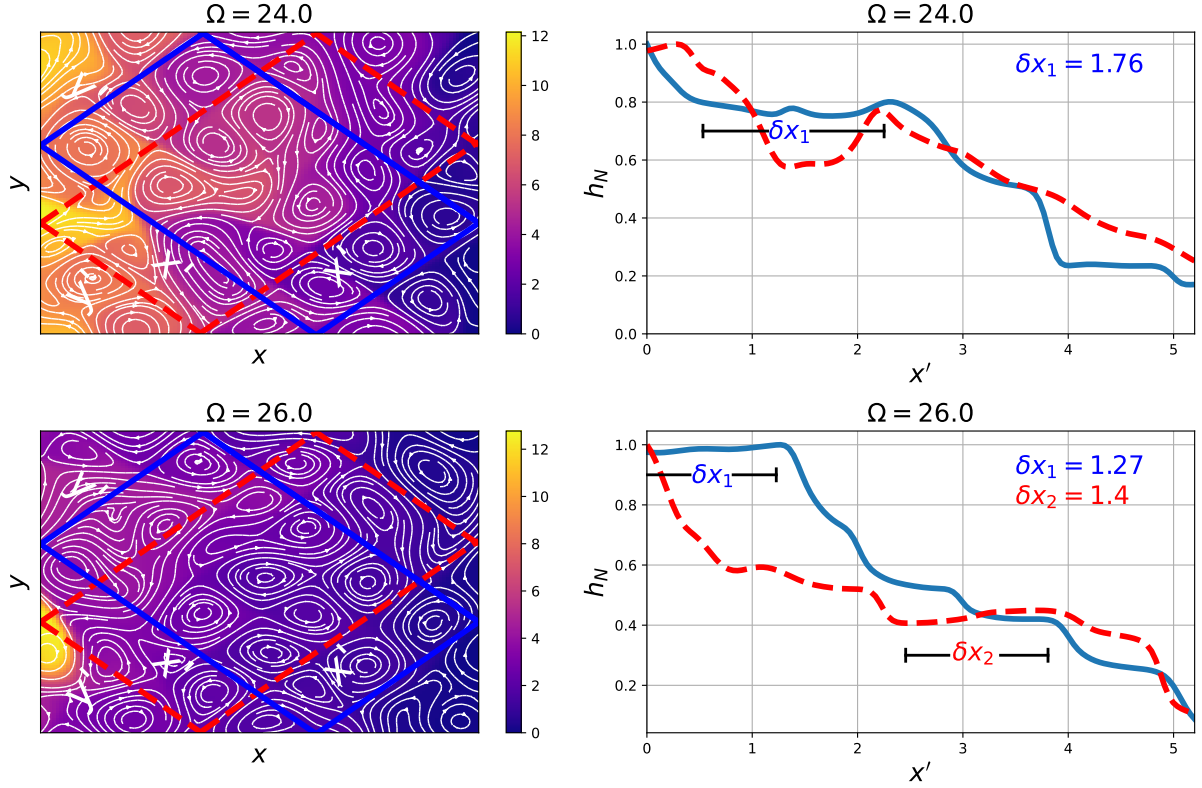


Figure 1.30. Snapshot of passive-scalar concentration for the case of $\Omega = 24.0$ and $\Omega = 26.0$. Similar to Fig. 1.23, we demonstrate the persistence of the staircase structure for $\Omega < 40$.

the boundaries and speed of the cells are maintained, the effective diffusivity and transport do not change. Since effective diffusivity is proportional to β , the transport of scalar concentration can change by modification of cell geometric properties (e.g., changing cell size d_x or d_y). We illustrate the effect of β on D^* in Fig. 1.28b.

These results have intriguing implications for experiments and theory. Our simulations show that the staircase structure is resilient in the regime of low-modest Reynolds numbers (i.e., steady fixed and spatio-temporal chaotic flow). For our model, low-modest Reynolds numbers indicate that flow properties are dominated by the forcing function (F_ω) as opposed to viscous forces. This regime is relevant to drift-wave turbulence, where effective Reynolds numbers range from 10 – 100. Here, the ratio of inverse correlation time to the Landau damping rate (i.e., nonlinear rate to the linear rate) can be used as a surrogate for the Reynolds number. We

	Vortex Field	Drift-Wave Turbulence (tokamak)
Inhomogeneity (free energy source)	∇n	$B_0, \nabla n,$ and ∇T
Reynolds number	$\Omega = 0 - 40$	$\text{Re}^* = 10^1 - 10^2$ (Landau Damping)
Flux	Scalar	Heat
Boundary Layer	Inter-cell boundary between cells	Shear layer (poloidal)

Table 1.4. Comparison of Vortex Array to Drift-Wave Turbulence in fusion devices.

emphasize that in this system the dissipation is due to collisionless processes, such as Landau damping (which occurs due to wave-particle resonance), and not collisional viscosity [71]. For a comparison of the fluctuating vortex array model and drift-wave turbulence, see Table 1.4.

In addition to staircase resiliency, our findings suggest that if shear strain flows are stronger than vortical flows, the spatial propagation of turbulent intensity will travel along regions of saddles, thus initiating the formation of the staircase barriers. By studying the propagation of the turbulent intensity, we observe that fluctuating cellular flow hinders the speed of avalanche propagation. It is important to note that transport of turbulent intensity (i.e., turbulent spreading) and avalanching are closely related [25].

To conclude, we find that the effective diffusivity derived for the fixed cellular array is a suitable approximation for marginally overlapping cells in magnetic confinement. This result applies if one only considers changes in the geometric properties of the $\mathbf{E} \times \mathbf{B}$ convection cells. Results such as $D^* \approx \sqrt{DD_{\text{cell}}}$ may be relevant to magnetic confinement scenarios when conditions are similar to those of this study. The result argues that effective diffusivity is not a simple addition, but a geometric mean.

Results presented in this paper open the door to more complicated numerical simulations and experimental work, which can expand and change our understanding of inhomogeneous mixing. The first and most simple avenue to explore is the parameter space of $\tau_S \geq \tau_V$. In this scenario, fluid momentum diffusion occurs at a rate faster than the growth of large scale flows.

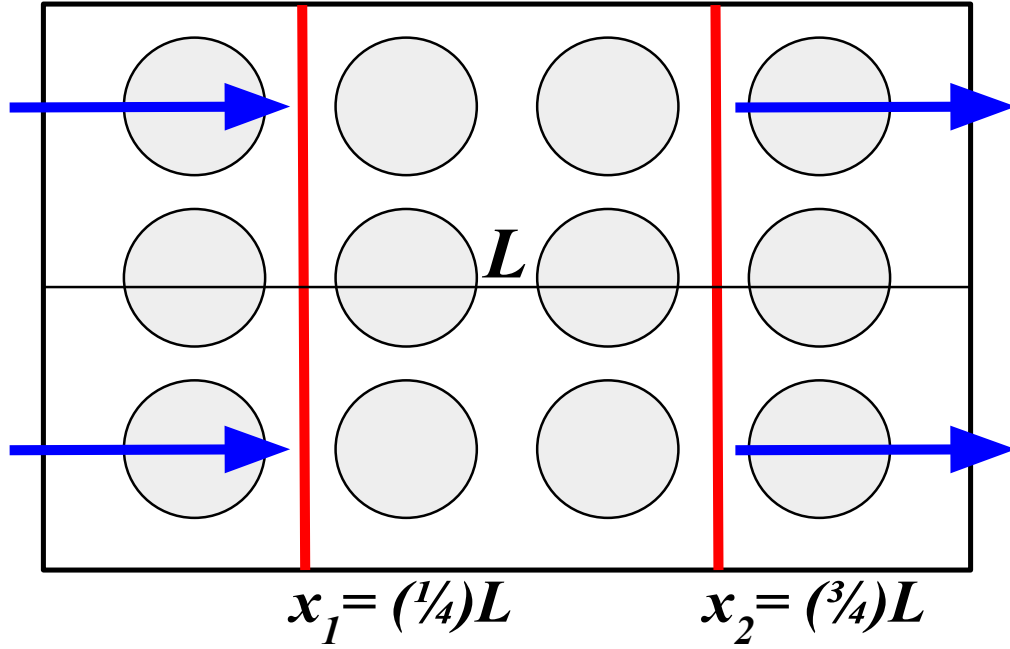


Figure 1.31. Illustration of region we use to calculate D^* .

Thus, scalar concentration dynamics may differ from those found in this paper. A logical next step to explore is the effects of *active scalar* dynamics on the cellular array and inhomogeneous mixing. For those interested, we provide an introduction to the *active scalar* problem in Appendix 6.4. Note that the active scalar problem is also relevant to the problem of liquid crystals, where waves in the director field behave like Alfvén waves.

Finally, due to the simplicity and generality of the problem, it is possible to test the theory presented here with actual experimental data. In the large linear magnetized plasma device (LAPD), a vortex array can be created through modifications of a cathode plasma source with designer masks that form multiple current channels in a cellular pattern, thus forming the staircase structure [72, 73, 74]. This experiment will allow us to test hypotheses and models of staircase resiliency presented in this paper. In addition, the results of this experiment will yield a unique set of observations that can be used to test staircase models and impact the interpretation of experiments in other magnetically confined plasmas.

Chapter 2, in full, is a reprint of the material as it appears in F.R. Ramirez and P.H. Dia-

mond. “Layered Patterns of Active Scalar Fields in a Two-Dimensional Magnetohydrodynamic System.” *Phys. Rev. E* 111, 055107 (2025). The dissertation author was the primary investigator and author of this paper.

Chapter 2

Layered patterns of active scalar fields in 2D MHD

2.1 Introduction

The formation of staircases (quasi-periodic layered patterns, where layers are uniform, homogeneous slices, thin in one dimension) [37], is ubiquitous in many physical systems [49, 75]. Staircases are arrays of mixing zones (“steps”) separated by mini-barriers (“jumps”), and are observed in diverse systems, including doubly diffusive convection [76, 77, 36, 78], oceanic mixing [79], rotating geophysical fluids [37], and magnetized plasmas [35, 1, 53, 54]. The ubiquity of staircases has led to increased interest in the mechanisms of their formation [55, 3, 54, 57]. Understanding how inhomogeneous mixing occurs is a focus of this investigation. Inhomogeneous mixing occurs when substances are distributed unevenly within a medium, creating distinct layers due to transport and mixing rates which vary in space. This type of mixing can locally sharpen a scalar gradient, producing jumps and steps in the scalar concentration or density profile and thus forming a staircase pattern.

Several mechanisms have been proposed to explain inhomogeneous mixing and the wide range of layering phenomena it produces. One is bistable mixing, where transport processes involve two mixing scales, one of which is usually emergent and depends on gradient and turbulence intensity [39, 38, 40, 55, 42, 80]. These processes typically manifest nonlinear flux-gradient curves with at least two stable branches. These correspond to the two states of

the local slope of the layered profile. Another idea is phase separation, similar to what occurs in spinodal decomposition phenomena [81, 82]. Phase separation – and thus layer formation – can be triggered by jamming [56]. Motility-induced phase separation is of particular interest. These approaches are commonly used to explain and interpret layering patterns observed in magnetically confined plasmas, discussed further in Appendix 6.5.

Many approaches to explaining inhomogeneous mixing invoke a feedback process between shear flow, turbulence, and their mutual interaction through modulation processes. However, layering does not necessarily require such a feedback process. Studies in Refs. [46, 48, 47] show that the simplest explanation of inhomogeneous mixing is realized when the turbulence field organizes into an array of adjacent or slightly overlapping (slightly $\equiv$ within boundary layer thickness) convective cells (see Fig. 2.1)¹. This configuration is expected when fluctuations are weakly excited but not strong enough to produce strong mixing. Note that this is different from overlapping cells, which eliminate distinct transport timescales as their boundaries intersect or blend (i.e., separatrices are destroyed), resulting in continuous convection and homogeneous mixing, which prevents layering. For a detailed explanation of the two arrangements, see Ref. [83]. Such an array may occur for conditions close to marginal stability. Here, the layering mechanism involves fast homogenization within cells and slow transport across cell boundaries. The interplay of two disparate timescales, is crucial. If the latter exceeds the former, mixing is inhomogeneous (see Fig. 2.2).

An example of the scenario described above is the problem of passive-scalar advection by a dense array of stationary convective cells [46]. In this simple model, the two disparate transport timescales are: rapid cell turnover ($\tau_H = d/U_0$) and slow diffusion through the cell boundary ($\tau_D = d^2/D$). The regulation of transport by these two timescales naturally causes

¹Note: Fig. 2.1 illustrates the passive scalar case; the $R_m^{1/3}$ scaling discussed later applies only to the active scalar scenario (see Appendix 6.6).

layering. The ratio of these two timescales is the Peclet number:

$$\text{Pe} = \frac{\tau_D}{\tau_H}, \quad (2.1)$$

and layering occurs for $\text{Pe} \gg 1$. This means that as passive scalar concentration elements travel through the array, cell fluid rotates, advecting scalar elements within cells but not across cell boundaries. Due to weak background diffusion (D), scalar elements can slowly scatter through the narrow gap (δ) between adjacent cells (see Fig. 2.3). The slowly-traversed-gaps act as transport barriers. For a fluctuating cellular array, the ratio of scattering length (across separatrix, see Fig. 2.4) to cell size (d) is estimated as

$$\left(\frac{\delta}{d}\right)^2 = \frac{D_T}{U_0 d} + \frac{1}{\text{Pe}}, \quad (2.2)$$

where $D_T = \tau_{c_T} \langle \tilde{v}_k^2 \rangle$ and U_0 represent the turbulent diffusivity (expressed as the product of a characteristic correlation time and the mean square velocity) and the cell turnover speed, respectively. For a detailed derivation, see Appendix 6.7. For the fixed cellular array, $D_T \rightarrow 0$, so Pe determines the scaling. Since $\text{Pe} \gg 1$, the scattering length is very small compared to the cell length ($\delta \ll d$), causing bottlenecks in scalar transport and so steepening gradients between cells. When fluctuations are present, Eq. 2.2 indicates that $D_T < U_0 d$ is required to maintain the staircase structure. In a study where the stationary cellular array conditions are generalized to a fluctuating vortex array (FVA), scalar staircase structures still form and remain resilient up to modest Reynolds numbers. However, steps become less regular on account of cell mergers [83]. No significant changes to passive scalar transport dynamics are observed. In Ref. [83], “resiliency” is established by maintaining (a) $\tau_D/\tau_H \gg 1$ and (b) a sufficiently high profile curvature, $\kappa \geq 1.5$, in a fixed-flux boundary condition passive scalar model.

Logically the next step is to now consider an *active* scalar field (A), which can feedback on layering dynamics [84, 85]. In this context, the dynamics of the active scalar are those of the

Feature	Passive Scalar	Active Scalar (A)
Interaction with flow	Governed by an advection-diffusion equation.	Couples to fluid dynamics.
Equations of Motion	No feedback on the flow.	Feedback to the flow, affecting its evolution.
Examples	Pollutant, Dye concentration, etc.	Magnetic potential, salinity, etc.

Table 2.1. Comparison between passive scalar and active scalar. For passive scalars, staircases form through advection and diffusion. In contrast, active scalars (magnetic potential) strengthen transport barriers by stretching and expelling (to boundary layers) the background magnetic field (see Appendix 6.6). This dynamic reinforcement distinguishes active scalar behavior and stabilizes the staircase structure.

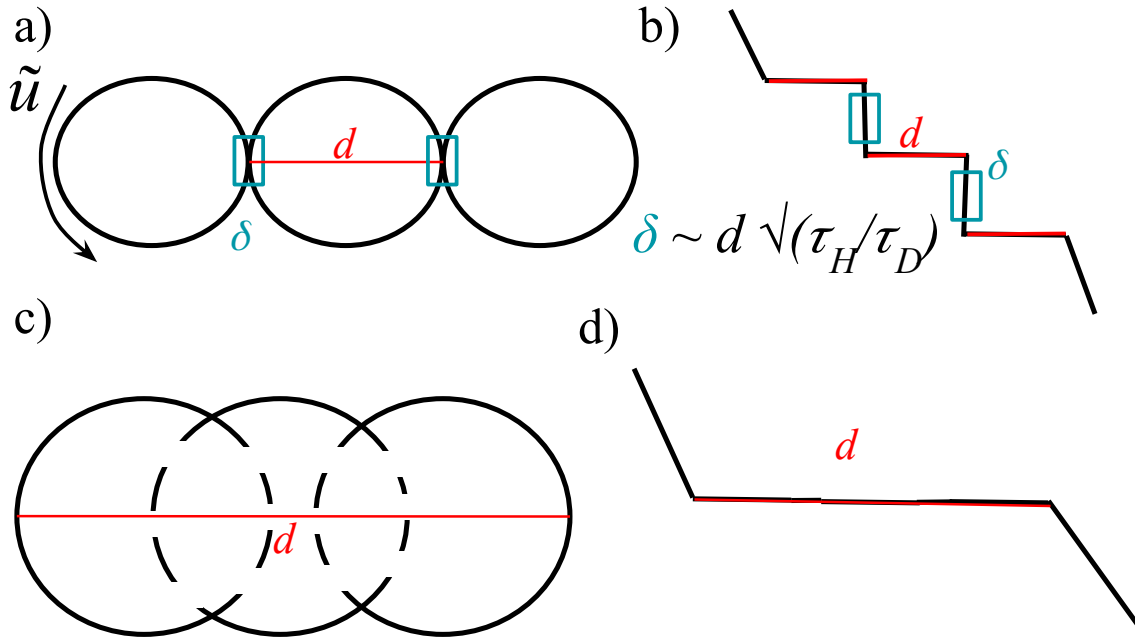


Figure 2.1. Illustration of (a) tangential cells, where separatrices remain intact, and (c) strongly overlapping cells, where cells merge beyond the boundary layer thickness, destroying the separatrices. For tangential cells, inter-cell transport is governed by slow diffusion through the boundary layers, leading to a (b) staircase profile. The diffusion length (δ) is determined by the cell length (d) and the square-root ratio of the cell turnover time to the background diffusion timescale. In contrast, for strongly overlapping cells, separatrix destruction results in (d) global mixing.

magnetic potential in 2D magnetohydrodynamics (MHD), which can be described by the active

Inhomogeneous Mixing (passive)

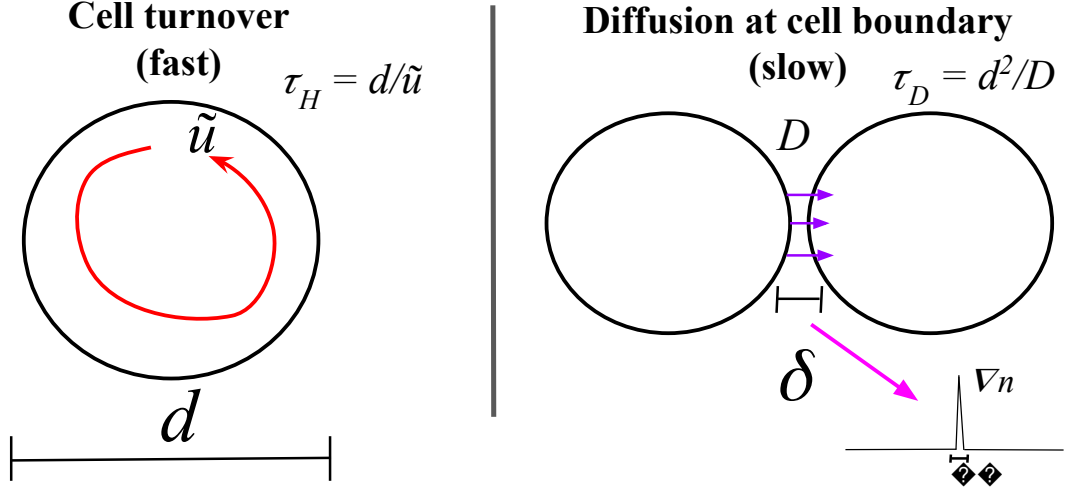


Figure 2.2. In a simple cell model without any feedback mechanism, inhomogeneous mixing arises from the interplay of two disparate transport timescales: rapid cell turnover (τ_H) and slow diffusion at the cell boundary (τ_D). In particular slow diffusion across inter-cell boundary steepens the scalar concentration gradient there.

scalar transport equation (A comparison of passive and active scalars is provided in Table 2.1):

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) A = \frac{1}{R_m} \nabla^2 A + F_A, \quad (2.3)$$

$$\mathbf{B} = \nabla \times A, \quad (2.4)$$

where R_m represents the magnetic Reynolds number. We focus on staircase formation in the magnetic potential (A). While we reference 2D MHD, it is a subset of the broader active scalar framework, which also encompasses phenomena like temperature in thermal convection, vorticity in 2D Ekman turbulence, and potential temperature in surface flows [86].

The magnetic potential is governed by the velocity from the 2D vorticity equation,

presented in nondimensional form as

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \omega = \frac{1}{\Omega} \nabla^2 \omega + M^2 (B \cdot \nabla \nabla^2 A) + F_\omega, \quad (2.5)$$

$$\nabla^2 \psi = \omega. \quad (2.6)$$

Here, we express the 2D vorticity equation utilizing the streamfunction ψ and vorticity $\omega = \nabla \times \mathbf{u}$, where $\mathbf{u}$ is given by $\mathbf{u} = \hat{z} \times \nabla \psi$. The spatially periodic force F_ω within the vorticity equation drives the cellular flow. We discuss F_ω and control parameters in Sec. 4.2. Here, A is coupled to flow dynamics via the $\mathbf{J} \times \mathbf{B}$ force. These equations constitute the forced 2D MHD problem [87, 88]. The Alfvén mach number (M) controls the coupling strength between the active scalar and the flow, defined as

$$M = \frac{v_A}{U_0}, \quad (2.7)$$

$$v_A = \frac{B_0}{\sqrt{\mu_0 \rho}}, \quad (2.8)$$

where v_A denotes Alfvén speed, derived from the magnetic field strength B_0 , the magnetic permeability μ_0 , and the density ρ . Here, μ_0 and ρ are standardized to one. Notably, B_0 is the variable parameter in M , which influences the active scalar’s effect on flow dynamics.

Previous studies show that the most significant dynamics occur during a quasi-stationary stage of slow decay of A and magnetic field, referred to as the “suppression” stage [89]. According to Zeldovich’s theorem, without external magnetic forcing, the suppression stage is limited due to resistive diffusion of $\mathbf{B}$ in 2D [90, 91]. We explore this in detail in Sec. 2.3.3 by introducing external active scalar forcing (F_A) to determine if forcing can prolong the suppression of magnetic potential transport. Note that F_A is set to 0 for Secs. 2.3.1 and 2.3.2.

During the suppression stage, the background magnetic field (B_0) restrains turbulent transport. Recent numerical studies and mean-field electrodynamic theory show that even a *weak* large-scale magnetic field can reduce turbulent transport of the active scalar [92, 93]. In 2D

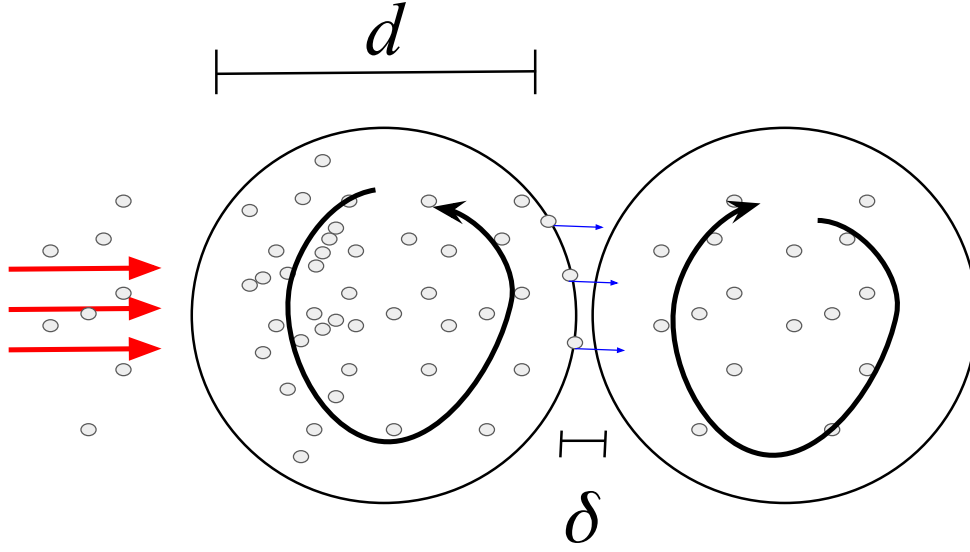


Figure 2.3. Illustration of passive-scalar advection by an array of stationary convective cells. Rapid fluid circulation within each cell homogenizes the scalar concentration inside. At cell boundaries, scalar transport is dominated by slow diffusion through narrow gaps, leading to much reduced transport across boundaries as compared to mixing within. This disparity creates sharp gradients at cell boundaries. Such inhomogeneous mixing patterns often result in staircase structures in the scalar field profile.

MHD, turbulent transport across cell boundaries is quantified using turbulent resistivity (η_T)

$$\eta_T \approx \frac{\eta_k}{1 + R_m \langle \mathbf{B} \rangle^2 / \langle u^2 \rangle}, \quad (2.9)$$

$$\eta_k = \sqrt{\langle u^2 \rangle} L. \quad (2.10)$$

Here, $\langle \mathbf{B} \rangle$ and $\langle u \rangle$ represent the mean magnetic field and flow velocity, respectively. This result stems from the influence of magnetic fields on turbulent flows, affecting how they scatter A . Note that R_m is big, so even in the limit of $v_A/U_0 \ll 1$, Eq. 3.22 predicts significant quenching of η_T . While R_m is not always large, most relevant astrophysical and laboratory cases involve large R_m (see Appendix 6.6). The trends of quenching as predicted by Eqs. 3.22 and 3.23 were verified by several numerical simulations [94, 89]. Using η_T , we can compare $\tau_{\eta_T} = d^2/\eta_T$ to τ_H to assess

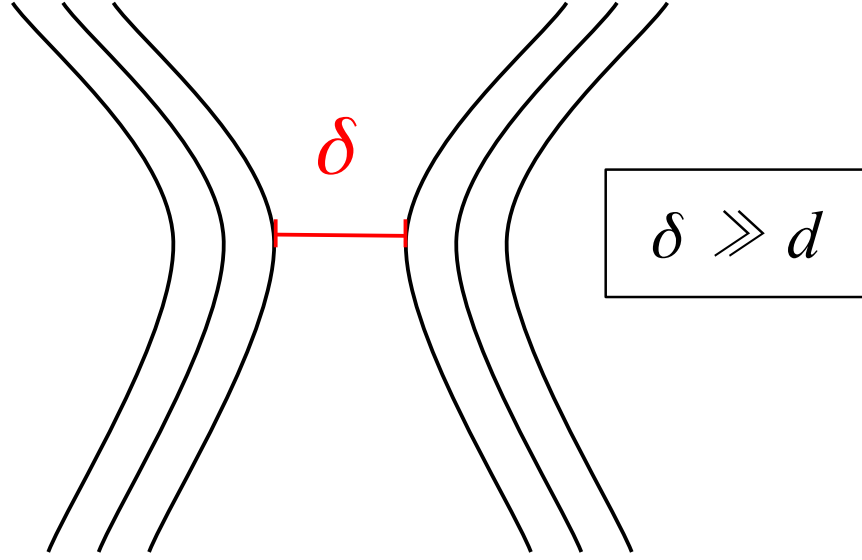


Figure 2.4. Illustration of transport across the boundary layer. Here, δ represents the scattering length of the scalar element. For passive scalar dynamics, the Peclet number (Pe) determines the scaling of scattering length to cell size. For an active scalar, the background magnetic field reduces the scaling, weakening transport across cell boundaries. For the derivation of scattering length in both scenarios, see Appendix 6.7.

staircase rigidity.

Another way to quantify rigidity requires a theory of transport across the boundary layer between cells in 2D MHD. Before establishing a scaling, we describe the interaction between fluid motion and the background magnetic field in this model of active scalar dynamics. Here, fluid motion stretches a weak background field to the vortex edges until the field lines reconnect. This feature of 2D MHD is known as flux expulsion [95]. A key point is that the strongest magnetic field accumulates at the vortex's edge. If the magnetic field at the boundary (b) is sufficiently strong, the induced stresses from the curved field lines cause cell disruption. This is referred to as the cell disruption regime [96]. For a detail discussion on these scenarios, see Appendix 6.6. The two MHD scenarios are distinguished by $\Sigma < 1$ (flux expulsion) and $\Sigma \geq 1$

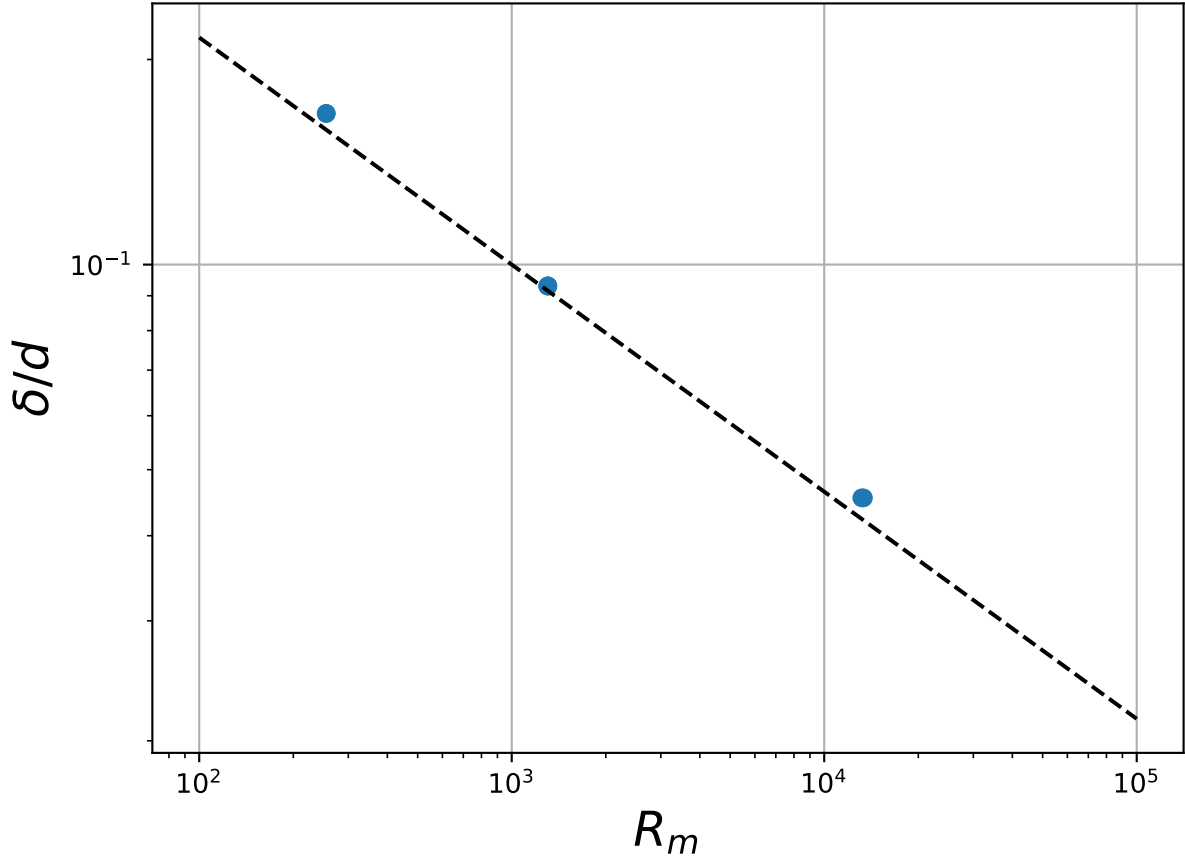


Figure 2.5. Comparison between theoretical scaling and numerical results: In this context, δ denotes the width of the cell boundary, and d the cell radius. The dimensionless emergent scale, represented by the ratio of layer width to cell radius, scales as $\sim R_m^{-1/3}$.

(cell disruption), where Σ is

$$\Sigma = M^2 R_m. \quad (2.11)$$

Σ is derived from the relative strength of the fluid convective term $(\mathbf{v} \cdot \nabla)\omega$ and the magnetic tension term $\mathbf{B} \cdot \nabla \mathbf{J}$.

Returning to the issue of “rigidity”, we examine how A affects inter-cell transport. For detailed derivations, see Appendix 6.7. We begin with the flux expulsion (“kinematic”) regime where B_0 is sufficiently low (specifically, where $v_A^2 < U_0^2/R_m$). Here, the magnetic field (b) concentrates at the cell boundary, so that $b \sim B_0 R_m^{1/3}$ (essentially $\sim \nabla A$ increases at cell

boundary) as A is expelled from the cell. In this case, the ratio of scattering length to cell size scales as:

$$\left(\frac{\delta}{d}\right)^2 \approx \frac{D_T/U_0 d}{1 + R_m^{5/3} v_A^2 / \langle \tilde{v}^2 \rangle} + \frac{\eta}{D\text{Pe}}, \quad (2.12)$$

where η is the magnetic resistivity. If $D_T \neq 0$, the denominator for the first term on the left-hand-side is usually far in excess of unity, so turbulent broadening is weaker for the magnetic/active case as compared to Eq. 2.2. The boundary layer is stronger since $\mathbf{B}$ reduces transport through it, maintaining layering (i.e., $\mathbf{B}$ locks in staircase structure). In the vortex disruption regime, $v_A^2 \geq U_0^2/R_m$, the first term in Eq. 2.12 becomes

$$\left(\frac{\delta}{d}\right)^2 \xrightarrow{v_A \gg \langle \tilde{v} \rangle} \frac{D_T/U_0 d}{R_m^{5/3} v_A^2 / \langle \tilde{v}^2 \rangle}. \quad (2.13)$$

Here, $\mathbf{B}$ again reduces the scattering length at the boundary layer, quenching inter-cell turbulent transport. The magnetic field effects on η_T reduce the rate of flux transport through the boundary layer, thus increasing the scalar's dwell time in the boundary as compared to the kinematic case. The main point is that the active scalar reinforces the staircase structure in both the flux expulsion and vortex disruption regimes. This result for the cell disruption regime is suprising, because one might expect that transport can no longer be limited once cell boundaries are no longer intact. Note that cell boundaries must be maintained for staircase structures to form. Previous studies on single-cell systems show that vortices do not survive in $\Sigma \geq 1$ scenarios [97, 98, 96]. *However, in a multiple cell system, residual cells may remain, implying that layering still can occur.* Thus, the active scalar field may produce a more robust staircase structure in both MHD regimes than its passive scalar counterpart, regardless of the value of Σ .

In this paper, we investigate structural fluctuations and feedback effects between a vortex array and an active scalar. We compare previous passive scalar layering to results found here and aim to address the following questions:

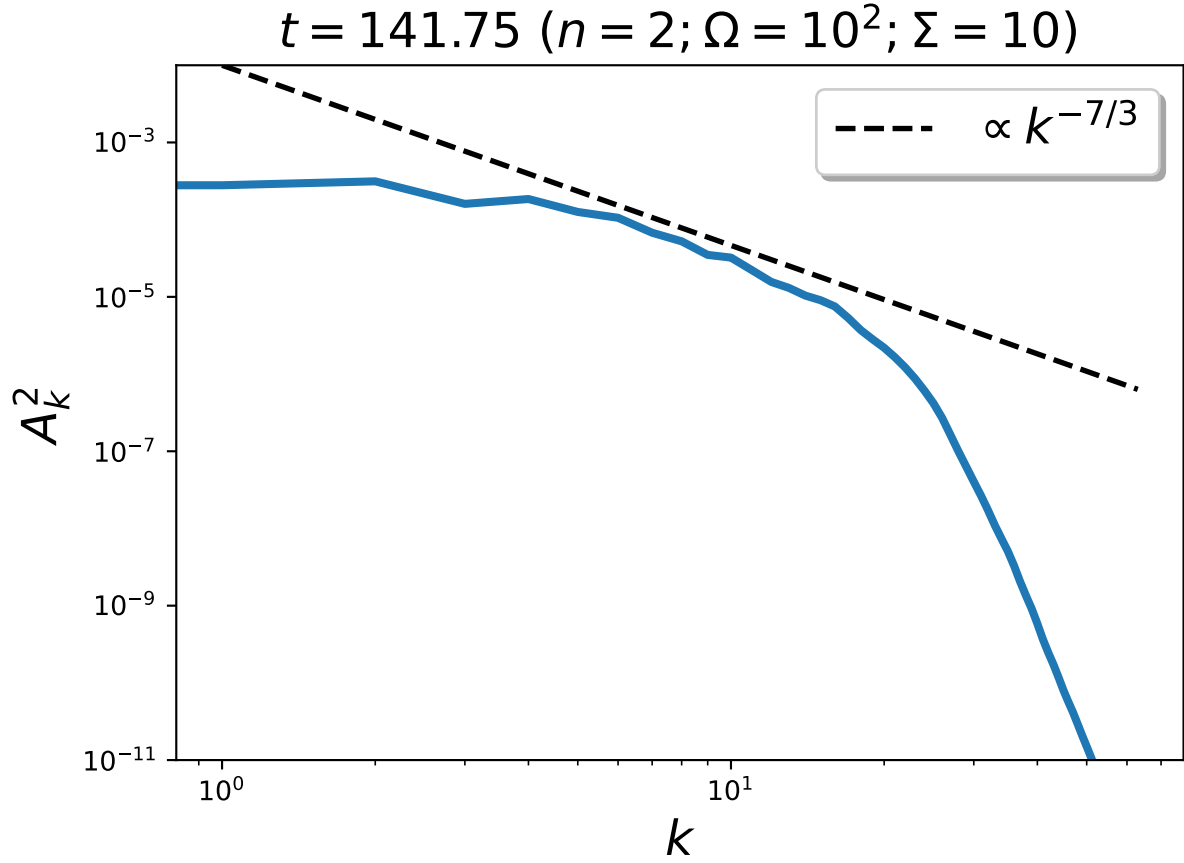


Figure 2.6. Comparison between theoretical scaling and numerical results: Simulation demonstrates 2D MHD turbulence supports a $k^{-7/3}$ spectrum of $\langle A^2 \rangle_k$, indicative of an inverse cascade of A^2 .

1. Will a magnetic potential staircase structure form in a 2D vortex lattice system with $\mathbf{J} \times \mathbf{B}$ forces and turbulence-induced mixing? How do MHD effects respond to changes in cell velocity, and how does this interaction impact staircase resiliency? How do active staircase dynamics differ from those of passive scalar staircases?
2. Will magnetic layered structures form in both the flux expulsion ($v_A^2 < U_0^2/R_m$) and cell disruption ($v_A^2 > U_0^2/R_m$) regimes? How do these cases differ?
3. How do quantities in both the flow and the active scalar determine staircase formation? What are the criteria for the formation of magnetic staircases, such as necessary values of profile curvature and the slow to fast timescale ratio?

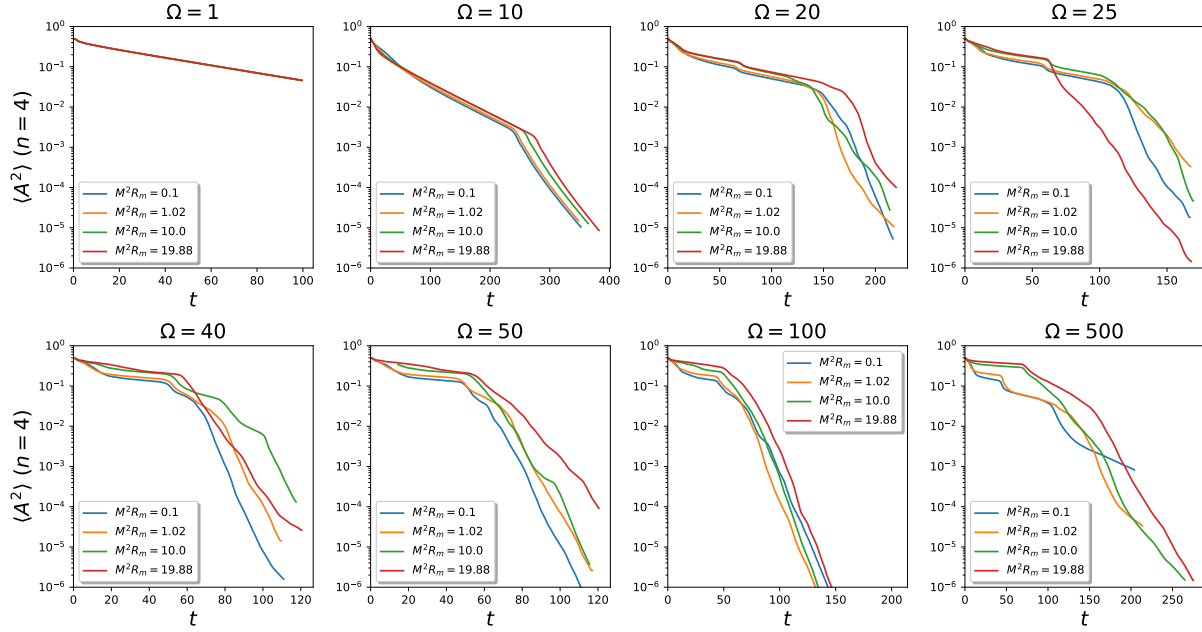


Figure 2.7. The time evolution of $\langle A^2 \rangle$ (mean-square potential) with wave injection $n = 4$ is depicted across various Ω and Σ values. The decay of the magnetic potential is gradual during the suppression stage, characterized by temporary suppression of turbulent transport by the background magnetic field, and rapid in the kinematic stage, where the magnetic field no longer influences the flow dynamics. Once the magnetic field dissipates sufficiently, turbulent flow accelerates magnetic field decay by stretching and twisting magnetic field lines. As Ω increases, magnetic field dissipation intensifies due to increase turbulent motion in the cell structure.

4. Given the decay of the magnetic field in 2D, can stochastic forcing of A maintain magnetic potential staircases and counteract the natural resistive decay of $\mathbf{B}$?

Throughout the paper, we delve into these questions.

We investigate the transport and resilience of magnetic staircase profiles in response to turbulence-induced variations in the flow structure. In addition, we compare our findings to previous studies of passive scalar layering. The paper is structured as follows: Sec. 4.2 describes the numerical properties of the model and discusses the parameters we explore. Sec. 2.3.1 examines flow properties, including energy evolution, mean squared magnetic potential, and the impact on vortex array integrity in both flux expulsion ($\Sigma < 1$) and vortex disruption ($\Sigma \geq 1$) regimes. Sec. 2.3.2 focuses on the formation of magnetic staircases, highlighting the roles of disparate timescales and profile curvature. Sec. 2.3.3 explores how stochastic magnetic potential

Array State	Ω Range
Frozen Array	$0 \leq \Omega < n\sqrt{2}$
Steady Distorted Array	$n\sqrt{2} < \Omega < 8$
Periodic Array	$8 < \Omega < 18$
Turbulent Array	$18 < \Omega$

Table 2.2. Different stages of the fluctuating vortex array. In this paper we consider small and large Ω any value within frozen array and turbulent array range, respectively. The turbulent state is characterized by irregular flow and the breakdown of the orderly vortex lattice into disordered behavior.

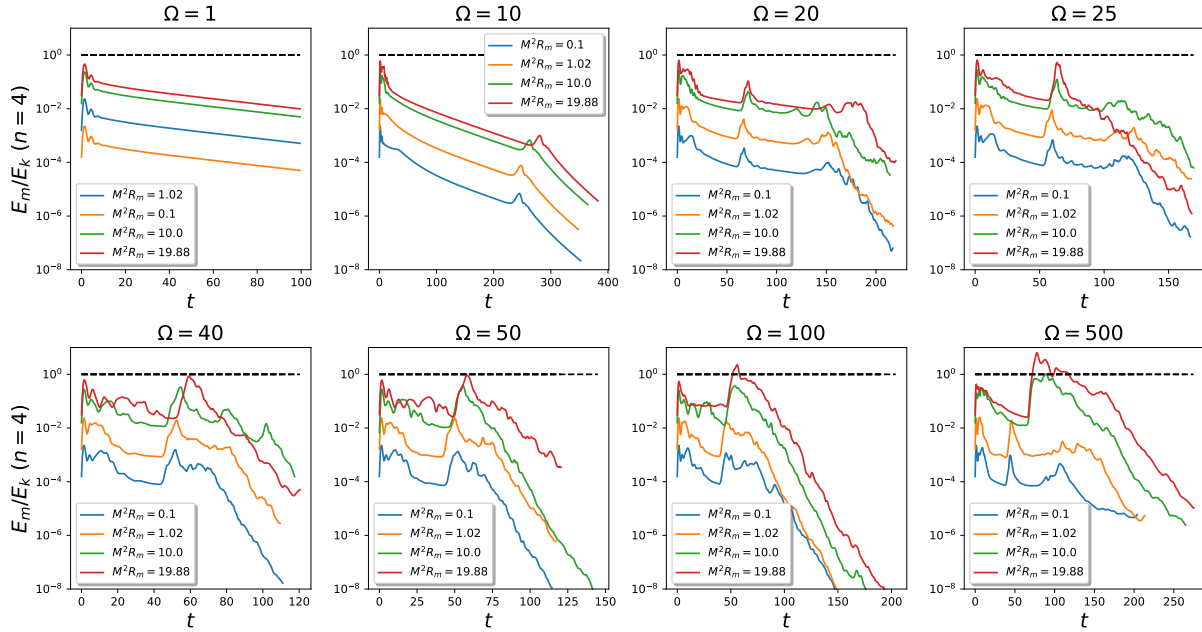


Figure 2.8. Time evolution of the ratio of magnetic energy (E_m) to kinetic energy (E_k) reveals that the suppression stage is transient and transitions to a kinematic decay stage after significant dissipation of the magnetic field (similar to Fig. 2.7). Layered structures are observed during the suppression stage, as demonstrated later in Figs. 2.19 and 2.20.

forcing influences the durability of magnetic staircases and the transport dynamics of active scalar concentrations in the FVA.

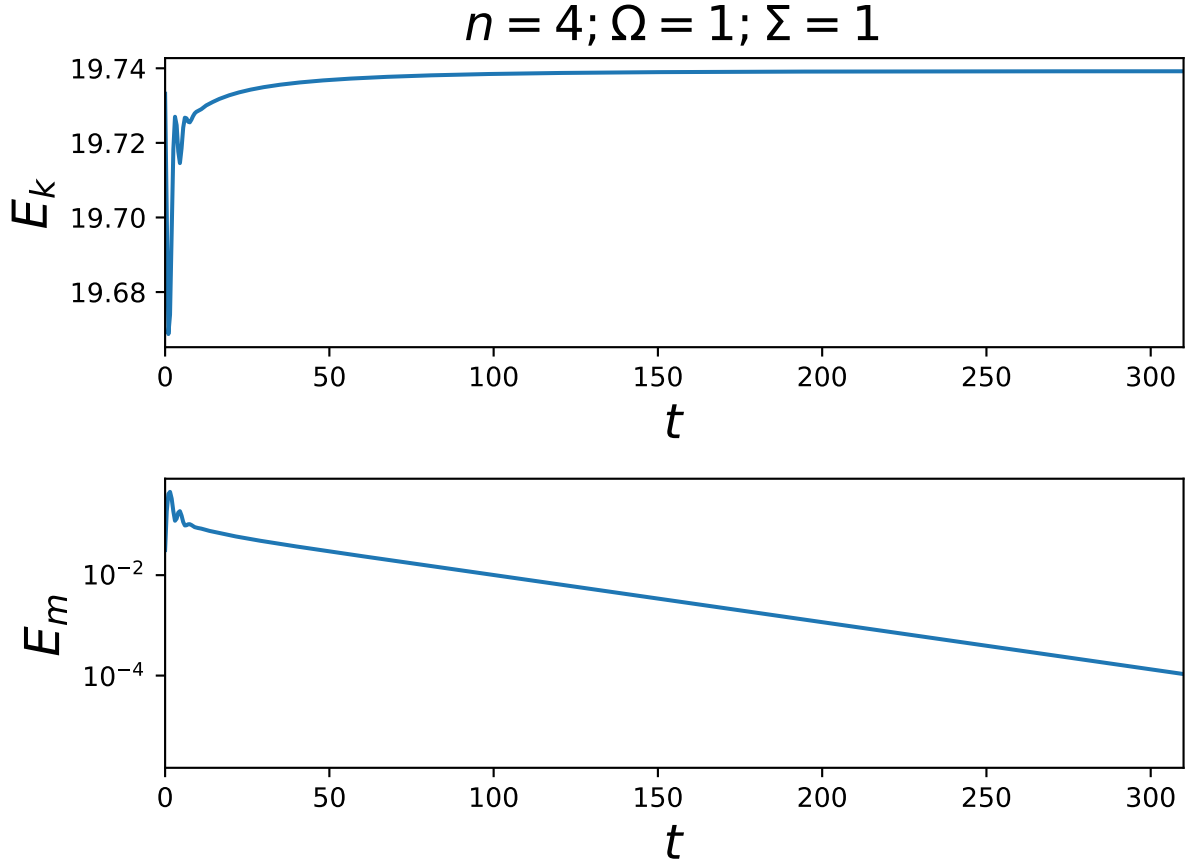


Figure 2.9. The energy time-trace of the frozen vortex array illustrates the predominance of F_ω in governing the flow. Here, the magnetic energy undergoes a brief stage of flux expulsion before transitioning into a phase of slow dissipation.

2.2 Model

The active-scalar transport equation (Eq. 2.3) couples with Eq. 2.5 to generate the FVA threaded with magnetic fields. We emphasize that forcing function F_A in Eq. 2.3 is set to zero for Secs. 2.3.1 and 2.3.2. For the non-zero definition of F_A , we direct the reader to Sec. 2.3.3. To study the effects of the active scalar on the FVA, the system is controlled by three parameters: the magnetic Reynolds number (R_m), the Reynolds number (Ω), and M . Here, R_m (based on domain size) and Ω are defined as the ratio of the magnetic diffusivity timescale ($\tau_\eta = n^2/k^2\eta$) and the viscous timescale ($\tau_\nu = n^2/k^2\nu$), respectively, to the cell circulation timescale ($\tau_H = n/kU_0$), with assigned values of $k = 10$, $U_0 = 1$, and $\eta = (n/k)/10^3$. Note that Ω represents a modified

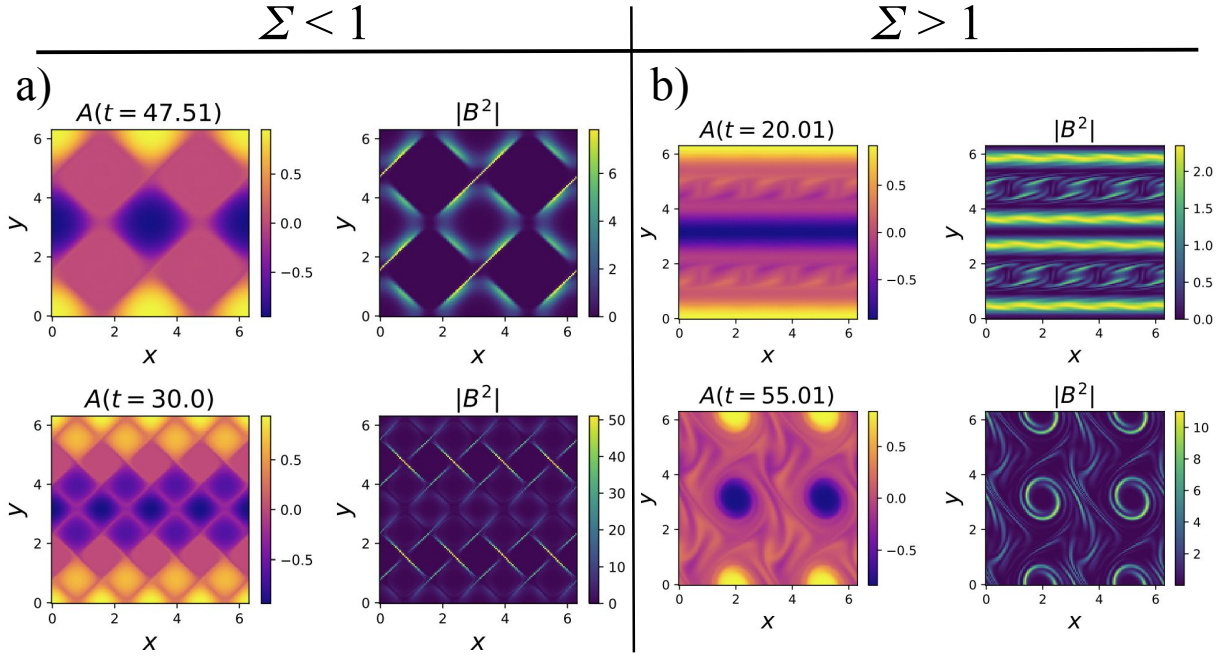


Figure 2.10. Parameter values: Top-left: $n = 2$, $\Omega = 1$, $\Sigma = 0.1$; bottom-left: $n = 4$, $\Omega = 1$, $\Sigma = 0.1$; top-right: $n = 4$, $\Omega = 100$, $\Sigma = 1$; bottom-right: $n = 4$, $\Omega = 500$, $\Sigma = 1$. Contours of A and $|B^2|$ for (a) flux expulsion ($\Sigma < 1$) and (b) vortex disruption ($\Sigma \geq 1$) regimes. In the flux expulsion regime, the magnetic field ($\sim \nabla A$) is expelled to the cell boundary, where the gradient of the active scalar concentration increases (Eq. 6.19). In (b), the magnetic scalar potential contours show that despite significant vortex disruption ($\Delta \sim 1$), layering persists (see Fig 2.11 for evidence of layering in the magnetic potential profile). Not all closed streamlines are disrupted, which is crucial to layering. Note that if the vortex array is not frozen, the layered structure will eventually collapse, due to the decay of the resistive field. Collapse is shown in Fig. 2.12.

Reynolds number², defined as $\Omega = n\text{Re}$, where n denotes the wave injection vector. The spatially periodic force F_ω within the NS equation is defined as

$$F_\omega = -n^3 [\cos nx + \cos ny] / \Omega. \quad (2.14)$$

Here, n is assigned values of 1, 2, and 4 for the conducted simulations. In our equations, F_ω drives the cellular flow at low values of Ω . Note that in our study, we omit the drag term in

²When $n \neq 1$, Ω is defined as $n(U_0/kv)$, differing from the conventional Reynolds number (Re) by a factor of n . Furthermore, since we adopt the concept of a melting vortex crystal framework from Ref. [60, 82], we retain the original notation for consistency with that work.

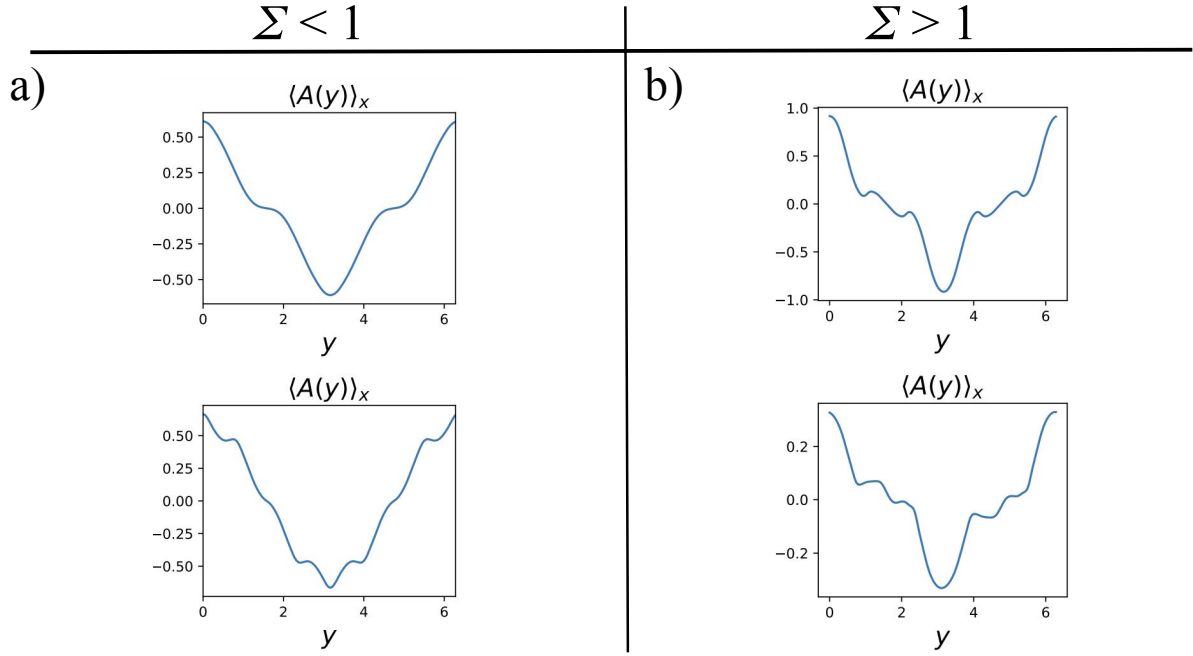


Figure 2.11. Magnetic potential staircase profile for (a) flux expulsion and (b) vortex disruption regimes. These profiles correspond to those in Fig. 2.10. Layering persists in both regimes. In the vortex disruption regime, residual cells maintain the staircase structure despite significant disruption to the cellular array.

Eq. 2.5, which is present in references [60, 61]. According to Ref. [60], the principal effect of the drag term is to delay the onset of transitions in the vortex array. To align with earlier linear stability analysis and DNS studies of the fluctuating vortex array problem [99], we ignore drag.

In this work, we conduct direct numerical simulations (DNS) of the NS and active-scalar equations (Eqs. 2.3 and 2.5), utilizing the Dedalus framework [100]. Our simulations adopt a pseudospectral approach with a 3/2 dealiasing rule and a fourth-order Runge-Kutta method for time stepping. The computational domain is a doubly periodic box with length $L = 2\pi$ and a spatial resolution of $N = 128$. The initial conditions for the vorticity and active scalar are given by

$$\psi(t=0) = \frac{U_0}{n} (\cos nx + \cos ny), \quad (2.15)$$

$$A(t=0) = A_0 \cos x, \quad (2.16)$$

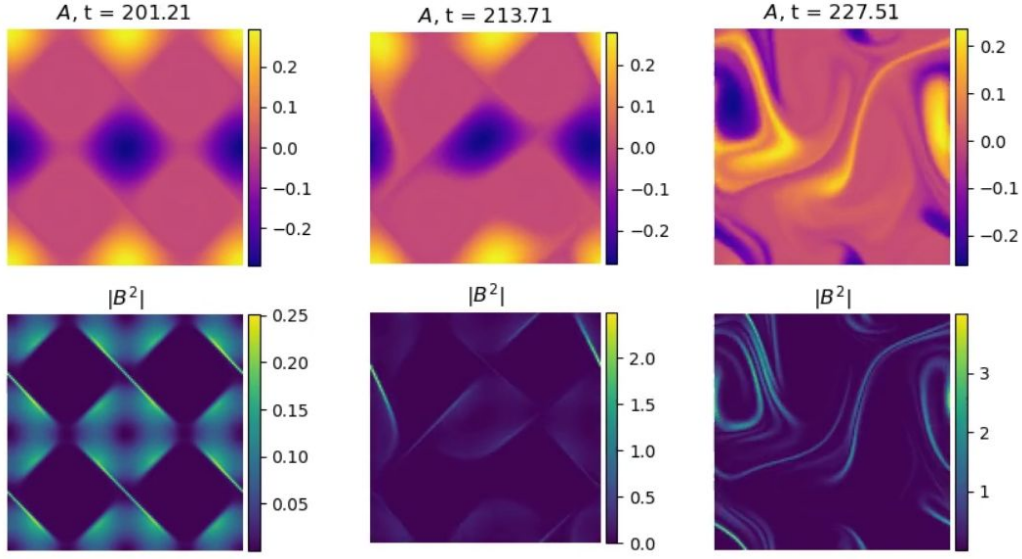


Figure 2.12. The evolution of the active scalar ($n = 2, \Omega = 500, \Sigma = 0.1$) shows that in the chaotic stage of the FVA, the magnetic field maintains the cellular array structure for a brief period. The magnetic field eventually dissipates, allowing cell fluctuations to dominate and leading to the disintegration of the vortex array and magnetic staircase structure.

with A_0 preset to one. The sinusoidal nature of the initial magnetic potential means the magnetic field has no mean over the domain $[0, 2\pi]$, as positive and negative parts cancel out. However, as shown in Fig. 6.1, restricting to the subdomain $[0, \pi]$ approximates the sinusoidal function as linear, creating an effective mean magnetic field in that region. This aligns with concepts in Ref. [97]. A similar setup is used in Ref. [89], which forms part of the foundation of our study.

The dynamics are driven by the force F_ω , exploring the stages of the fluctuating vortex array (FVA) by increasing the value of Ω . To compare with previous passive scalar studies, our study spans effective Reynolds numbers from 1 to 500, with a focus examination of spatiotemporal chaos within the cellular lattice. Note that at $\Omega = 0$, the FVA is static. This model facilitates the exploration of a broad range of excitations typical of the modest levels of turbulence, where effective Reynolds numbers range from 10 to 100. For relation to drift-wave turbulence and more information on staircases in magnetic confinement devices, see Appendix 6.5.

To ensure accurate temporal evolution, our simulations employ numerical methods consistent with those in prior studies [99, 60, 61]. We systematically capture, categorize, and validate the various stages in the evolution of the FVA. For a detailed discussion of these stages, see Ref. [60]. As established in previous research, a grid size of $N = 128$ is sufficient, as finer grids of $N = 256$ and $N = 512$ show no significant changes in results. To confirm this for our study, we ran select cases at $N = 512$ and compared energy-trace and magnetic potential contour plots to those of $N = 128$. The higher-resolution cases exhibited trends consistent with the lower-resolution simulations ($N = 128$). In addition, we validate our computations by reproducing established results, including the magnetic hydrodynamics (MHD) mean squared magnetic potential inverse cascade scaling ($k^{-7/3}$) [101, 102] and the flux expulsion boundary layer scaling with magnetic Reynolds number in a single cell ($n = 1$) [Eq. 6.18]. Comparative plots of theoretical predictions and simulation outcomes are shown in Figs. 2.5 and 2.6.

Simulation Parameters	
Ω	Σ
1	[0.1, 1, 10, 20]
10	[0.1, 1, 10, 20]
20	[0.1, 1, 10, 20]
25	[0.1, 1, 10, 20]
40	[0.1, 1, 10, 20]
50	[0.1, 1, 10, 20]
100	[0.1, 1, 10, 20]
200	[0.1, 1, 10, 20]

Table 2.3. Parameter values used in simulations, with fixed $R_m = 10^3$.

In this study, we primarily explore active scalar staircase dynamics in FVA stages that occur well after the onset of cellular fluctuations, identified by $\Omega_s = n\sqrt{2}$ (determined via linear stability analysis) [99]. We are particularly interested in conditions where Ω exceeds

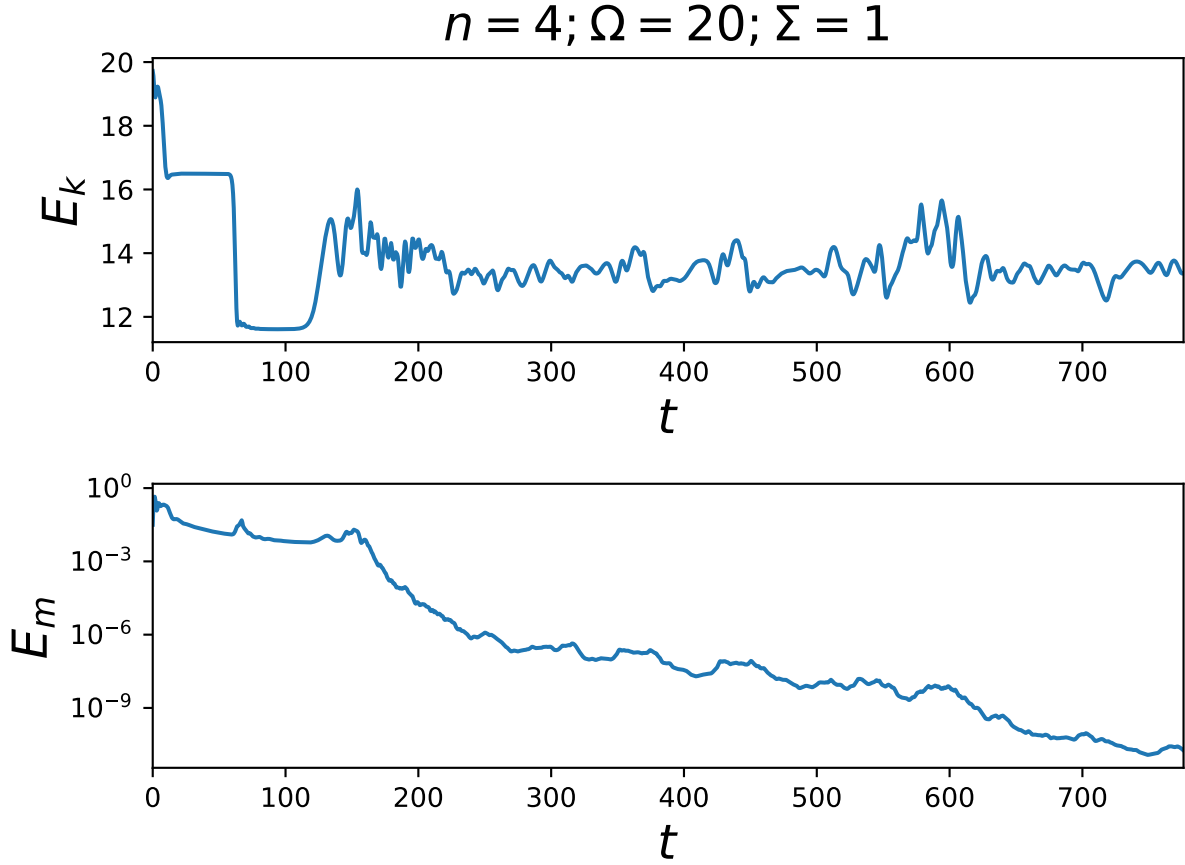


Figure 2.13. Energy time trace depicting the chaotic vortex array stage. This example showcases the suppression stage, governing fluctuations in the flow’s kinetic energy. As the magnetic energy dissipates sufficiently (around $t \sim 150$), the kinetic energy evolution transitions from an initially stable pattern to a chaotic and random one, with the magnetic field’s influence becoming negligible. Note that this transition to chaos subsequently accelerates the magnetic energy decay.

Ω_s , leading to cell perturbations that cause their convergence and merging. We consider FVA stages characterized by a locked array structure, such as the frozen ($\Omega < \Omega_s$) or steady distorted array ($\Omega_s < \Omega < 8$), to be of lesser interest due to previous studies of passive scalar transport. Those studies confirmed that static cellular arrays promote layering under certain conditions (e.g., close boundaries with fast cell turnover velocity and slow background diffusion across cell boundaries) [46, 47, 83]. Instead, our focus shifts to FVA stages that feature dynamic and time-varying distortions – specifically, those that oscillate either periodically or quasiperiodically ($8 < \Omega < 18$), as well as stages marked by spatiotemporal chaos, and turbulence ($18 < \Omega$).

In these latter stages, the spatial organization of the vortex array distorts, with flow patterns emerging and vanishing in an irregular fashion. A table of these transitions is provided in Table 2.2. Our objective in this paper is to examine the behavior of the active scalar (A) amidst such turbulent fluctuations.

2.3 Results

2.3.1 Characterizing the flow and magnetic potential field

We examine inhomogeneous mixing by initializing the active scalar according to Eq. 2.16. We keep the magnetic Reynolds number fixed at 10^3 , while B_0 and Ω are varied. Σ values of 0.1, 1, 10, 20, covering both the flux expulsion and the vortex disruption regimes are explored for both $n = 2$ and $n = 4$. For a comprehensive list of the parameter values examined, refer to Table 2.3.

To begin our study, we first examine the evolution of the mean-squared magnetic potential ($\langle A^2 \rangle$) and kinetic (E_k) and magnetic (E_m) energy for both $\Sigma \leq 1$ and $\Sigma > 1$ cases, while increasing Ω . Kinetic and magnetic energy are defined as follows:

$$E_m = \frac{1}{2} \int_S M^2 |\mathbf{B}|^2 d^2x, \quad (2.17)$$

$$E_k = \frac{1}{2} \int_S |\mathbf{u}|^2 d^2x, \quad (2.18)$$

where S represents the area of the two-dimensional grid. The evolution of the mean-squared magnetic potential and ratios of E_k and E_m are presented in Figs. 2.7 and 2.8. Both figures exhibit similar characteristics over time for various P_m and Σ values.

Observe that for $\Omega > 1$, both $\langle A^2 \rangle$ and E_m/E_k manifest a brief period of very slow decay (suppression stage), followed by a distinct rapid decay stage. As Ω increases, cells fluctuate more intensely. Such fluctuations accelerate magnetic decay. We present qualitative results of this in Sec. 2.3.1. The decay of the magnetic potential is consistent with Zeldovich's theorem,

which proves that in 2D MHD, absent external forcing of A , the magnetic field will inevitably decay. As a result, for Ω values where the array isn't frozen, the magnetic field maintains the cellular array structure only for a limited time.

For the cases corresponding to the frozen vortex array ($\Omega = 1$), the flow behavior is significantly influenced by the forcing function (F_ω). As a result, the cellular array structure remains fixed throughout, with only the magnitude of A decreasing overtime (i.e., layering persists, but the magnitude of A tends to decay to zero over time) [See Figs. 2.7 and 2.9].

Flux expulsion regime ($\Sigma < 1$)

In order to understand the behavior of both $\langle A^2 \rangle$ and E_m/E_k for different magnetic Prandtl numbers, we present Figs. 2.10 and 2.12, which show contours of the magnetic potential and magnetic field in the flux expulsion regime ($\Sigma \leq 1$). In this regime, the magnetic back reaction on flow is weak. In Fig. 2.10, for the case of $\Omega = 1$, we demonstrate the increase of the concentration of magnetic field at the cell boundaries and the homogenization of the active scalar within the vortices during the suppression stage, for $n = 2$ and $n = 4$. To confirm the presence of a layered structure, we analyze the scalar concentration averaged in x , as a function of y ,

$$\langle A(y) \rangle_x = \frac{1}{L} \int_0^L A(x, y) \, dx. \quad (2.19)$$

Our analysis of Fig. 2.11 confirms that magnetic layering is a distinct feature of the flux expulsion and frozen vortex array regime for both $n = 2$ and $n = 4$. However, as the flow behavior is driven by the cellular forcing function, the cellular array structure remains fixed throughout. This confirms there is no back reaction of A on the flow. In this situation, the staircase structure persists. However, the magnitude of the staircase steps decreases over time due to the decay of A , as shown in Fig. 2.7.

To investigate the longevity and stability of the vortex array and staircase structure in the flux expulsion regime, we study scenarios where $\Omega \geq 1$. In Fig. 2.12, we present the evolution of

an $n = 2$ vortex array for the case of $\Omega = 500$. Initially, flux expulsion sustains the array structure. However, as the magnetic field weakens, the wrapped field disintegrates, and the system loses its cellular structure. In Fig. 2.13, we demonstrate how magnetic fields constrain kinetic energy fluctuations up to $t = 200$ for the case of $\Omega = 20$ in a $n = 4$ array. Once the magnetic field dissipates sufficiently, the flow transitions to a turbulent state with no apparent structure.

An alert reader here will notice that the specific observations discussed in the previous paragraph correlate with the shorter lifetime of the suppression stage for increasing Ω values, as shown in the evolution of $\langle A^2 \rangle$ and E_m/E_k (See Fig. 2.14). To elucidate this decrease in duration of suppression or staircase lifetime, we present Fig. 2.15. The figure demonstrates that as the value of Ω increases (relative to the case of $\Omega = 1$), additional k modes are excited and become more prominent in the kinetic energy of the flow. Consequently, fluctuations in the array stretch magnetic field lines, so accelerating magnetic decay – a process intensified for larger Ω values.

Vortex disruption regime ($\Sigma \geq 1$)

Now, we shift our attention to the vortex disruption regime ($\Sigma > 1$). Here, magnetic back reaction on the flow is significant. To characterize and quantify the effects of Ω and M on the vortex array, we measure vortex array disruption by utilizing the Okubo-Weiss field (W) [62, 63], defined as

$$W = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 - \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)^2 \quad (2.20)$$

$$= W_+ - W_- . \quad (2.21)$$

The terms W_+ and W_- represent the normal and shear components of the rate of strain, and the vorticity, respectively. The Okubo-Weiss field measures the relative strength of shear strain to vorticity, characterizing the local fluid flow topology. Specifically, it helps identify areas of rotation versus deformation, with a vortex defined as a region of negative W .

Utilizing the definition of W , we measure vortex disruption in our simulations employing

the vortex disruption parameter (Δ) [96], defined as

$$\Delta = 1 - \frac{\int_{S'} W \, d^2x}{\int_{S'} W_0 \, d^2x}, \quad (2.22)$$

where W_0 represents W for the undisturbed vortex array. Δ is a dimensionless parameter that quantifies the extent of vortex disruption. For the $n = 4$ case, we evaluate the area (S') across four vortices. The fraction term in Eq. 2.22 measures the relative strength of the vortices to the initial W value of the undisturbed vortices. A Δ value of 0 indicates no disruption, meaning the vortices retain their original structure and strength. A Δ value of 1 signifies complete disruption, where the vortices have lost their structure and the flow has transitioned into a mixing regime. Intermediate values of Δ provide a measure of partial disruption, reflecting the degree to which the vortices have been affected. By analyzing Δ , we can determine whether the flow remains stable ($\Delta = 0$) or has transitioned into a turbulent state ($\Delta = 1$). This parameter is useful for understanding the stability and mixing behavior of the vortex array.

To place our findings in the context of previous vortex array studies, we investigate the response of the disruption parameter to injection at the $n = 4$ wave number. Fig. 2.16 reveals that significant vortex disruption occurs at $\Omega = 10$, for both MHD regimes. The value of Δ escalates as Ω approaches 500. For $\Sigma \leq 1$, we find that cell disturbance occurs primarily due to cell perturbations. For $\Sigma > 1$, disruption of the vortex array is caused not only by cell perturbations but also by the background magnetic field via $\mathbf{J} \times \mathbf{B}$ forces. In Fig. 2.16, we observe that the maximum disruption parameter values for the vortex array approach unity for $\Sigma > 1$ at $\Omega > 50$. While one might think that $\Delta \sim 1$ in the vortex disruption regime implies that a staircase structure will not form, our observation suggests otherwise. Despite the influence of the magnetic field on the original array, Figs. 2.10 and 2.11 demonstrates that active scalar layering persists. The figure manifests discernible layering patterns in both the contours and averaged profile of the active scalar concentration, irrespective of the apparent integrity of the cells in the vortex array.

To understand the persistence of layering in regimes where $\Sigma > 1$, we examine the

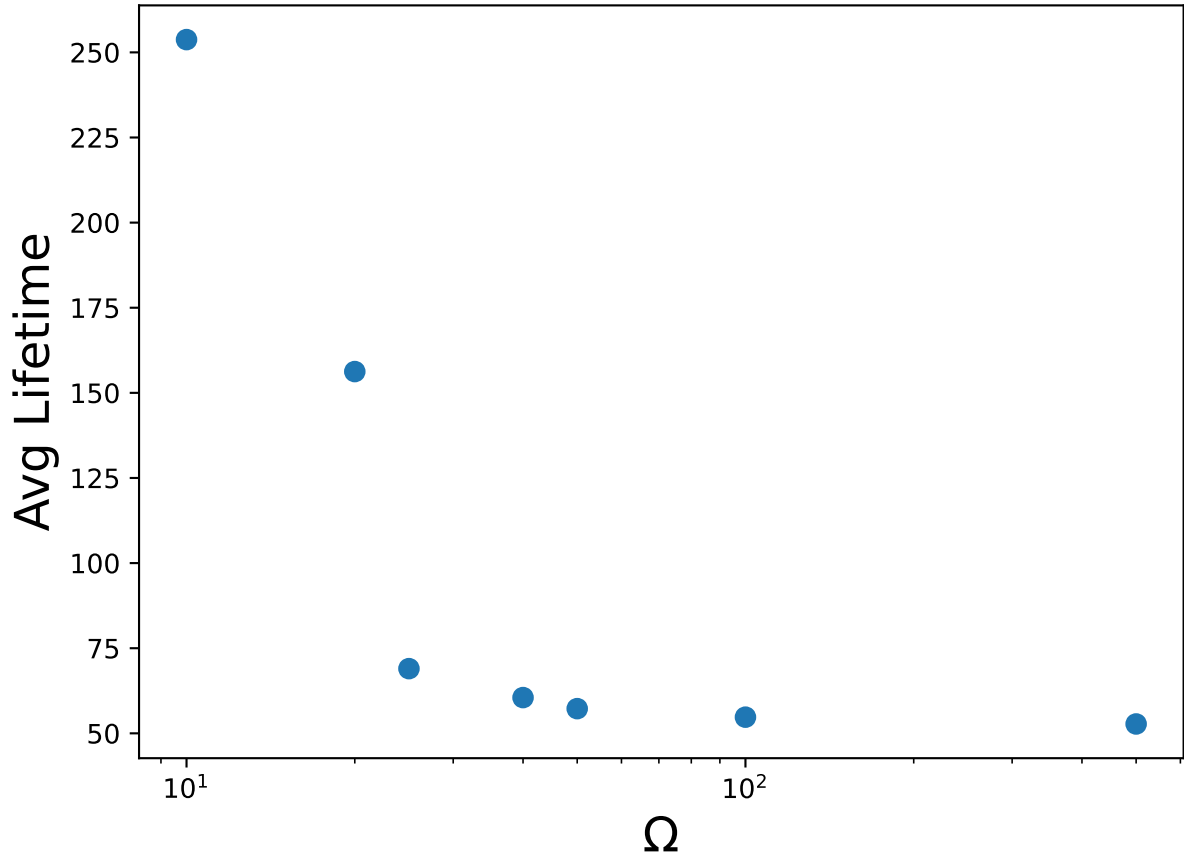


Figure 2.14. The average lifetime of the suppression stage decreases as flow viscosity decreases for Σ values ranging from 0.1 and 20. Significant transitions in suppression lifetime occur at Ω values of 10 and 20, caused by the excitation of $k \neq 4$ modes as Ω increases (see Fig. 2.15). These excitations lead to cell perturbations, distorting the vortex array.

Okubo-Weiss field (constructed using streamlines) in Fig. 2.17 at the same time as shown in Fig. 2.10. Here, staircase structures are similar, but flow structure differs. The Okubo-Weiss parameter helps identify vortical regions, and the streamlines indicate whether cell streamlines are closed, which is essential for layering. We use a measure commonly used to identify vortical regions [103, 104]. Vortical regions are determined by $W < -0.2\sigma$, where σ represents the standard deviation of W . In Fig. 2.17, the original form of the vortex array is no longer apparent. However, during the initial disruption, some small closed streamlines remain intact. As time progresses and the magnetic field weakens, viscous forces become more pronounced, leading to cell movement and mergers, observable at $t = 55$. Our study shows that in a multiple vortex

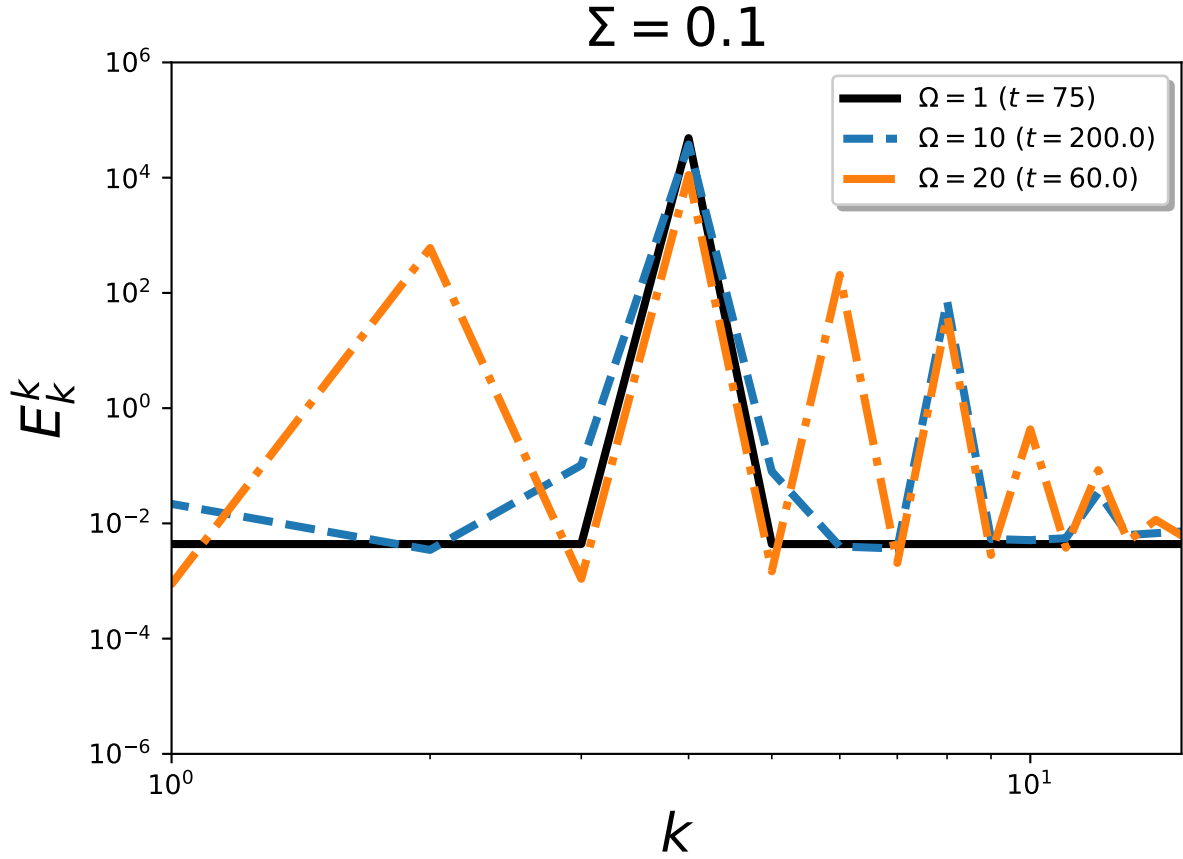


Figure 2.15. The kinetic energy spectrum of the flow at $\Sigma = 0.1$ is examined for various Ω values. At $\Omega = 1$, the flow is predominantly influenced by the wave injection number ($n = 4$). However, as flow viscosity decreases, additional modes emerge. These modes become increasingly prominent as Ω increases.

system, closed streamlines can still exist in the dynamic regime. Thus, layering emerges as a feature common to *both* the flux expulsion and vortex disruption regimes. Counter-intuitively, layering persists in the vortex disruption regime.

As a partial summary: In this subsection, we observe that layered structures emerge during the suppression stage, followed by a rapid decay stage where the flow becomes predominantly hydrodynamic. The lifetime of these layered structures decreases as Ω increase. The excitation of different modes as Ω increases induces cellular perturbations, generating electric currents and accelerating magnetic field decay. Our analysis indicates that inhomogeneous mixing persists in both the flux expulsion and vortex disruption regimes. In the vortex disruption regime, not

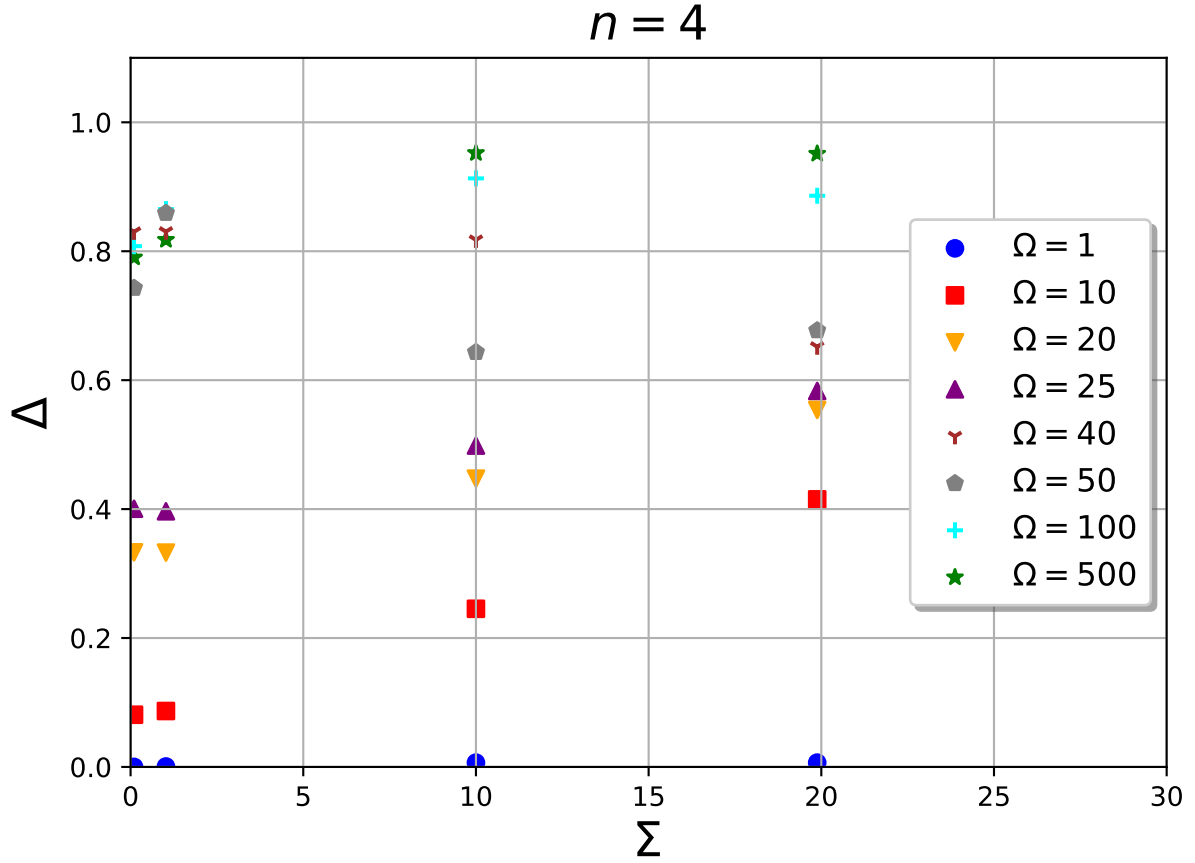


Figure 2.16. Plot of maximum disruption parameter as a function of Σ . For $n = 4$, the value of Δ is measured for a set of four vortices together. We find that for $\Sigma > 1$, the value of Δ approaches 1 for larger values of Ω . In the vortex disruption regime, we observe that not all closed streamlines are completely disrupted within the vortex array structure.

all closed streamlines are fully destroyed, allowing layering to persist. Layering of A is a characteristic feature of fluctuating vortex arrays in *both* 2D MHD regimes for Ω values ranging from 1 to 500 (see Table 2.4). In the next section, we delve into staircase formation within the FVA model, exploring the dynamics and key aspects of active scalar layering throughout the staircase lifetime. We compare results to previous studies of passive scalar staircases.

2.3.2 Magnetic staircase morphology

Having characterized the flow behavior and established that layering persists in regimes of flux expulsion and vortex disruption, we focus on the evolution of the disruption parameter,

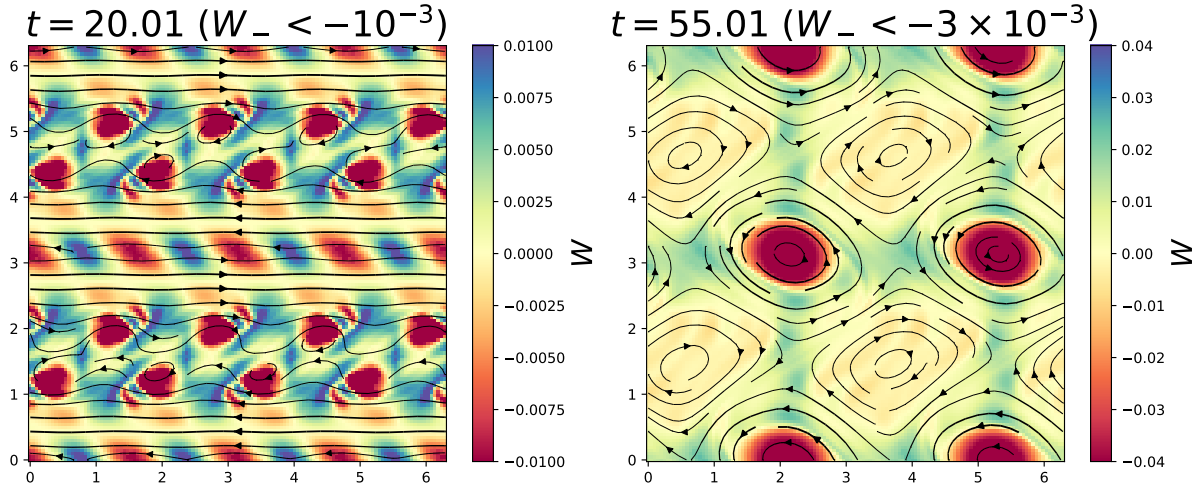


Figure 2.17. Contours of the Okubo-Weiss field for $\Omega = 500$ ($n = 4$, $\Sigma = 10$) at various times demonstrate that even within the regime of vortex disruption, the array with $n = 4$ still maintains closed streamlines. This observation illuminates the persistence of scalar layered structure depicted on the right-side of Fig. 2.10.

Ω	Σ	$\Delta_{\max}$	Staircase Structure
[1, 500]	$\Sigma < 1$	0.8	Y
[1, 500]	$\Sigma \geq 1$	0.98	Y

Table 2.4. Summary of the range of Reynold numbers (Ω), Σ values, and their corresponding maximum vortex disruption parameter ($\Delta_{\max}$). Note that the value of $\Delta_{\max}$ is influenced by both the magnetic field and turbulence-induced fluctuations. The table also indicates the presence or absence of a magnetic staircase structure in each regime.

profile curvature, and the key disparate timescales during magnetic staircase formation. We begin with Fig. 2.18, which illustrates the probability density function (PDF) of A for the frozen vortex array shown in Fig. 2.10. The PDF qualitatively exhibits quasi-periodic layered structures, with the staircase steps indicated by the edges of bright yellow regions (highlighted by pink arrow marker). For $\Omega = 13$ ($\Sigma = 1$), we observe a gradual reduction in the intensity of the staircase steps due to slow magnetic dissipation. The PDF(A) visually represents the layered structure's evolution in the simulation runs, highlighting the impact of magnetic dissipation and cell perturbations on the stability and persistence of the staircase structure.

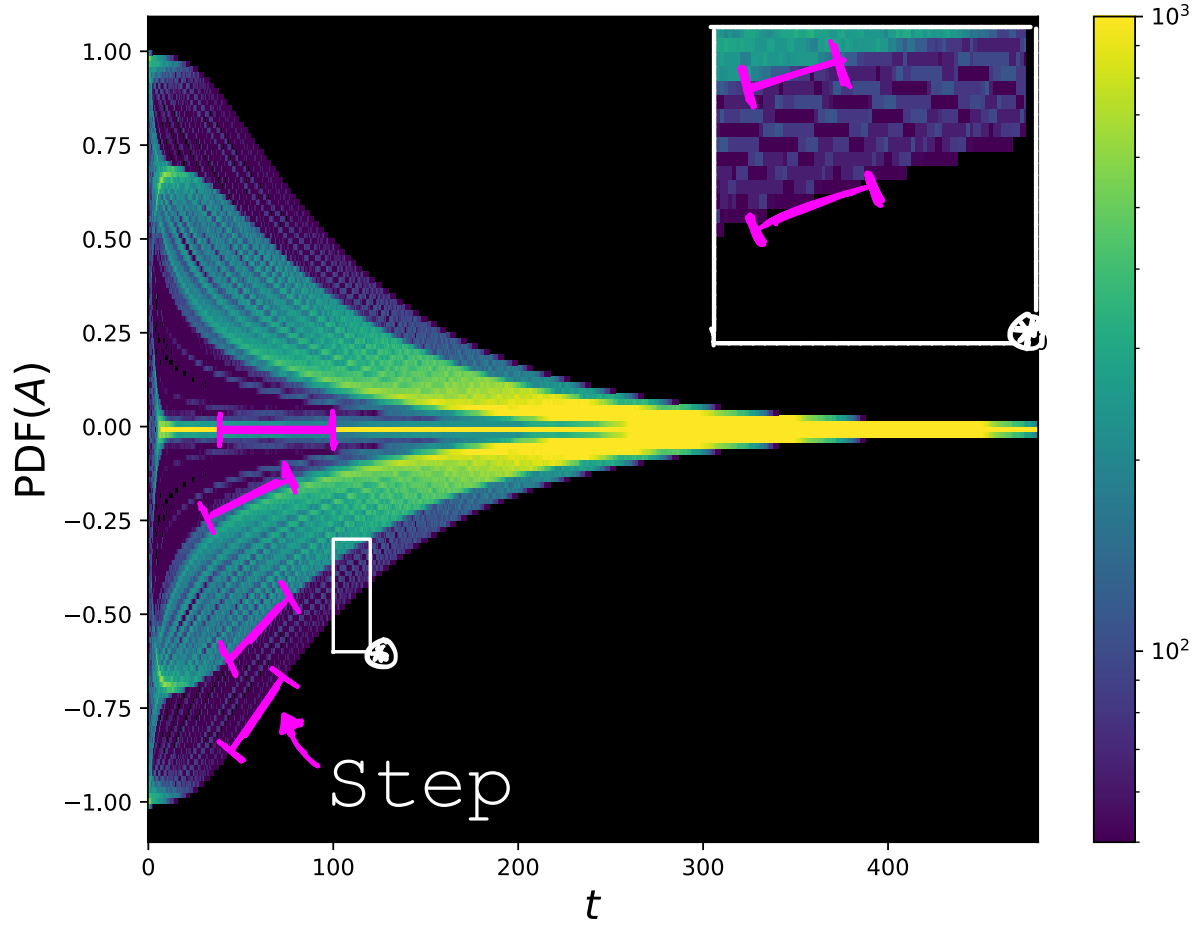


Figure 2.18. PDF of A as a function of time for $n = 4; \Omega = 1; \Sigma = 1$. The edges of intense yellow (light gray) layers represent staircase steps, with seven steps in this example, traced by bright pink bars in the color version. A zoomed-in section is included to emphasize the boundary edge, marking the staircase step. The $\text{PDF}(A)$, as qualitative measure, shows that the step amplitude tends to decrease toward zero, consistent with Zeldovich’s theorem.

Magnetic staircase dynamics

Having identified the behavior of magnetic staircase structures in the PDF of A for the frozen vortex array, we proceed to examine the evolution of the PDF of A and the disruption parameter (Δ) across select simulation runs. These cases represent both common trends, such as flow stability, and unique characteristics, such as mergers, observed for various models. In Fig. 2.19, we observe staircase steps in the PDF merging and diminishing in intensity over time. Notably, when staircase steps are distinct in the PDF, the disruption parameter indicates stability,

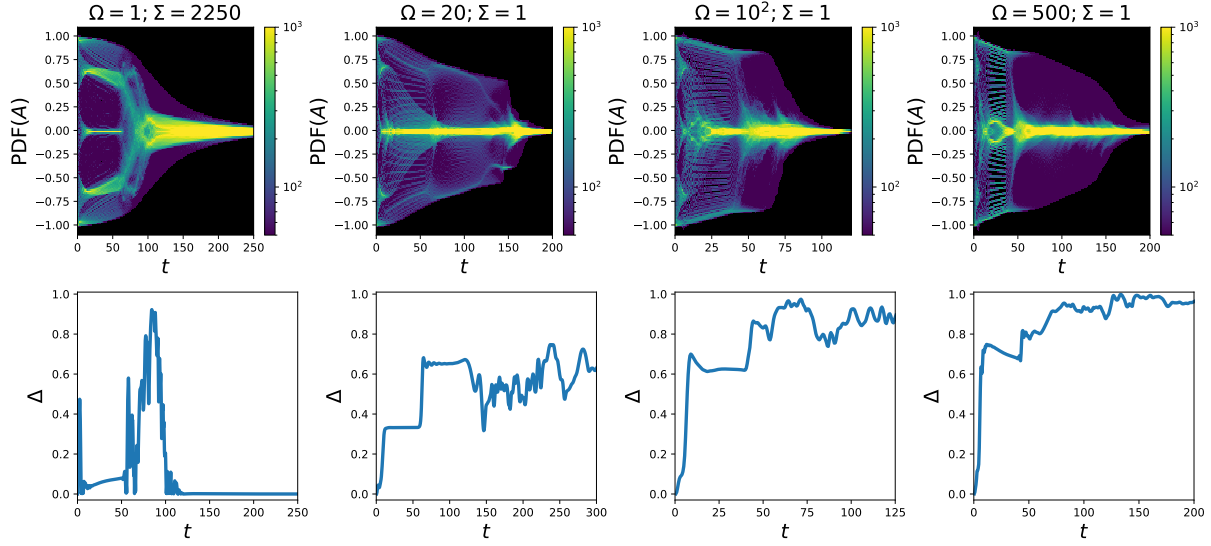


Figure 2.19. Evolution of PDF(A) and Δ for $n = 4$. For the PDF of A, the edges of bright yellow (light gray) layers represent staircase steps, similar to those in Fig. 2.18. The plot of Δ shows the disruption of the vortices over time. When staircase structures are present, Δ remains flat, indicating minimal or no fluctuations. However, once the magnetic field dissipates, the layered structure collapses, and the value of Δ begins to fluctuate.

with minimal fluctuations or remaining nearly constant ($d\Delta/dt \approx 0$).

For instance, consider the case of $\Omega = 1$ ($\Sigma = 2250$): initially (at $t < 50$), the magnetic field disrupts the cellular array, yet layering persists, as shown by the low level of fluctuations in Δ . However, as the magnetic field decays, the cellular array forcing becomes dominant again, leading the flow back to a state of $\Delta = 0$. When examining the case of $\Omega = 500$ ($\Sigma = 1$) before $t = 45$, a similar trend and stability in Δ are observed. However, after $t = 45$, the weakening of F_ω due to large Ω (recall that forcing of ω is divided by Ω) allows cell fluctuations to play a significant role, preventing Δ from returning to zero. These examples, among others shown in Fig. 2.19, illustrate that layering typically occurs at the beginning of the simulation runs and tends to persist in regions where Δ remains relatively stable or changes very slowly, with small fluctuations.

Beyond analyzing the PDF of A and Δ , a key figure of merit for layering is the curvature of the magnetic potential profile (κ). κ correlates with the global layering structure and quantifies the staircase step length and the number of steps within the profile. Here, κ is a macroscopic

measure that relates to the slow to fast timescale ratio, a microscopic measure discussed in detail later in Sec. 2.3.2. The curvature is defined as

$$\kappa = \int_0^L \frac{h''(y)}{(1 + h'(y)^2)^{3/2}} dy, \quad (2.23)$$

$$h(y) = \langle A(y) \rangle_x. \quad (2.24)$$

Fig. 2.20 shows the evolution of profile curvature alongside a snapshot of the magnetic field lines, marked by a pink (light-gray) point indicating the snapshot timing of $|\mathbf{B}^2|$. We observe that layering coincides with regions of steady κ , indicating minimal fluctuations, similar to earlier plots of Δ . Layering occurs (according to contour snapshots of the magnetic field in Fig. 2.20) for stable κ values around $\mathcal{O}(10^{-3})$ when $\Omega \geq 10$, where fluctuations start to become more prominent. During stages where the array structure is frozen ($\Omega < 10$), the staircase structure persists, but the step magnitude (jump height) decreases over time. As A decreases, κ values steadily drop beyond 10^{-3} . The precise critical (or minimal) value that κ decreases to for $\Omega < 10$ is not determined in this study.

In addition to identifying a general trend in the profile curvature, we establish that under conditions presented here, a staircase of 4 to 7 steps exists in an $n = 4$ FVA coupled to an active scalar at stable values of $\kappa \sim \mathcal{O}(10^{-3})$. Although we do not determine exact values of κ corresponding to number of steps, we observe that cell (steps) mergers correlate with a decrease in κ . This is particularly notable in the $\Omega = 20$ and $\Omega = 500$ cases, where a decrease in κ suggests a reduction in the number of staircase steps. This decrease is confirmed visually in snapshots of $\Omega = 20$ and $\Omega = 500$ (1-2 intermediate steps in Fig. 2.11[b]), which show fewer layered regions as compared to the $\Omega = 1$ case (3 intermediate steps Fig. 2.11[a]).

The evolution of κ is continuous but may appear punctuated due to abrupt step and cell mergers, which cause sharp transitions. This effect is illustrated in Fig. 2.20 for the case $[\Omega = 20; \Sigma = 1]$. During $10 < t < 50$ and $60 < t < 150$, κ remains stable as the magnetic field preserves

the cellular array structure. However, between $50 < t < 60$, cell mergers occur, leading to sudden jumps in κ . Since mergers occur on a much shorter timescale than the period over which the magnetic field maintains the array, the evolution of κ can appear discontinuous.

The impact of step mergers can be observed in the PDF of A . There, the bright yellow lines merge as time proceeds during the suppression stage (see Fig. 2.19). In contrast, the $\Omega = 1$ case, which does not have mergers during the suppression stage, shows bright yellow lines continuing without intersecting.

Although connecting κ values to the number of staircase steps does not significantly contribute to the main topic of the paper, it is important to note that fewer regions of corrugation occur for smaller stable κ values. This trend agrees with profile curvature values observed in previous passive scalar studies. While there is not a simple connection of this study to passive scalar κ values due to the difference in boundary conditions, we find in both cases that as the number of steps decrease, profile curvature also tends to decrease. This finding is useful for establishing a relationship between profile curvature and the number of steps or regions of corrugations.

Process of staircase formation: Timescale ratios

Having established the presence and persistence of layering in a vortex array and the behavior of key quantities such as Δ and κ for various combinations of $\Omega = [1, 500]$ with $\Sigma = [0.1, 20]$, we now delve into the process of staircase formation, with a focus on pertinent timescales. In Fig. 2.21, we present the evolution of various timescales for the case of $\Omega = 20$ ($\Sigma = 0.1$). Note that variability in timescales across simulations primarily arises with τ_v , τ_A , and τ_{η_T} due to varying B_0 and Ω .

In our analysis, we define τ_A as a function of $\langle B \rangle$, considering the evolving nature of the magnetic field in the flow. In addition, we define τ_{η_T} as $(n^2/k^2\eta_T)$ for reference (for definition of η_T , see Eq. 3.22). Fig. 2.21 shows that $\mathbf{B}$ significantly impacts both the transport of the active scalar and vortex array dynamics. During the suppression stage, $\tau_A < \tau_v$, confirming

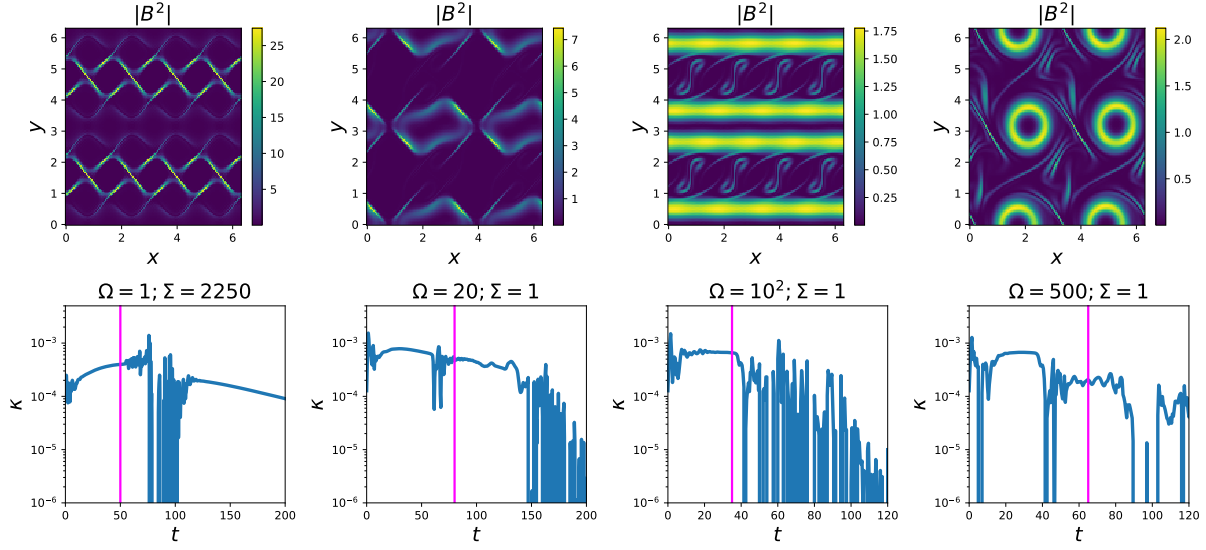


Figure 2.20. The evolution of $|B^2|$ and κ for $n = 4$ reveal insights into the layering process. κ is an integral measure that evolves as the profile of A changes, providing a quantitative assessment of step mergers and structural evolution. Stable regions in the evolution of κ correspond to periods where layering is present, as highlighted in pink. Preliminary spikes in the curvature coincide with the magnetic field adjusting to restrain the array. For non-frozen vortex arrays, profile curvature hovers at $\mathcal{O}(10^{-3})$. At this κ value, layering consists of 5 to 7 steps.

that the magnetic field effectively dampens cellular fluctuations and reinforces the cell array structure. Here, the τ_v timescale governs fluid momentum diffusion across a scale length (n/k) , corresponding to cell movement during array fluctuations. As the simulation progresses, $\tau_v < \tau_A$ due to the decay of $\mathbf{B}$. This causes the magnetic staircase structure (or vortex array) to collapse due to cell perturbations. In Sec. 2.3.3, we expand and confirm this argument regarding τ_A and τ_v .

Another consistent observation across all simulations is the relative position of τ_H and $\tau_{\eta T}$

Scalar Type	Passive	Active
κ	$\mathcal{O}(10^0)$	$\mathcal{O}(10^{-3})$
$\tau_{\text{slow}}/\tau_{\text{fast}}$ timescale	$\tau_D/\tau_H \propto \mathcal{O}(10^1)$	$\tau_{\eta T}/\tau_H \propto \mathcal{O}(10^3)$

Table 2.5. Comparison of profile curvature (κ) and the ratio of slow and fast timescales in passive and active scalar simulations. The table highlights that the ratio of slow to fast timescales is 10^2 times greater in magnitude for active scalar simulations, indicating a more rigid staircase structure than for passive scalar cases. Note that differences in profile curvature may be due to different boundary conditions.

as the fastest and slowest timescales, respectively, in the presence of layering. We find that the ratio between the two timescales is 10^3 when layering occurs (i.e., $\tau_H \ll \tau_{\eta_T}$). Previous studies, such as those in Refs. [46, 83], have emphasized that layer formation requires two disparate mixing timescales, a slow and fast one, as we observe in our simulations. *Comparing the ratio of the slow and fast timescales in our active scalar system to those in passive scalar dynamics, we find that relative to the passive scalar case of Ref. [83], the active scalar system amplifies this disparity by approximately 10^2 (i.e., $\tau_{\eta_T}/\tau_D \sim 10^2$). This indicates that the background magnetic field in the active scalar system significantly strengthens the staircase structure relative to the passive scalar case.*

Similar to previous observations, mixing within the flow occurs once the magnetic field has decayed sufficiently, leading to the approach of τ_{η_T} to τ_H (equivalent to $\eta_T = \eta_k$). At this stage, the transport of A across cell boundaries occurs at the rate of τ_H , eliminating the distinction between slow and fast mixing timescales and ultimately leading to the breakdown of the active scalar staircase structure.

As a partial summary: We establish that layering persists even without a fully coherent vortex array; that is, $\Delta = 0$ is not required for layering to occur. This is illustrated in Fig. 2.20. For non-frozen vortex arrays ($\Omega \geq 10$), the value of κ consistently stays at $\mathcal{O}(10^{-3})$ in simulations where layering is present. Furthermore, layering induces transient stabilization in flow and active scalar behavior, as evidenced by minimal changes in the time-evolution plots of Δ and κ . Finally, we establish two criteria for staircase formation. First, $\tau_A < \tau_v$ is required for the magnetic skeleton to dampen cellular perturbations caused by viscosity. We also confirm the need for coexistence of disparate slow and fast timescales for layering to occur, namely that $\tau_H \ll \tau_{\eta_T}$. When these conditions are no longer satisfied, the magnetic staircase structure ceases to exist. We compare essential staircase structure criteria between passive and active scalars in Table. 2.5.

In the forthcoming section, we will explore the impact of global stochastic magnetic potential forcing on the lifetime of inhomogeneous mixing – whether it reinforces layering or

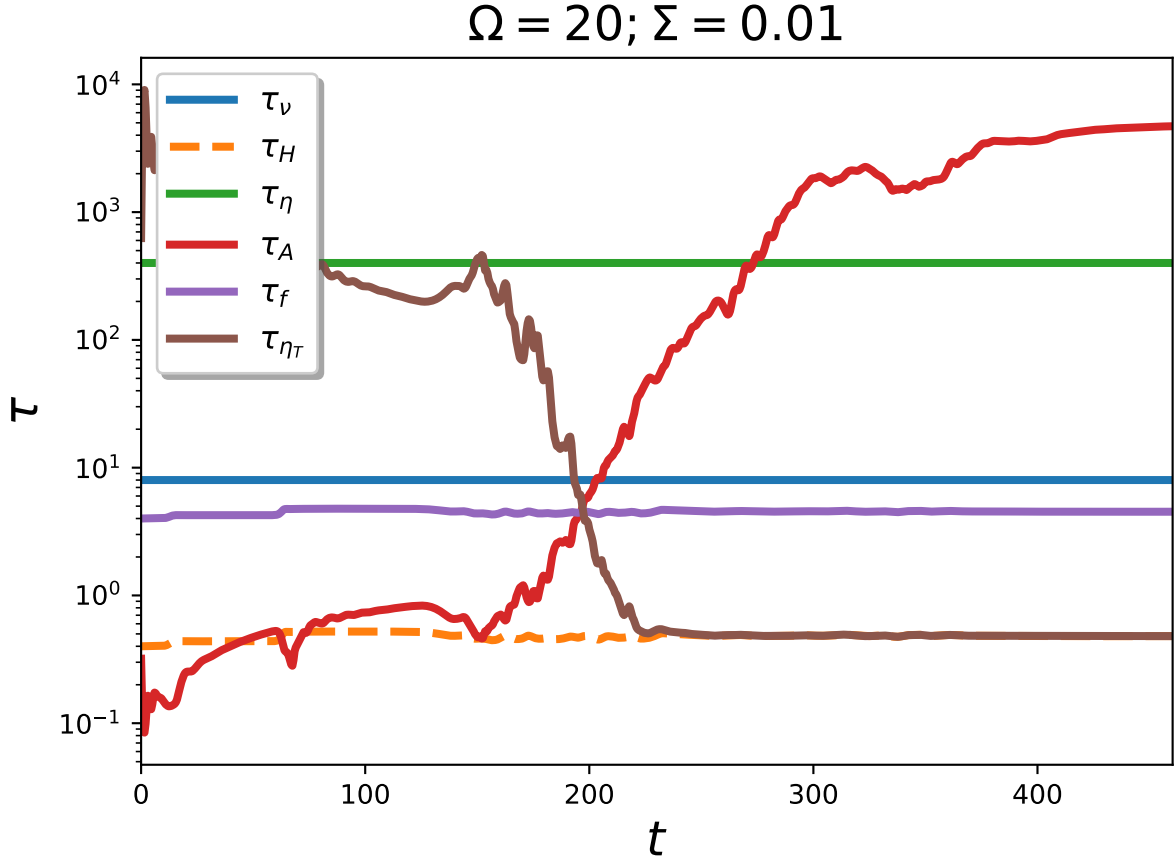


Figure 2.21. We monitor the evolution of various timescales, considering the case of $\Omega = 20$ ($n = 4$, $\Sigma = 0.1$). Notably, both τ_A and τ_{η_T} change significantly over time. The magnetic field ($\mathbf{B}$) plays a crucial role in both response and suppression dynamics. During the suppression stage, τ_A responds more rapidly than the effects governed by τ_v , simultaneously restraining turbulent transport, as evidenced by the evolution of τ_{η_T} . The suppression stage concludes when τ_{η_T} transitions to τ_H . This is accompanied by τ_A sufficiently exceeding τ_v .

allows for reformation of staircase structures during the kinematic decay stage. Our goal is to determine whether it is possible to extend the lifetime of staircases structure by applying external forcing to prevent the decay of A . In addition, we will examine active scalar transport across the system with stochastic magnetic forcing to understand its influence on transport and how it compares to cases without forcing.

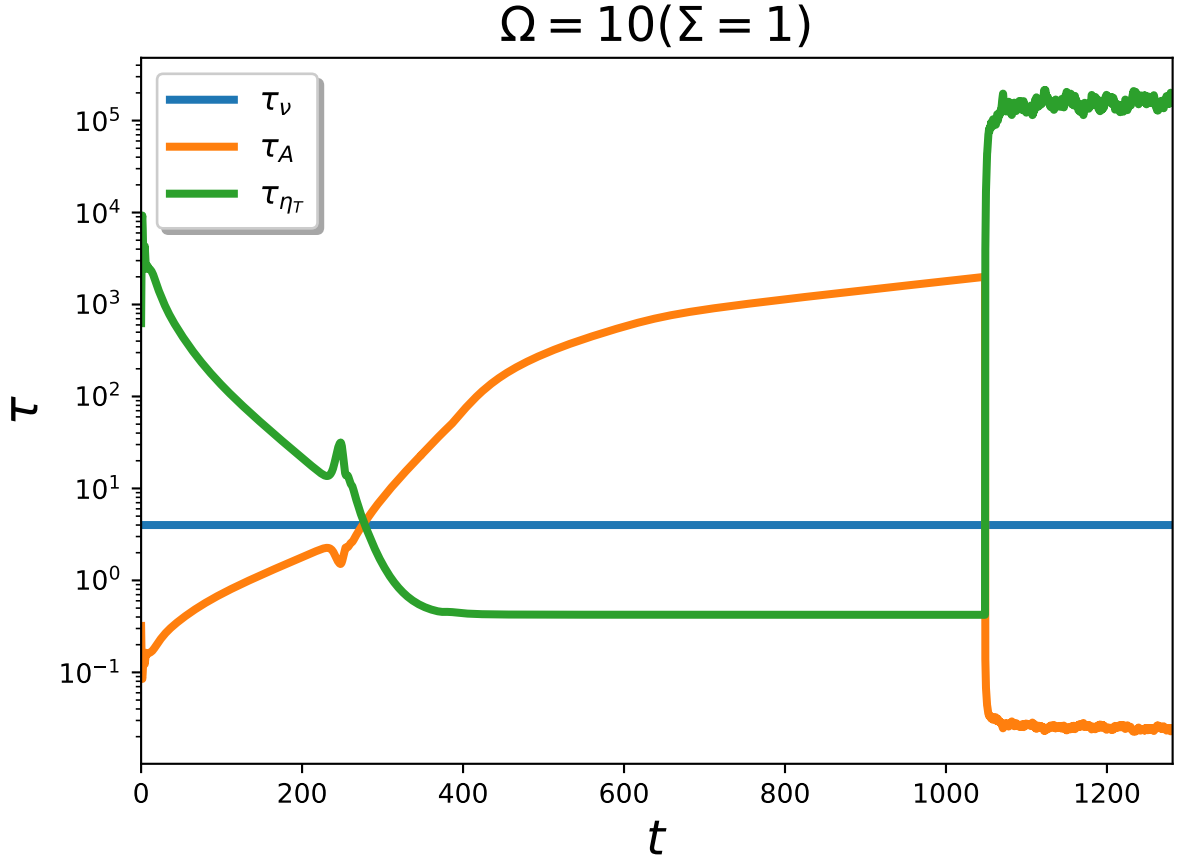


Figure 2.22. Timescale evolution corresponding to Fig. 2.23 elucidates why we observe the vortex array reverting back to a structure similar to that found in the frozen vortex array stage. This plot demonstrates that upon turning on external forcing of A , τ_A becomes smaller than τ_v , indicating that the magnetic field responds faster than fluid momentum diffuses.

2.3.3 Stochastic magnetic forcing

We have demonstrated that staircase structures without external magnetic forcing can exist and be sustained on a slow τ_{η_T} timescale through a feedback process between the flow and the magnetic field. However, the inherently limited lifetime of the magnetic field in 2D is a recurring theme in this paper. In this subsection, we investigate the impact of external magnetic potential forcing on the model's longevity and transport mechanisms.

To this end, we introduce an additional term, $F_A = F_{amp}\hat{F}_A$, to the right side of Eq. 2.3, representing global stochastic white noise magnetic potential forcing. We apply a Gaussian

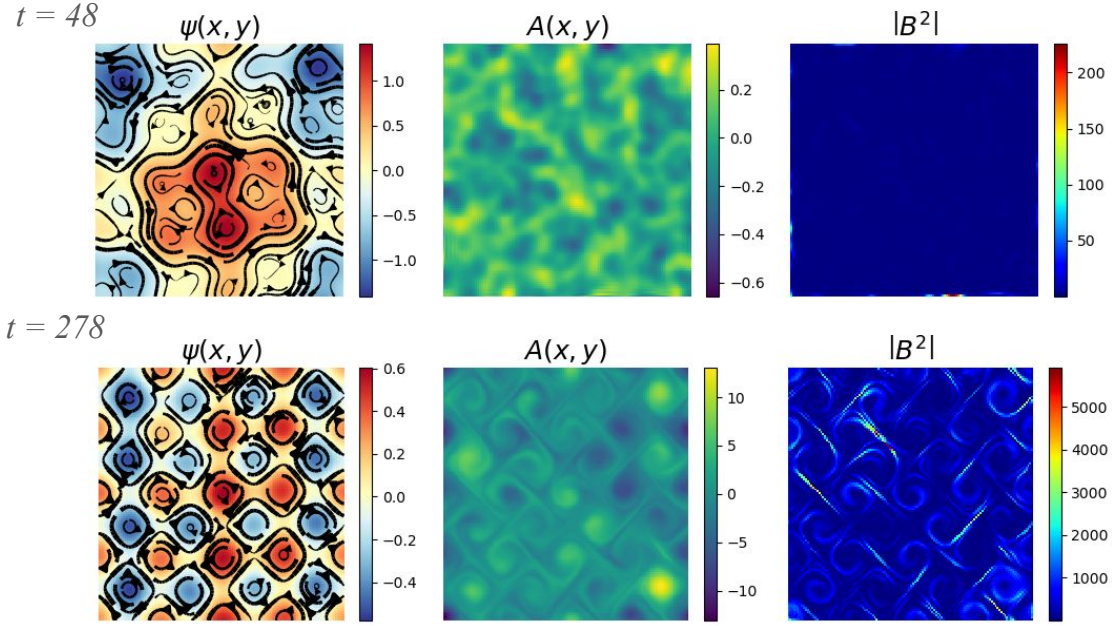


Figure 2.23. Contour snapshots of $\Omega = 10(n = 4, \Sigma = 1)$ in the kinematic decay stage, where we restart with stochastic external forcing. Simulations reveal that the gradual application of stochastic magnetic potential forcing drives the fluctuating vortex array back to an almost frozen vortex array.

filter to the white noise to attenuate high-frequency components that could result in unrealistic outcomes while preserving the overall shape of the noise. Through various simulations, we examine the influence on flow suppression for different magnitudes (F_{amp}) of the stochastic forcing, specifically applied to different Σ values across a range of Reynolds numbers.

First, we present an approximate calculation to establish expectations for η_T from simulations with external forcing. Considering Eq. 2.3 with the initial conditions

$$A = B_0 y + \tilde{A}, \quad (2.25)$$

$$\mathbf{B} = B_0 \hat{x} + \tilde{\mathbf{B}}, \quad (2.26)$$

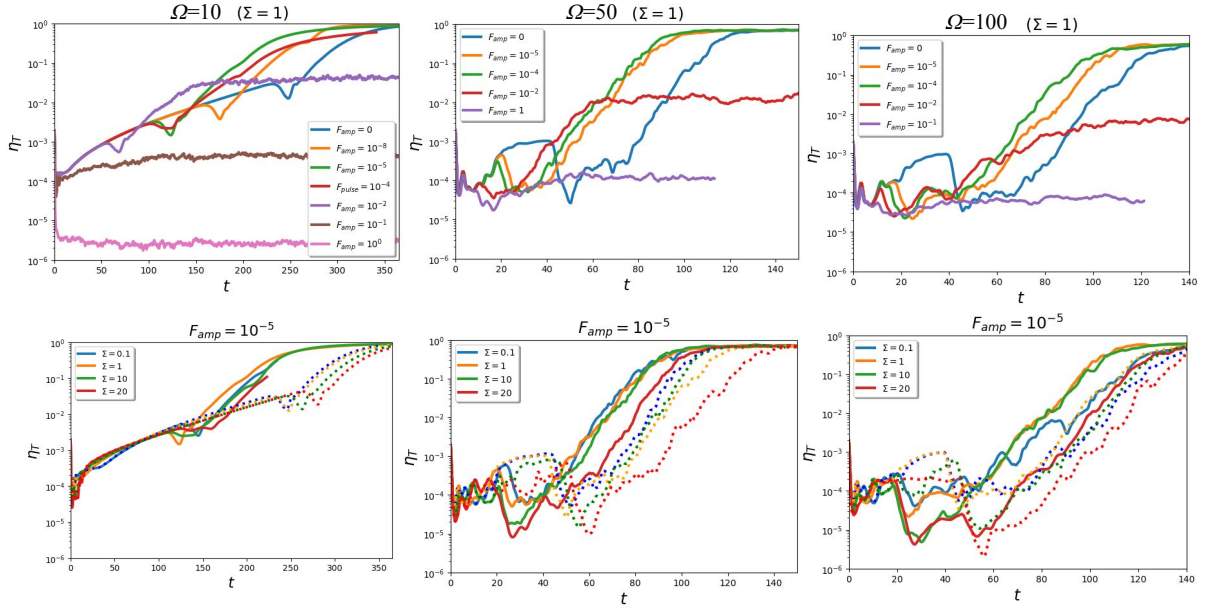


Figure 2.24. The evolution of turbulent resistivity (η_T) for $n = 4$ is depicted in the top panel for various forcing amplitudes at $\Sigma = 1$ across a range of Ω values, while the bottom panel shows η_T for a forcing amplitude of 10^{-5} across different Σ values, each corresponding to the top Ω value. These plots illustrate that for sufficiently small forcing amplitudes ($F_{amp} < 10^{-2}$), turbulence quenching can be enhanced, albeit at the expense of a shorter suppression lifetime. However, for forcing amplitudes exceeding 10^{-2} , turbulent quenching can be sustained or even improved indefinitely, as long as the forcing is sustained.

we express the two-dimensional MHD equation for the magnetic potential fluctuation as:

$$\frac{\partial \tilde{A}}{\partial t} + \nabla \psi \times \hat{\mathbf{z}} \cdot \nabla \tilde{A} = -\tilde{u}_y \langle B \rangle + \eta \nabla^2 \tilde{A} + \tilde{F}_a, \quad (2.27)$$

where we define $\langle A \rangle$ as $\langle A(y) \rangle$. Here, $\tilde{F}_a$ represents the general expression for the (random) magnetic stirring force, with $d\tilde{A}/dt = \tilde{F}_a$, where $\tilde{A}$ denotes the random perturbation of A . Multiplying by $\tilde{A}$, assuming the fluctuation correlation length is smaller than the scale of $\langle B \rangle$ variation, and integrating over space ($\langle \rangle$), we obtain:

$$\frac{\partial \langle \tilde{A}^2 \rangle}{\partial t} + \langle \nabla \cdot \mathbf{u} \tilde{A}^2 \rangle = 2 \left[-\langle \tilde{u}_y \tilde{A} \rangle \langle B \rangle + \eta \langle \tilde{A} \nabla^2 \tilde{A} \rangle + \langle \tilde{A} \tilde{F}_a \rangle \right]. \quad (2.28)$$

For stationary systems with periodic boundary conditions, the left-hand side of Eq. 3.39 vanishes.

Integrating by parts on the right-hand side gives:

$$\langle \tilde{B}^2 \rangle = \frac{-\langle \tilde{u}_y \tilde{A} \rangle \langle B \rangle + \langle \tilde{A} \tilde{F}_a \rangle}{\eta}. \quad (2.29)$$

We can write $\langle \tilde{u}_y \tilde{A} \rangle$ in the form of a Fick's law ($\langle \tilde{u}_y \tilde{A} \rangle = -\eta_T \partial \langle A \rangle / \partial y = -\eta_T \langle B \rangle$), yielding:

$$\langle \tilde{B}^2 \rangle = \frac{\eta_T \langle B \rangle^2 + \langle \tilde{F}_a^2 \rangle \tau_a}{\eta}. \quad (2.30)$$

Here, τ_a is the auto-correlation time of the (random) magnetic stirring force $\tilde{F}_a$. We can insert this into the following representation of η_T , derived from a quasilinear expression for the turbulent flux ($\Gamma = -\langle \delta A \psi_y \rangle + \langle A_y \delta \psi \rangle$),

$$\eta_T = \left(\langle \tilde{u}^2 \rangle - \langle \tilde{B}^2 \rangle \right) \tau, \quad (2.31)$$

where τ represents the average correlation time of the turbulence. By inserting Eq. 3.41 into Eq. 3.42, we arrive at the following approximation for η_T with random external forcing

$$\eta_T \sim \frac{\left(\langle \tilde{u}^2 \rangle - \langle \tilde{F}_a^2 \rangle \tau_a \right) \tau}{1 + R_m \frac{\langle B^2 \rangle}{\langle u^2 \rangle}}. \quad (2.32)$$

Here, the negative sign in front of $\langle \tilde{F}_a^2 \rangle$ indicates that η_T decreases when external forcing is applied to Eq. 2.3. In the next few subsections, we will explore several scenarios to determine whether turbulent transport is indeed suppressed ($\eta_T \ll \eta_k$) with the application of external magnetic potential forcing.

Spontaneous staircase formation

Before examining how different external stochastic magnetic potential forcing amplitudes affect the vortex array and magnetic staircase structures, we investigate whether layering can

spontaneously reoccur during the kinematic decay stage, where the flow is nearly hydrodynamic. In Fig. 2.22, we restart the case of $\Omega = 10$ with $\Sigma = 1$ with magnetic potential forcing and analyze the evolution of τ_A and τ_{η_T} during the decay stage. Irrespective of the value of F_{amp} , the plot shows that around $t \sim 10^3$, τ_{η_T} reverts to a long timescale, indicating that $\eta_T \ll \eta_k$, thus signaling a possible return of inhomogeneous mixing. Additionally, τ_A becomes smaller than τ_v , implying that the magnetic field responds at a rate faster than turbulence-induced mixing can occur (i.e., $\mathbf{B}$ dampens cell perturbations caused by kinematic viscosity). As further evidence, we present Fig. 2.23, which shows distinct cellular array structures with significantly strong magnetic fields at the boundaries. Therefore, when restarting with magnetic forcing, the fluctuating distorted array gradually returns to its original cellular array structure over time.

Impact of external forcing on suppression lifetime

Next, we examine the influence of stochastic magnetic forcing on the suppression lifetime. In Fig. 2.24, we present the results of selected simulations. Note that we measure η_T directly from flow velocity and magnetic field time traces, following Ref. [92]. The top row illustrates the evolution of η_T for different F_{amp} values, consistently applied at $\Sigma = 1$ across various Ω values. For $0 < F_{amp} < 10^{-2}$, η_T initially decreases early in the simulation but eventually reverts back to η_k , as in the zero amplitude case. To sustain η_T , we explore the impact of pulsing external A . By pulsing a forcing amplitude of 10^{-4} every $t = 10$, we extend the suppression stage duration, matching the $F_{amp} = 10^{-5}$ case without pulsing. Therefore, pulsed forcing offers a feasible approach to prolonging suppression for small forcing amplitudes.

We also conduct simulations with an amplitude of $F_{amp} = 10^{-5}$ for similar Ω values and different Σ values, exploring both the flux expulsion and the vortex disruption regimes. Consistent with the observations in the top row of Fig. 2.24, we find that an amplitude of $F_{amp} = 10^{-5}$ reduces the value of η_T for a brief period (specifically $100 < t < 150$ for $\Omega = 10$ and $20 < t < 50$ for both $\Omega = 50$ and $\Omega = 500$). *It is important to note that, as in cases discussed earlier with $F_{amp} < 10^{-2}$, we do not observe layering once $\eta_T/\eta_k \rightarrow 1$.*

While the small forcing amplitudes studied thus far decrease η_T , they do not indefinitely extend the lifetime of inhomogeneous mixing. However, for $F_{amp} \geq 10^{-2}$, the lifetime of layered structures is extended. We first examine the case with $\Omega = 10$ in Fig. 2.24. At $F_{amp} = 10^{-2}$, η_T does not fully convert to η_k but remains steady at around $\sim 10^{-2}$. This pattern persists as F_{amp} increases. Both $F_{amp} \sim 10^{-1}$ and $F_{amp} \sim 1$ exhibit consistently smaller values of turbulent resistivity compared to η_k and the corresponding zero forcing case. Simulation results qualitatively agree with the estimate in Eq. 3.43, which predicts that greater forcing amplitude can further decrease the value of η_T . These results indicate that external random magnetic potential forcing can enhance layering of A by maintaining a smaller η_T relative to η_k .

Partial summary: In this subsection, we explore how stochastic external forcing of magnetic potential affects suppression lifetime and provide an approximation to η_T in such scenarios (see Eq. 3.43). Our simulations show that forcing amplitudes of $0 \leq F_{amp} \leq 1$ reduce turbulent transport during the suppression stage, aligning with theoretical predictions. However, for forcing amplitudes less than 10^{-2} , turbulent resistivity eventually evolve to η_k over time. Enhancing confinement for $F_{amp} < 10^{-2}$ is feasible through pulsing or short-term external forcing. For forcing amplitudes greater than 10^{-2} , $\eta_T \ll \eta_k$ is maintained. Thus, sufficiently high forcing amplitudes or pulsing methods can extend the lifetime of active scalar staircase structures in a FVA.

2.4 Conclusion

In this paper, we compare the formation of layered patterns in a forced 2D MHD system including feedback between magnetic potential (active scalar) and flow (see Fig. 2.25) – to corresponding passive scalar studies [83]. The study focuses on a fluctuating cellular array threaded by magnetic field lines. The principal results are:

1. Staircases persist in both the flux expulsion and vortex disruption limits. Surprisingly, in the latter case, residual vortex cells homogenize A .

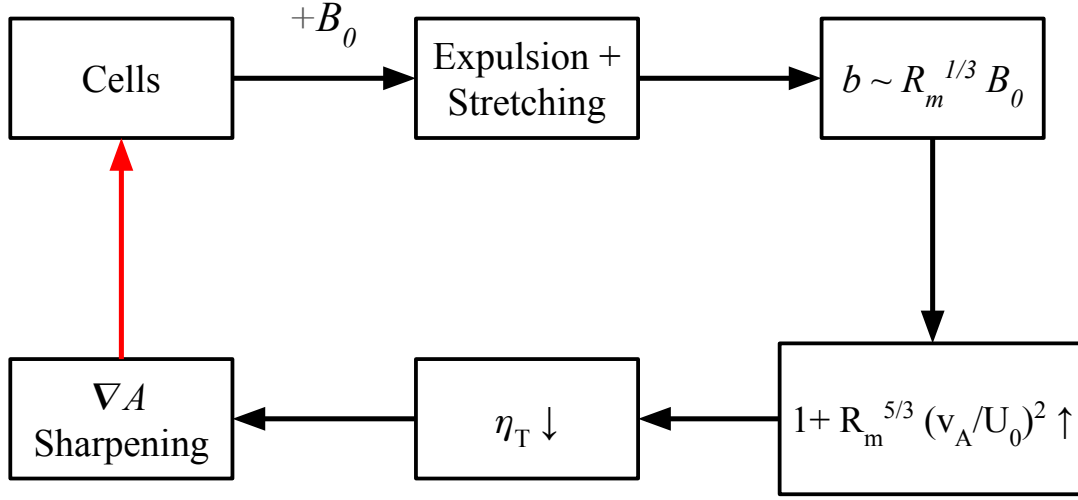


Figure 2.25. In the flux expulsion regime ($\Sigma < 1$), a feedback loop exists between cells and the magnetic potential. As cells expel magnetic field lines, the magnetic field increases at the boundaries. This increase reduces turbulent transport across cell boundaries, sharpens the gradient of A , and strengthens the cell array structure. This process is known as transport reduction feedback.

2. R_m -dependent quenching of the turbulent diffusion of A by weak magnetic fields increases the disparity between the cell circulation time and the time for inter-cell transport of magnetic potential. The background magnetic field increases the timescale disparity by a factor of $\sim 10^2$, reinforcing the staircase relative to its passive scalar counterpart. This is a new feedback mechanism, which promotes staircase formation.

For a more detailed summary, see the partial summaries in the result sections and the overall summary provided below.

During a quasi-stationary suppression stage, staircase structures form and persist in regimes of low to modest Reynolds numbers ($0 \leq \Omega \leq 500$). These cases encompasses both flux expulsion ($\Sigma < 1$) and vortex disruption ($\Sigma \geq 1$) limits [see Figs. 2.10 and 2.11]. The lifetime of the magnetic potential staircase is ultimately limited by resistive $\mathbf{B}$ decay. In both, turbulent

resistivity measurements (see Fig. 2.21) show that the gradient of the magnetic potential at the cell boundaries increases. This effectively limits active scalar cross-boundary transport and damps cell perturbations, stabilizing the cellular structure.

A remarkable result found in this paper is the persistence of magnetic potential staircase structure in the vortex disruption regime, where the back reaction on the flow is significant. As shown in both Figs. 2.10(b) and 2.17, active scalar staircase profiles form in various flow structures despite disturbances to the vortex array. The persistence of the staircase structure occurs because not all closed streamlines are disrupted by the large background magnetic field ($v_A^2 \geq U_0^2/R_m$). Earlier single-vortex studies do not predict the persistence of inhomogeneous mixing in the vortex disruption regime, making our findings unexpected.

It is important to note that unlike passive scalar layering, which is simple and primarily determined by vortex size and timescale ratio, active scalar layering is nontrivial, as transport barriers depend on background magnetic field strength. As shown in Eqs. 6.18 and 6.19, the ratio of background to boundary magnetic field strength sets the boundary layer width relative to cell length. For flux expulsion, boundary layers vary with magnetic field strength, leading to changing layer spacing over time. This variability is captured in the profile curvature plots (see Fig. 2.20), which illustrate the dynamic staircase formation and the relationship between cell length and boundary width. Layering also persists in the vortex disruption case, making it another nontrivial instance.

Comparing our findings to previous passive scalar layering studies, we observe that even a weak back reaction on the flow due to the active scalar strongly reinforces (i.e., tends to lock in) the staircase pattern. We measure this trend by using the ratio of turbulent resistivity (τ_{η_T}) to cell circulation time (τ_H), analogous to the ratio of passive scalar diffusion across cell boundaries (τ_D) to τ_H . Notably, τ_{η_T}/τ_H remains constant at $\mathcal{O}(10^3)$ on average during the suppression stage, for Ω values ranging from 0 to 500. In contrast, passive scalar studies show a decrease in the ratio of slow-to-fast timescales from 30 to 17.5 for Ω values from 0 to 35. The ratio contrast between the two cases is a factor of 10^2 . This finding highlights that active scalar staircase

structures can support sharper gradients than passive scalar staircase structures. The result aligns with boundary layer scales and transport theory discussed in Appendix 6.7. For a comparison of active and passive scalar results, see Table 2.5.

To expand on results discussed above, we observe that for the magnetic staircase structure, both the vortex disruption parameter (Δ) and the curvature of the magnetic potential profile (κ) stabilize, exhibiting steady evolution with minimal fluctuations. During the suppression stage of the FVA, a staircase structure consisting of 4 to 7 steps corresponds to κ values of $\mathcal{O}(10^{-3})$. Although κ values differ between studies of active and passive scalar staircases (due to different boundary conditions), both studies suggest that a decrease in profile curvature implies fewer steps (or regions of corrugations) [see Fig. 2.20]. In addition, this decrease in κ occurs as Ω increases, which is also observed in passive scalar studies.

As mentioned previously, in the absence of external magnetic forcing, the staircase structure exists only for a finite lifetime during the initial quasi-stationary suppressive stage. To investigate whether we can prolong the structure's lifetime or generate it, we explore the effects of external stochastic magnetic potential forcing (F_A) [Eq. 2.3]. In both cases, we find that $\tau_{\eta_T} \ll \tau_H$ and $\tau_A < \tau_v$ re-emerge in timescale evolution studies, allowing magnetic staircase structures to form while damping cell perturbations (see Fig. 2.22). For $F_{amp} < 10^{-2}$, turbulent resistivity decreases, rendering the magnetic staircase structure more rigid during its initial formation. It eventually reverts to η_k over time, regardless of external magnetic forcing. For forcing amplitudes greater than 10^{-2} , the external magnetic forcing maintains $\eta_T \ll \eta_k$ throughout the system's evolution. These results are qualitatively consistent with our estimate of η_T with external (random) forcing included, as presented in Eq. 3.43. This suggests that with external magnetic potential forcing, the value of η_T decreases, so reducing turbulent transport of magnetic potential. For F_{amp} values greater than 10^{-2} , we argue that the magnetic potential is injected at a rate faster than the magnetic field can decay, continuously sustaining transport suppression of A across cells.

Our results have implications for experiments and theory concerning systems with layered

structures, as mentioned in the introduction and in Appendix 6.5. Although our model differs from others (e.g., PV staircases, double-diffusive staircases, and staircases in forced stratified turbulence), it demonstrates a novel feedback mechanism that drives layering and reinforces transport barriers, contributing to theories of staircase formation in various systems. We find that when perpendicular active scalar field lines (e.g., $\delta B_{\perp}$) thread through convective cells, the layered structure's resilience is enhanced, and not all convective cells are disrupted, regardless of the field strength. In future work, we plan to apply fixed flux boundary conditions to this model, to compare quantitatively to previous studies of passive scalar transport. We expect that the steady introduction of magnetic flux will form and reinforce the layered structure. Based on studies of external magnetic potential forcing, we also anticipate that higher external flux of magnetic potential may also affect the longevity of the staircase structure. These topics will be pursued in future research.

Chapter 3, in full, is a reprint of the material as it appears in F.R. Ramirez and P.H. Diamond. "Staircases of Passive and Active Scalar Concentration in Cellular Flow." *Phys. Plasmas* 32, 072302 (2025). The dissertation author was the primary investigator and author of this paper.

Chapter 3

Staircases of passive and active scalar concentration in cellular flow

3.1 Introduction

Layering is a remarkable phenomenon where by a quantity which is mixed, spontaneously separates into distinct layers, rather than homogenizing. Layering usually requires anti-diffusion, in which up-gradient transport occurs (at least in some channel or some field) [49]. The familiar negative viscosity phenomenon is a case of anti-diffusion. Profiles of layered quantities often form *staircases* [see Fig. 3.1(a)], in which regions of strong mixing or homogenization form ‘steps’, interspersed between sharp gradients or ‘jumps’. The sharp gradient layers are, of course, transport barriers, so we see that layering may be thought of as the formation of a quasi-periodic lattice of micro-barriers. Here, we use the term micro-barrier to distinguish staircase barriers from an isolated, large transport barrier [see Fig. 3.1(b)]. Layering and staircase formation are ubiquitous [75]. Examples include:

1. density staircases in stirred, stably stratified fluids [51].
2. staircase formation in double diffusion systems [76, 77, 36, 78] – i.e., as in thermohaline convection, of relevance to climate applications, and which can occur in a cafe latte [105].
3. potential vorticity (PV) staircases in geophysical fluids [37], planetary atmospheres etc.

The bands of Jupiter and other giant planets are a particularly compelling example of a PV

staircase.

4. $\mathbf{E} \times \mathbf{B}$ staircases in magnetized plasmas [35, 1, 53, 54]. These closely resemble geophysical PV staircases. $\mathbf{E} \times \mathbf{B}$ staircases involve the coordination of temperature or density gradients with $\mathbf{E} \times \mathbf{B}$ shear layers. $\mathbf{E} \times \mathbf{B}$ staircases will be discussed further in this paper (see Sec. 3.3).
5. layered structures in Cahn-Hilliard-Navier-Stokes fluids. Here layering results from phase separation, as in spinodal decomposition [81, 82].

This list is finite. Other examples exist. We note that readers might associate the cellular structure studied here with Rayleigh–Bénard (RB) convection, a well-known buoyancy-driven system. However, it is important to distinguish the two. In RB convection, scalar transport (typically temperature) is enhanced by large-scale vertical rolls and plumes that rapidly mix the domain once instability sets in. In contrast, the model presented here involves externally forced recirculating cells that trap scalar quantities and suppress mixing between cells. Transport across cell boundaries occurs slowly via diffusion or weak turbulent scattering. This separation of timescales—between fast circulation within cells and slow cross-cell transport—is central to staircase formation and is absent in RB convection. Thus, while both systems involve cellular patterns, their roles in regulating transport are fundamentally different.

The central question, of course, is how do these amazing layered structures emerge spontaneously from mixing and turbulence? One universal element of staircase production is *inhomogeneous mixing*, which can lead to spatial variation in profile gradients – i.e., separated regions of flat spots and of steep gradients in the profile [37]. This should be contrasted to global gradient relaxation, the expected – and indeed, the usual – outcome of mixing processes. As to the mechanisms of inhomogeneous mixing, three prominent possibilities stand out. These are: *bistable mixing* [38, 55, 42, 80], *phase separation* [81, 82], and *homogenization* [70] and the concurrent *boundary gradient sharpening* in closely packed or tangential cells [48, 47].

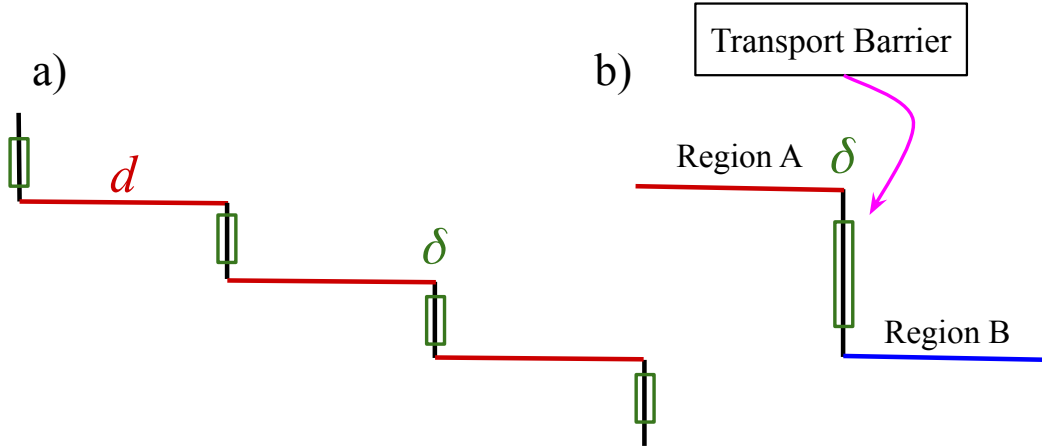


Figure 3.1. Illustration of (a) a staircase structure and (b) a transport barrier (i.e., creates a strong separation between two regions). Staircase profiles consist of alternating regions of strong mixing or homogenization ("steps") and sharp gradients ("jumps"). These sharp gradient layers serve as transport barriers.

In bistable mixing, the underlying transport mechanism involves two mixing lengths [41, 34, 43, 44], one set by the excitation mechanism and one which is *emergent*, and dependent upon gradient and turbulence intensity [39, 40]. A classic example is a stirred, stably stratified fluid. There $1/l_{mix} \simeq 1/l_e + 1/l_{oz}$ so $l_{mix} = l_e l_{oz} / (l_e + l_{oz})$. Here l_e is the externally prescribed stirring scale and l_{oz} is the emergent Ozmidov scale, defined by the cross-over of the eddy turnover rate and the buoyancy frequency. Here $l_{oz} \sim \tilde{v}/N$. Note $l_{oz} \sim \tilde{v}/(g\partial\rho/\partial z)^{1/2}$, which is both gradient and fluctuation intensity dependent. Also note that as density gradients sharpen, $l_{mix} \rightarrow l_{oz}$, the emergent nonlinear scale. Such processes typically exhibit nonlinear flux-gradient curves, where two stable branches co-exist. Transport bifurcations then occur between the stable branches, thus explaining the coexistence of two profile states (flat gradient, sharp gradient) in the staircase.

Phase separation can form layers and regions of different phases, as in a spinodal decomposition. Here jamming, triggered by the mechanism of motility induced phase separation

(MIPS) is of particular interest. In MIPS, speeds decrease in regions of high density, thus leading to jams, transport barriers and phase separation. Interestingly, the criterion for the onset of MIPS in 1D is identical to Lighthill and Whithams' criterion for a backward shock (i.e., jamming) in the kinematic model of traffic flow [106]. Jamming can also occur through the effects of response time delay in extended traffic flow, and more general transport models. This jamming model has been shown to lead to formation of 'jam trains' [55, 56, 57].

A third mechanism for inhomogeneous mixing is homogenization within quasi-tangential cells or phase space islands [46, 48, 47]. "Quasi-tangential" refers to either tangential or nearly tangential cells with intact bounding separatrixes (i.e., these don't overlap). The homogenization process sharpens gradients at the cell boundary, leading to the formation of transport barriers between cells. These problems support many scales – the boundary layer thickness ($\delta = d/\text{Re}^{1/3}$ for cells – d is radius, Re is Reynolds number), the cell size d and the spacing between cells Δ . $\Delta \geq d$ assures inhomogeneous mixing, $\delta \ll d$ assures maintenance of separatrix integrity. Note that this case also fits the mold for bistable mixing, as it supports two length scales – the prescribed one being the vortex size d and the emergent one being the edge vorticity gradient scale $d/\text{Re}^{1/3}$. This paper is concerned with the homogenization mechanism in tangential cells. Both passive scalar and active scalar layering are investigated. For the latter case, the active scalar is magnetic potential A , the system is an ensemble of vortex cells coupled to $\mathbf{B}$ -fields, as in 2D MHD, and the element of 'activity' is magnetic reinforcement of inter-cell transport barriers by the process of flux expulsion [97, 98, 95].

Layering and staircase formation are common phenomena. Thus, it is not surprising that many common questions arise. First, there is *resiliency* – i.e., are staircase structures delicate or are they robust to noise and external variations? Is there a quantitative figure-of-merit which measures resiliency? In a similar vein, are there measures of 'degree of inhomogeneity in mixing', which predict the formation of layers? And what are the characteristics of dynamics which tend to reinforce layering? To what extent do such processes make the staircase more robust?

In this paper, we explore questions concerning staircases and layering in the context of what must surely be the simplest possible models. These are first the fluctuating vortex array, derived from a melting vortex crystal model [60], along with an advected scalar concentration. The system is driven by a forcing which excites the vortex flow. We then introduce scalar activity and feedback by extending the first model via:

- replacing the arbitrary passive scalar concentration with the 2D magnetic potential A .
- incorporating the $\mathbf{J} \times \mathbf{B}$ feedback due to A evolution in the crystal vorticity equation. This renders the scalar active.
- threading the vortex array with an in-plane magnetic field which ‘fuels’ the active scalar evolution. This replaces the external injection of scalar concentration in the first.

In both cases, the vortex cells are (at least initially) quasi-tangential, so layering occurs naturally. Vortex cells tend to homogenize scalar concentration (either active or passive), while sharpening scalar gradients at the boundary. The latter is a means for transport barrier formation *between* mixing regions. Of course the combination of *homogenization in cells along with barrier formation between cells* gives *staircases* – the focus of this study. While earlier publications presented numerical results for passive and active scalar transport separately, this work seeks to develop a unified physical picture that encompasses both systems. We do so by constructing a model in which scalar dynamics are described by a coupled random walk process, capturing both rapid intra-cell mixing and slower, stochastic cross-cell transport. This unifying model enables a deeper understanding of how staircase structures emerge, persist, and ultimately degrade in the presence of fluctuations, and highlights the shared physics underlying both passive and active scalar layering.

The key physics issue in both models is the mechanism of transport barrier formation and maintenance in the inter-cellular layers. Barrier maintenance in the presence of vortex noise and finite amplitude fluctuations is particularly relevant to the question of resiliency. We develop

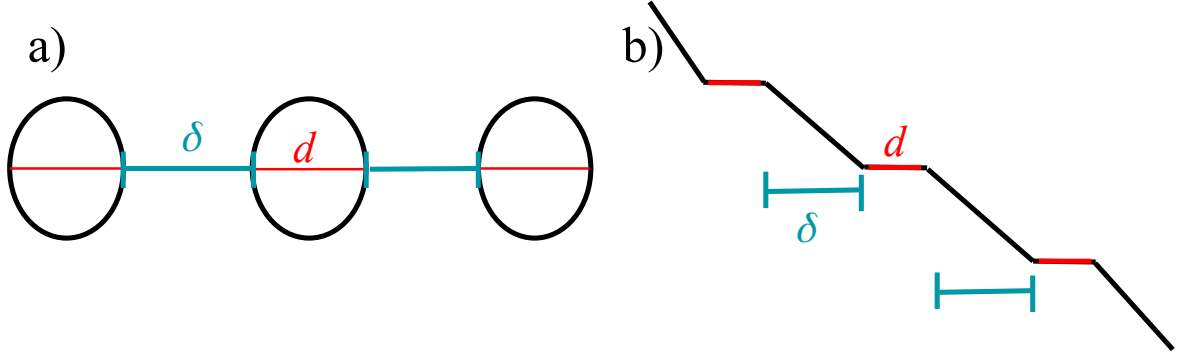


Figure 3.2. Illustration of (a) well-separated cells. Here, inter-cell transport is due only to ambient diffusion, and profile structure is (b) isolated flat spots due to localized homogenization.

a simple multi-component random walk model which predicts δ/d – the ratio of cell boundary layer thickness δ to vortex cell size d – as a function of fluctuating vortex array parameters. In the active scalar case – a fluctuating vortex array coupled to a mean magnetic field *which threads the array* – the mean field is stretched and expelled from the individual cells [95]. **B** tends to accumulate in the inter-cell boundary layers, where it can further quench the transport of magnetic potential between the cells. Here the active scalar transport tends to reinforce and strengthen the inter-cell boundary layers. In this case, the strengthened magnetic field tends to quench scalar transport by the well known mechanism of diffusivity suppression – related to conservation of mean square magnetic potential in 2D MHD [92, 93]. Of course, resistive diffusion ultimately triumphs in 2D MHD, so structure lifetime is limited, though the latter can be long for large magnetic Reynolds number R_m [90, 91]. To offset this, we also examine the layering dynamics of a stochastically forced magnetic potential.

The remainder of this paper is organized as follows: Sec. 3.1.1 presents a heuristic

approach to transport barriers and layer formation in cellular flow. Special focus is placed on the effect of cellular fluctuations on inter-cell boundary layers. Sec. 3.2 presents results for the fluctuating vortex array. Sec. 3.2.1 identifies Peclet number and staircase curvature as figures-of-merit for staircase resiliency and explores their variation with Reynolds number. Sec. 3.2.2 discusses the active scalar model, and studies staircase persistence on the presence of expelled magnetic fields. The behavior of the magnetic potential staircase in the limit of vortex bursting is examined, as well, Sec. 3.2.2 discusses the case of external forcing of magnetic potential. Sec. 3.3 discusses applications to magnetic confinement physics. Sec. 3.4 presents a discussion and conclusions.

3.1.1 Heuristics of transport barriers and layer formation in cellular flow

The spatial organization of cells is of prime importance to mixing, and thus structure formation, in flows. One possible structure is the staircase. Here we discuss the heuristics of staircase formation in flows assembled from an array of cells.

Given an array of similar cells, one can envision three generic possibilities. These are:

1. well-separated cells, an uninteresting limit of weak cell-to-cell interaction, as shown in Fig. 3.2(a). In this case, inter-cell transport is due only to ambient diffusion, and profile structure is isolated flat spots due to localized homogenization [Fig. 3.2(b)].
2. strongly overlapping cells – also an un-interesting limit where cells overlap, bounding separatrixes are destroyed, and mixing is homogeneous and *global*. This case is depicted in Fig. 3.3.
3. tangential cells – the interesting case – where cells are tangent or overlap only slightly (slightly $\equiv$ within boundary layer thickness), but cell separatrixes survive [Fig. 3.4(a)]. For tangential cells, two time scales enter the determination of transport namely: a *fast* timescale d/U_0 , for the circulation time in a cell, and a *slow* time scale characteristic of

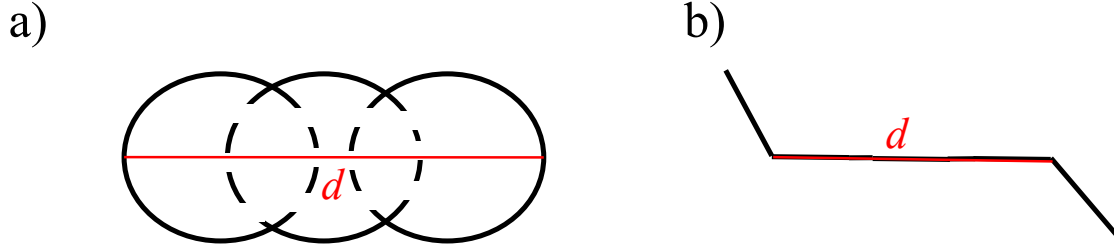


Figure 3.3. Illustration of (a) strongly overlapping cells. Here, cells overlap, bounding separatrices are destroyed, and mixing is (b) homogeneous and global.

(collisional) diffusion d^2/D . *Cell-to-cell transport is governed by slow diffusion through inter-cell boundary layers, which are effective transport barriers. A profile staircase thus forms, with steps – the cells – interleaved with jumps which are the boundary layers [Fig. 3.4(b)].*

Staircase formation in a cellular flow is thus seen as a consequence of a tangential cell configuration at high Peclet number $Pe \gg 1$,

$$Pe = \frac{\tau_D}{\tau_H} = \left(\frac{d^2}{D} \right) \frac{U_0}{d} = \frac{dU_0}{D}, \quad (3.1)$$

so that cell circulation time and cell diffusion are well separated. $Pe \gg 1$ ensures rapid mixing within cells but slow scattering and transport across inter-cell boundaries. This leads to

inhomogeneous mixing of a passive scalar density n , which evolves according to,

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)n = D\nabla^2 n, \quad (3.2)$$

so regions of homogenization (controlled by fast advection) within cells are bounded by transport barriers (controlled by slow diffusion) between cells. Note that the condition of time scale separation $\text{Pe} \gg 1$ is equivalent to requiring that the inter-cell boundary layer thickness (δ) be thin as compared to the cell size (d), i.e.,

$$R = \frac{\delta^2}{d^2} \ll 1. \quad (3.3)$$

For a fixed array of packed tangential vortices $R = 1/\text{Pe}$. This follows from $\delta \sim (Dd/U_0)^{1/2}$, where the timescale is set by the scalar particle transit time in the boundary layer, and the diffusion is collisional. Note that $\delta^2 \sim Dd/U_0 \sim DU_0d/U_0^2$ implies $\delta \sim (U_0dD)^{1/2}/U_0$ – i.e., the layer thickness is set by the geometric mean of the collisional and dynamic diffusivity.

The fixed packed array is an instructive limit but one of limited relevance. In a dynamic physical system, the cells themselves scatter, due to interactions, noise etc. Thus, the boundary layer thickness should reflect the effects both (collective) of cellular scattering as well as molecular diffusion. Here, we give a simple estimate of the boundary layer size which incorporates both effects. Taking the boundary layer thickness as determined by both cellular scattering and ambient scalar diffusion

$$\delta = \delta_{x_T} + \delta_{x_M}. \quad (3.4)$$

Here δ_{x_T} is the scattering excursion due to cell fluctuations (i.e., turbulence) and δ_{x_M} is the excursion due to molecular diffusion. Since these two stochastic processes are statistically

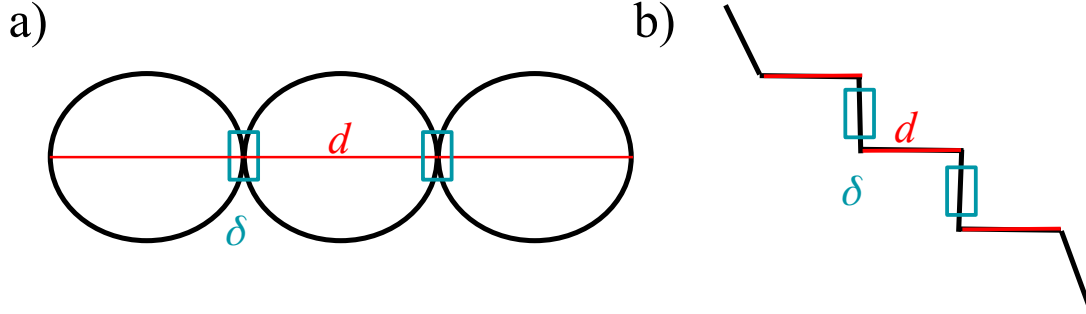


Figure 3.4. Illustration of (a) tangential cells. Here, cells are tangent or overlap only slightly, but (b) cell separatrixes survive.

independent, $\langle \delta x_T \delta x_m \rangle = 0$ and we have

$$\langle \delta^2 \rangle = \langle \delta x_T^2 \rangle + \langle \delta x_M^2 \rangle \quad (3.5)$$

$$= \int^t dt' \int^t dt'' \langle v_T(t') v_{-T}(t'') \rangle + \int^t dt' D. \quad (3.6)$$

v_T is the turbulent velocity. The turbulent velocity field has autocorrelation time τ_c yielding

$$\delta^2 = \int^t dt' \tau_c \langle \tilde{v}_T^2 \rangle + \int^t dt' D \quad (3.7)$$

$$= (D_T + D)t. \quad (3.8)$$

Now, taking $t = d/U_0$ ($U_0 \equiv$ rotation velocity in cell), the dwell time in the boundary layer, we

have

$$\frac{\langle \delta^2 \rangle}{d^2} = \frac{D_T}{dU_0} + \frac{1}{\text{Pe}}. \quad (3.9)$$

Eq. 3.9 gives the normalized boundary layer width in the presence of turbulence. Note for $D_T = 0$ (no turbulent scattering), we recover $\langle \delta^2 \rangle^{1/2} \sim \delta \sim d/\sqrt{\text{Pe}}$. Three regimes are apparent:

1. very weak turbulence $D_T/U_0d \ll 1/\text{Pe}$ with $\text{Pe} \gg 1$. Here $\delta \sim d/\sqrt{\text{Pe}}$, as for the fixed cellular array, and cellular turbulence is irrelevant. Barriers are maintained.
2. turbulent and molecular scattering are relevant for $D_T \gtrsim U_0d/\text{Pe}$ with $\text{Pe} \gg 1$. In this regime, we expect staircase structure to persist, but degrade for increasing D_T . Note that maintaining the staircase requires $D_T < U_0d$.
3. strong turbulent scattering $D_T/U_0d \gg 1/\text{Pe}$, up to $D_T/U_0d \sim 1$. In this limit, we expect the staircase to be destroyed.

Note that $D_T \geq U_0d$ is conceptually equivalent to the Lindemann criterion for vortex array melting, as both involve scattering lengths comparable to or exceeding a vortex crystal cell size.

Most of the cases studied in this paper fall into regimes 1) and 2) – where D_T is either negligible or modest (i.e., $D_T \sim U_0d/\text{Pe}$ with $\text{Pe} > 1$). Since the boundary layer is the ‘active’ region of the cell where irreversible scattering occurs, one can estimate transport coefficients by $D_{\text{eff}} \sim U_0\delta$. Thus, we see:

1. for a fixed cellular array $D_{\text{eff}} \sim U_0d/\sqrt{\text{Pe}} \sim \sqrt{U_0dD}$.
2. for the modest fluctuation regime $D_{\text{eff}} \sim (D_T/U_0d + 1/\text{Pe})^{1/2} U_0d$. In case 2, cell scattering enhances the prediction of D_{eff} relative to that for the fixed vortex array.
3. for the strong fluctuation regime: $D_{\text{eff}} \sim (D_T/U_0d)^{1/2} dU_0 \sim (D_T U_0d)^{1/2}$.

In the upcoming result section we explore in further detail the scattering of both passive and active scalar concentration in a fluctuating cellular array.

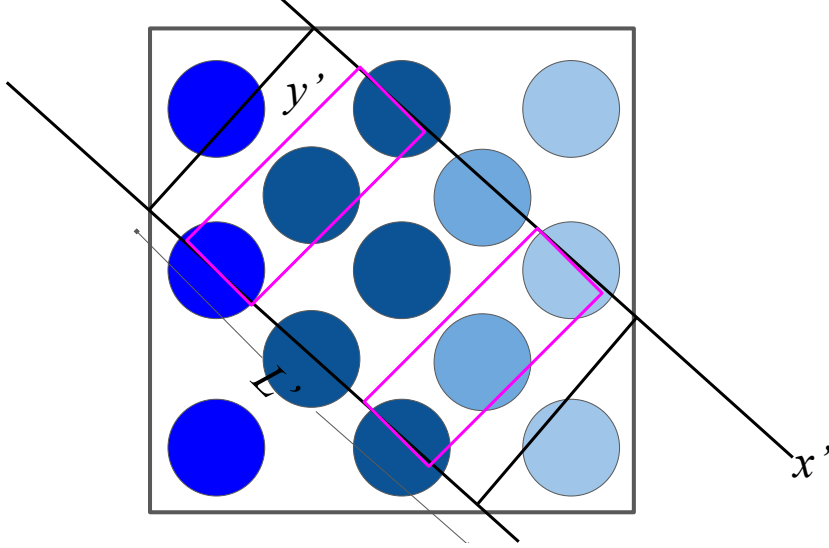


Figure 3.5. Illustration of the incline scalar concentration averaging.

3.2 Results (fluctuating vortex array)

3.2.1 Passive scalar (fixed flux model)

The fixed cellular array described is overly idealized and imposes unrealistic constraints, making it an inadequate representation of natural physical systems. In reality, most physical systems exhibit inherent variability, fluctuations, and intrinsic dynamics. As a result, cell patterns are more accurately represented by a fluctuating vortex array rather than a static structure. We focus on a driven variable cellular array flow, examining the transport of both passive and active scalars within this dynamic context.

To study transport in a cellular array with intrinsic fluctuations, we adopt the "Melting Vortex Crystal" model introduced in [99, 60]. Transport of the passive scalar (Eq. 3.2) is governed by the velocity field derived from the 2D vorticity equation, expressed in nondimensional form as:

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \omega = \frac{1}{\Omega} \nabla^2 \omega + F_\omega - \alpha \omega, \quad (3.10)$$

$$\nabla^2 \psi = \omega. \quad (3.11)$$

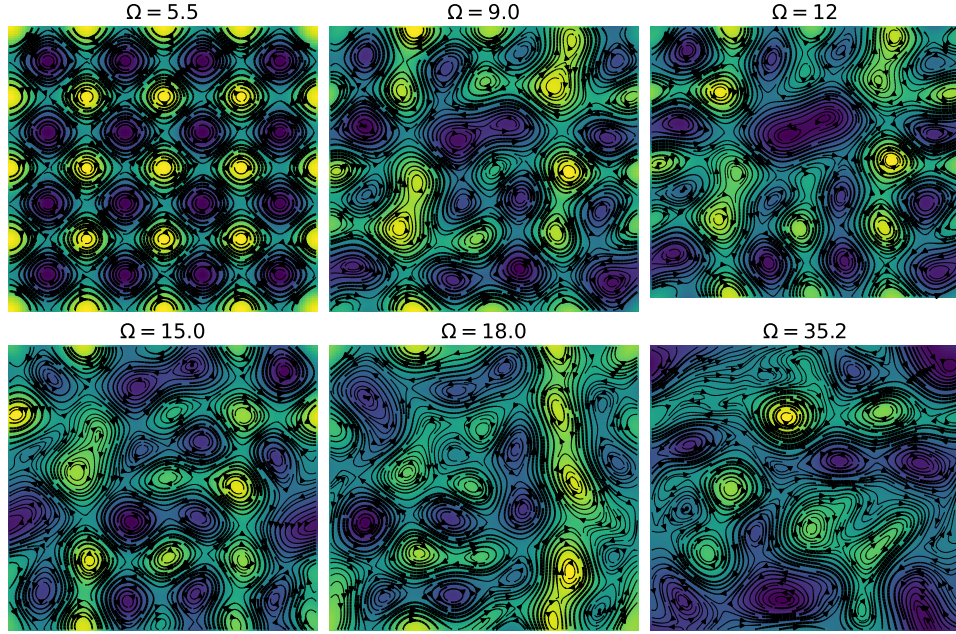


Figure 3.6. Contour plots, illustrating different stages of the fluctuating vortex array. As Ω increases, there is a merger of vortices along with distortions of the cellular flow array. Growth in Ω leads to scattered vortices and large scale flow structures.

The vorticity equation uses the streamfunction ψ and vorticity $\omega = \nabla \times \mathbf{u}$, with velocity defined as $\mathbf{u} = \hat{\mathbf{z}} \times \nabla \psi$. The spatially periodic force F_ω in the vorticity equation is given by

$$F_\omega = -n^3 (\cos nx + \cos ny) / \Omega. \quad (3.12)$$

Here, n represents the injection wave vector and is set to $n = 4$ for all simulations in Secs. 3.2.1 and 3.2.2. Two control parameters govern Eq. 3.10: the nondimensional Ekman friction coefficient (α) and Ω . The parameter Ω is a modified Reynolds number, defined as the ratio of the viscous timescale $\tau_v = n^2/k^2\nu$ to the cell circulation timescale $\tau_H = n/kU_0$, with assigned values $k = 10$ and $U_0 = 1$. Ω controls the strength of the forcing and determines small-scale flow structure, while α damps large-scale low-frequency modes. For Secs. 3.2.1 and 3.2.2, the periodic forcing term F_ω generates a stationary cellular array for $\Omega < 5.5$. As Ω increases, the structure of the cellular array evolves. An illustration of how Ω affects the array is shown in Fig. 3.6.

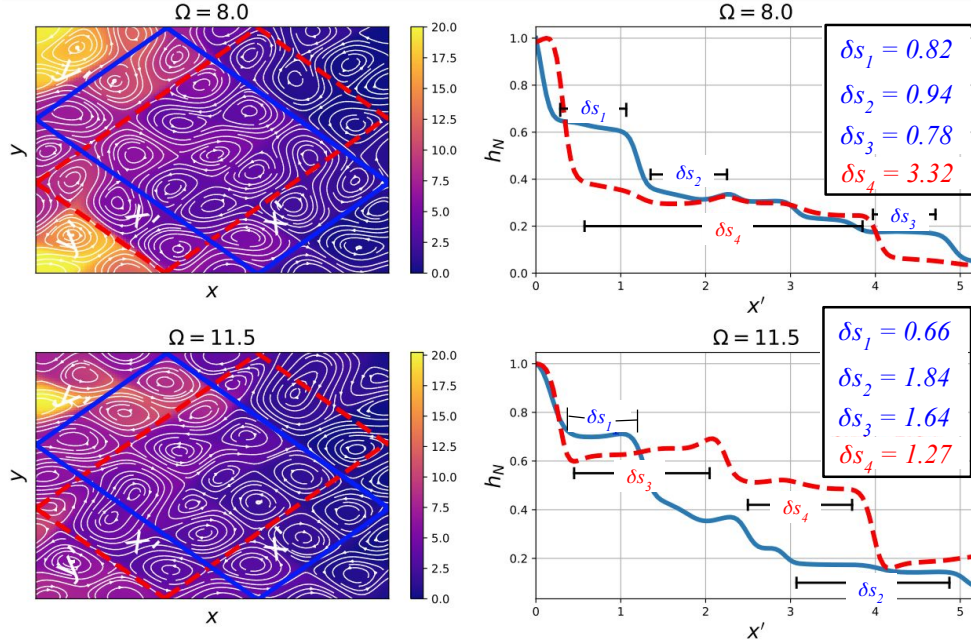


Figure 3.7. Snapshot of passive-scalar concentration. On the left figures, the streamlines display mergers/connections of vortices. These mergers are reflected on the normalized $h(x')$ profiles on the right, by an elongated staircase step on the solid blue and dashed red line. As the cells fluctuate from the zero temperature state, the staircase structure remains robust. We indicate steps as regions where $dh'/dx' \sim 0$. These are clearly separated by distinct jumps (i.e., regions of corrugations). [Reproduced with permission from Phys. Rev. E 109, 025209 (2024). Copyright 2024 American Physical Society.]

To examine how Ω influences passive scalar transport and staircase formation, we first characterize the staircase profile. This is defined as the average scalar concentration:

$$h(x') = \langle n(x') \rangle = \frac{1}{L'} \int_0^{L'} n(x', y') dy', \quad (3.13)$$

where x' and y' are the x - and y -axes in two orientations ($\theta = \pm\pi/4$) relative to x . The length L' is $0.82L$ (see Fig. 3.5). For a zero-temperature (i.e., fluctuation-free) cellular array, the average scalar concentration profile forms a staircase profile identical to that for the fixed cellular array [46]. This occurs because the vortices remain locked in the cellular array pattern.

We repeat the process for different stages of the evolution of the fluctuating vortex array. As cells scatter from the zero-temperature state, they begin to merge, forming larger structures.

In Fig. 3.7, we present two cases: $\Omega = 8$ and $\Omega = 11.5$. The scalar concentration contour plots show that cell streamlines connect across several structures. These connections correspond to mergers in staircase steps when observed in the average scalar concentration profile. So vortex mergers lead to elongated steps. In the plots of Fig. 3.7 and 3.8, we highlight the increase in step size (δs). The comparison between the frozen cellular array staircase and $\Omega = 11.5$ manifests a clear increase in δs .

We also calculate the curvature of the scalar concentration profile. As observed in Figs. 3.6 and 3.7, the number of steps decreases as the vortex array evolves due to mergers. To quantify the effect of step mergers, we compute the curvature of the scalar concentration profile:

$$\kappa = \int_0^{L'} \frac{h''(x')}{[1 + h'(x')^2]^{3/2}} dx', \quad (3.14)$$

where $h(x')$ is defined in Eq. 3.13. Figure 3.9 shows that increasing Ω leads to a decrease in κ . However, κ does not vanish, remaining above ~ 1.5 for large values of Ω . The curvature κ quantifies the step scale and texture of the staircase: large values correspond to short, sharp steps, while small values reflect broader, merged structures. As highlighted in Fig. 3.9, κ serves as a useful macroscopic metric for tracking staircase structure across Ω , with its decline signaling the coarsening of layers due to vortex mergers.

Interestingly, both κ and the Peclet number decrease with increasing Ω , as shown in Fig. 3.9. While Pe provides a microscopic measure of advection–diffusion balance, κ captures the macroscopic organization of the staircase profile. The consistent trend in their behavior suggests a connection between the two quantities: as the flow becomes more turbulent and Pe decreases, the staircase structure coarsens due to mergers, reflected by the decrease in κ . This relationship highlights the interplay between micro- and macroscopic mechanisms governing staircase formation and persistence.

To this point, we have shown that staircase structure persist at modest Reynolds numbers for the passive scalar case. However, to define “resiliency” precisely, we must establish a

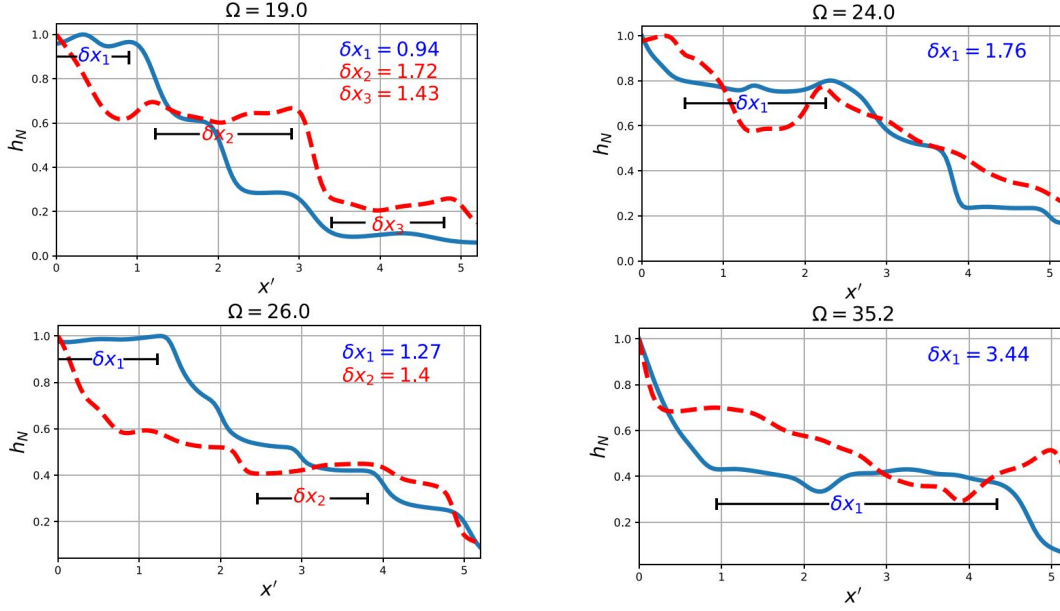


Figure 3.8. Additional passive scalar staircase profiles for various values of Ω , demonstrating that staircase structure persists for $\Omega < 40$.

set of more broadly applicable criteria [83]. This necessity arises because resiliency is not a universally understood or quantifiable property. Without clear and measurable criteria, it becomes challenging to evaluate the robustness of staircase structures across different physical systems or flow conditions. Resiliency must account for the ability of the structure to withstand perturbations, maintain its characteristic structure, and persist under varying levels of turbulence. Establishing criteria also provides a framework for predicting when and how staircase structures may fail or collapse. By identifying key parameters and thresholds, we ensure that resiliency can be rigorously tested and meaningfully applied to a wide range of scenarios.

To establish the first criterion, we refer to the discussion in Sec. 3.1.1, where the formation of a staircase structure within a cellular array is linked to maintaining two distinct timescales: rapid cellular mixing (τ_H) and slow scattering across cell boundaries (τ_D). Specifically, the condition $\tau_D \gg \tau_H$ is essential. To evaluate this, we measure the Peclet number (Pe) across different stages of the fluctuating vortex array. Since the maximum flow velocity and background diffusivity ($D = 1.7 \times 10^{-2}$) are fixed, Pe depends solely on the dimensions of the cells. Using

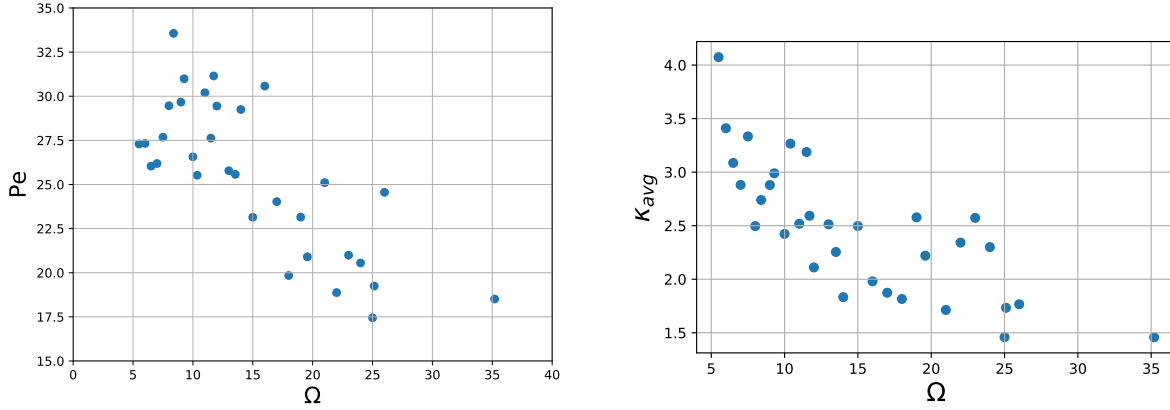


Figure 3.9. Peclet much greater than one, is necessary for transport barriers in the process of scalar mixing. In addition, a staircase should maintain a sufficiently high curvature (equivalent to sustaining a sufficient number of steps). κ correlates with the global layering structure and quantifies the staircase step length and number of steps within the profile. Note that large values of κ and Pe correspond to $\Omega = 5.5$ and lowest values to $\Omega = 35.2$. [Reproduced with permission from Phys. Rev. E 109, 025209 (2024). Copyright 2024 American Physical Society.]

the streamfunction (ψ) in Eq. 3.10, we estimate the average cell size and compute Pe . In Fig. 3.9, we observe that for $\Omega > 25$, Pe decreases to approximately 17.5. Nonetheless, the $Pe \gg 1$ condition holds for the range $0 < \Omega < 40$. Thus, we conclude that $Pe > 1$ is a necessary condition for the formation and maintenance of transport barriers in scalar mixing. Furthermore, maintaining $Pe > 1$ across a relevant range of Reynolds numbers provides a robust criterion for defining staircase "resiliency." This is reasonable, as $Pe \gg 1$ maintains the disparity between circulation and diffusion time, which in turn preserves the integrity of cells and barriers.

A second criterion for staircase resiliency is the maintenance of sufficiently high curvature (κ), which corresponds to sustaining a requisite number of steps over the relevant range of Ω values. While the precise threshold for κ is not definitively determined, our studies on the passive scalar case suggest that $\kappa \geq 1.5$ is sufficient to characterize a staircase. As shown in Fig. 3.9, κ does not vanish even at the highest Reynolds numbers studied, though it decreases as the staircase coarsens due to cell mergers. This coarsening is driven by increased Ω , which perturbs cell positions, leading to overlap and eventual merging. It is worth noting that κ may depend on the dimensions of the simulation domain. For instance, in this study, the box length is 2π ,

and a larger domain could accommodate more steps, potentially resulting in a higher κ . Thus, κ appears to be an extensive quantity, influenced by the size of the system.

Therefore, staircase resilience across a range of Ω (i.e., Reynolds number) values requires the following criteria to be met:

1. $Pe > 1$, so cell barriers (i.e., separatrixes) are formed and maintained.
2. $\kappa \geq 1.5$, so a requisite number of steps persist.

Criterion (1) is grounded in first principles (as discussed in the introduction), while criterion (2) is empirical, derived from trends observed in our numerical simulations. Specifically, the threshold $\kappa \geq 1.5$ reflects the minimum curvature required for staircases to maintain distinct layers and resist step merging.

Beyond the primary resiliency criteria, we also investigated whether a direct relationship exists between the flux intensity of scalar concentration and the staircase profile curvature κ . Specifically, we examined whether increased flux across the system correlates with changes in curvature, reflecting broader or narrower staircase steps. However, our numerical analysis did not reveal any clear or systematic dependence between the scalar flux and the curvature (see Fig. 3.10). Although this was a negative result, it is worth reporting here for completeness. It suggests that in this fluctuating cellular flow system, curvature is predominantly determined by the dynamics of cell scattering and mergers rather than by the instantaneous intensity of scalar transport. These findings, which were not presented in previous studies, provide further clarification on the dominant factors shaping staircase structures in passive scalar transport.

In addition, to track the degradation of the staircase structure, we recall the Lindemann criterion [107], which measures cell displacement relative to cell size:

$$L = \frac{\delta x_{cell}}{d}. \quad (3.15)$$

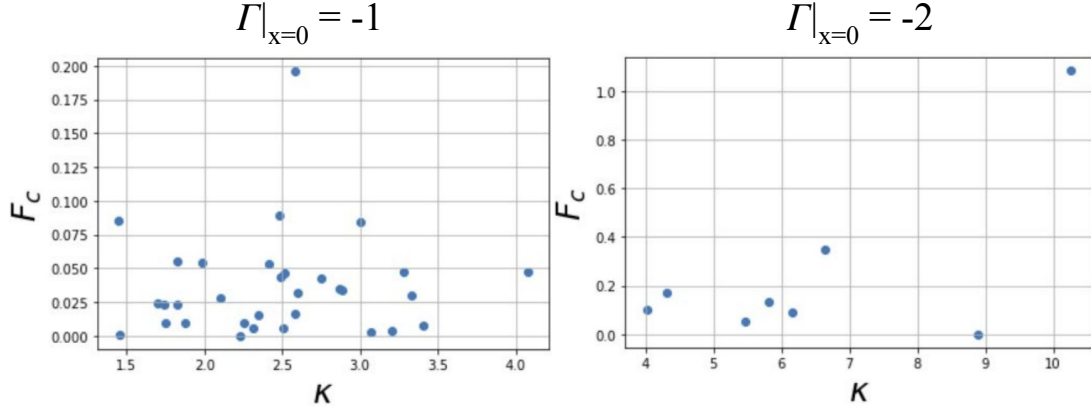


Figure 3.10. Plot of staircase profile curvature κ versus scalar flux F_c measured across two internal cross-sections ($x = \pi/4$ and $x = 7\pi/4$) within the simulation domain. The figure shows results for two different imposed boundary flux conditions at $x = 0$: the original case with $\Gamma = -1$ (left), and a stronger flux case with $\Gamma = -2$ (right). Despite the increased boundary flux, no discernible trend is observed between F_c and κ . The right panel shows a similar comparison, but with the background diffusivity reduced in the $\Gamma = -2$ case. Lower diffusivity increases the Peclet number, resulting in a more rigid staircase structure (i.e., larger κ). However, even in this case, no clear functional relationship emerges between flux intensity and curvature, aside from a general decrease in F_c with decreasing κ . Further investigation is needed to assess the robustness and interpretation of these observations.

As above, we can estimate L for timescale d/U_0 by

$$L^2 \sim \frac{D_T}{dU_0}, \quad (3.16)$$

where $L \sim 1$ corresponds to the strong scattering regime. Staircase structure collapses when $L \rightarrow 1$, indicating significant disruption to the cellular array. In our study, the staircase remains intact for $\Omega < 40$, meaning L has not yet reached unity. However, we predict that as the flow becomes more turbulent (i.e., as Ω increases), the structure will eventually collapse due to vortex scattering. Further investigation is needed.

As a partial summary, we identify that staircase structures form and persist for a broad

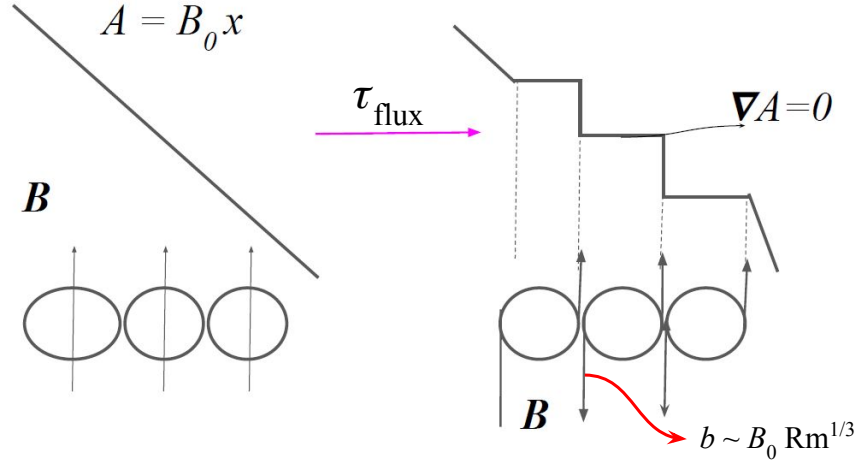


Figure 3.11. Consider an initial magnetic potential profile which is linear. A vortex array will homogenize A within cells, thus expelling magnetic fields to the inter-cell boundaries, and so also producing a staircase in A .

range of Reynolds numbers and are resilient to fluctuations present in a cellular array. We also establish a set of criteria in order to constrain the meaning of “resiliency”. These two criteria are $Pe > 1$ and $\kappa \geq 1.5$. We also propose profile curvature as a macroscopic measure of staircase formation, which complements the microscopic criteria $Pe > 1$.

3.2.2 Active scalar (periodic model)

No external forcing

The next step is to now consider an active scalar field (A), which can feedback on layering dynamics. In this context, the dynamics of the active scalar are those of the magnetic potential in 2D magnetohydrodynamics (MHD), which can be described by the active scalar transport equation:

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) A = \frac{1}{R_m} \nabla^2 A + F_A, \quad (3.17)$$

$$\mathbf{B} = \hat{\mathbf{z}} \times \nabla A, \quad (3.18)$$

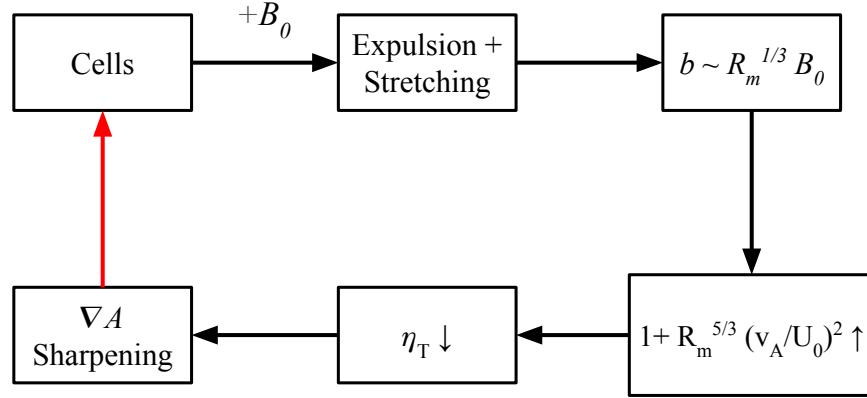


Figure 3.12. In the flux expulsion regime, a feedback loop exists between cells and the magnetic potential. As cells expel magnetic field lines, the magnetic field increases at the boundaries. This increase reduces turbulent transport across cell boundaries, sharpens the gradient of A , and strengthens the cell array structure. This process is known as transport reduction feedback.

where R_m represents the magnetic Reynolds number, defined as the ratio of the magnetic diffusivity timescale $\tau_\eta = n^2/k^2\eta$ to the cell circulation timescale (τ_H). We focus on staircase formation in the magnetic potential (A). Here, F_A denotes an external forcing term applied to the magnetic potential. Later in Sec. 3.2.2, we use F_A to drive magnetic potential in the system and examine its impact on the lifetime and structure of the magnetic potential staircase. Conversely, for Sec. 3.2.2, we set $F_A = 0$ and focus on the evolution of the system with a fixed initial value of A . This approach allows us to investigate how the lifetime of the magnetic staircase structure is affected in the absence of external forcing of A .

Similar to the passive scalar case, the magnetic potential is governed by the velocity from the 2D vorticity driven equation, though with an additional term

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \omega = \frac{1}{\Omega} \nabla^2 \omega + M^2 (B \cdot \nabla \nabla^2 A) + F_\omega. \quad (3.19)$$

Here, A is coupled to flow dynamics via the $\mathbf{J} \times \mathbf{B}$ force. Thus, the equations are those for the forced 2D MHD problem [87, 88]. In such problems, MHD systems differ fundamentally from

non-MHD systems, where scalar concentrations are advected without significantly influencing the flow. In 2D MHD, feedback between the magnetic potential and flow introduces dynamics. For instance, the magnetic potential can alter flow properties, leading to phenomena like vortex disruption or the suppression of transport (see Figs. 3.11 and 3.12). These feedback effects, absent in purely passive scalar systems, are central to understanding how magnetic staircases form and persist. We will explore these scenarios in greater detail after discussing the relevant control parameters and the regimes of interest.

The Alfvén mach number (M) controls the coupling strength between the active scalar and the flow, defined as

$$M = \frac{v_A}{U_0}, \quad (3.20)$$

$$v_A = \frac{B_0}{\sqrt{\mu_0 \rho}}, \quad (3.21)$$

where v_A denotes the Alfvén speed, derived from the magnetic field strength B_0 , the magnetic permeability μ_0 , and the density ρ . Here, μ_0 and ρ are standardized to one. Notably, B_0 is the variable parameter in M , which influences the active scalar’s effect on flow dynamics.

In 2D, Zeldovich’s theorem states that, without external magnetic forcing, the magnetic field decays slowly (no dynamo effect) due to resistive diffusion of $\mathbf{B}$ [90, 91]. This decay occurs in two stages: a rapid decay phase and a quasi-stationary “suppression” stage characterized by slow decay of A and the magnetic field. During this suppression stage, the background magnetic field (B_0) inhibits turbulent transport [89]. Indeed, numerical studies and mean-field electrodynamic theory show that during the suppression stage even a *weak* large-scale magnetic field can reduce turbulent transport of the active scalar [94, 89]. In 2D MHD, turbulent transport

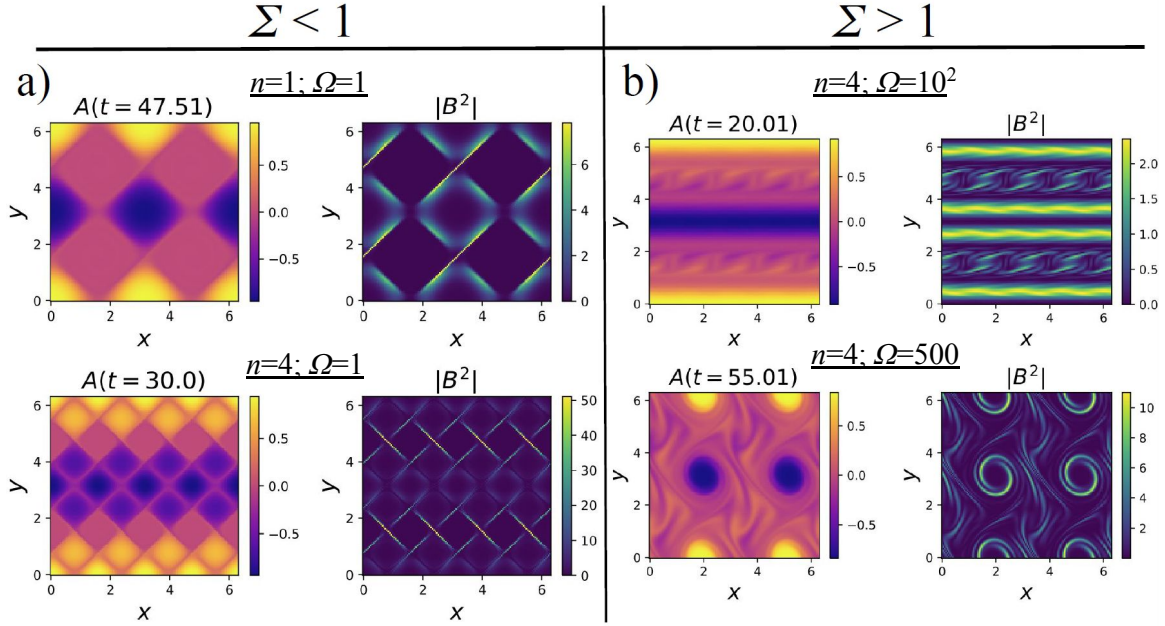


Figure 3.13. In the flux expulsion regime ($\Sigma < 1$), the system is relatively steady, so we show two cases with different injection wave numbers: $n = 1$ (top-left) and $n = 4$ (bottom-left). In this regime, magnetic field lines are expelled to the cell boundaries, sharpening the gradient of A and reinforcing the staircase structure. In contrast, the vortex disruption regime ($\Sigma > 1$) exhibits time-dependent behavior. We present two different time snapshots for $n = 4$ simulations with $\Sigma = 10$: the top-right frame corresponds to $\Omega = 10^2$, and the bottom-right to $\Omega = 500$. Not all closed streamlines are disrupted, which is crucial to layering. Note that if the vortex array is not frozen, the layered structure will eventually collapse, due to the decay of the resistive field. [Reproduced with permission from Phys. Rev. E 111, 055107. Copyright 2025 American Physical Society.]

across cell boundaries is quantified using turbulent resistivity (η_T):

$$\eta_T \approx \frac{\eta_k}{1 + R_m \langle \mathbf{B} \rangle^2 / \langle u^2 \rangle}, \quad (3.22)$$

$$\eta_k = \sqrt{\langle u^2 \rangle} L. \quad (3.23)$$

Here, $\langle \mathbf{B} \rangle$ and $\langle u \rangle$ represent the mean magnetic field and flow velocity, respectively. This result stems from the influence of magnetic fields on turbulent flows, affecting how they scatter A . Note that R_m is big even in the limit of $v_A/U_0 \ll 1$, Eq. 3.22 predicts significant quenching of η_T . Using η_T , we can compare $\tau_{\eta_T} = d^2/\eta_T$ to τ_H to assess staircase rigidity.

Another way to quantify rigidity requires a theory of transport across the boundary layer between cells in 2D MHD. Before establishing a scaling, we describe the interaction between fluid motion and the background magnetic field in this model of active scalar dynamics. Here, fluid motion stretches a weak background field to the vortex edges until the field lines reconnect. This phenomenon of 2D MHD is known as flux expulsion. The strongest magnetic field accumulates at the vortex's edge. If the magnetic field at the boundary (b) is sufficiently strong, the induced stresses from the curved field lines can cause cell disruption [97, 98, 96]. The two MHD (expulsion and disruption) scenarios occur for $\Sigma < 1$ (flux expulsion) and for $\Sigma \geq 1$ (cell disruption), respectively, where Σ is

$$\Sigma = M^2 R_m. \quad (3.24)$$

$\Sigma = 1$ is derived from the relative strength of the fluid convective term $(\mathbf{v} \cdot \nabla)\omega$ to the magnetic tension term $\mathbf{B} \cdot \nabla \mathbf{J}$.

Returning to the issue of “rigidity”, we examine how the magnetic field affects inter-cell transport of A within a cellular array. Specifically, we consider how the magnetic field influences the mean square scattering length ($\langle \delta_A^2 \rangle$) (i.e., for transport of A) when B_0 is threaded through a vortex array. The presence of the magnetic field modifies the scaling of the scattering length as follows:

$$\frac{d}{dt} \langle \delta_A^2 \rangle \approx -\frac{\langle \tilde{v}_r \tilde{A} \rangle}{\nabla \langle A \rangle} + \eta \quad (3.25)$$

$$= \frac{\langle \tilde{v}_r^2 \rangle \tau_c}{1 + R_m v_A^2 / \langle \tilde{v}^2 \rangle} + \eta \quad (3.26)$$

$$= \frac{D_T}{1 + R_m v_A^2 / \langle \tilde{v}^2 \rangle} + \eta, \quad (3.27)$$

where η is the magnetic resistivity. The first term accounts for the turbulent diffusion of A , which is suppressed by the presence of the magnetic field. The second term represents resistive

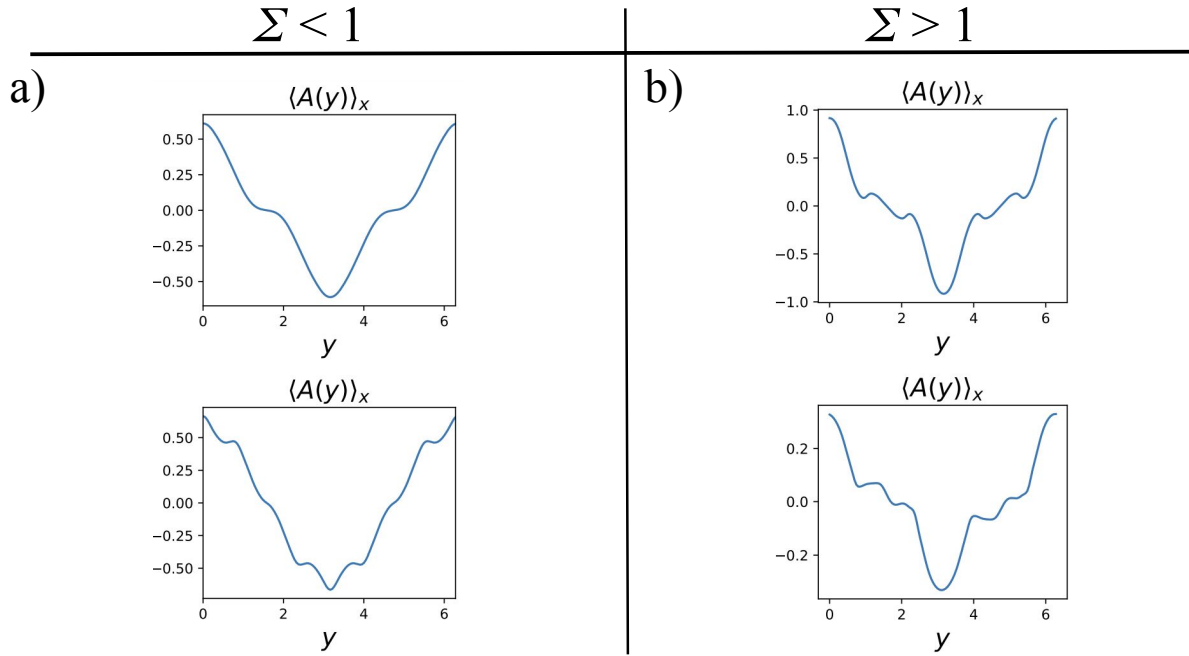


Figure 3.14. Magnetic potential staircase profile for (a) flux expulsion and (b) vortex disruption regimes. These profiles correspond to those in Fig. 3.13. Layering persists in both regimes. In the vortex disruption regime, residual cells maintain the staircase structure despite significant disruption to the cellular array. [Reproduced with permission from Phys. Rev. E 111, 055107. Copyright 2025 American Physical Society.]

diffusion, which becomes significant for weak turbulent diffusion. Integrating with respect to time (using the transit time through the layer as the timescale) and normalizing by the cell size d , as before, we obtain:

$$\frac{\langle \delta_A^2 \rangle}{d^2} = \frac{D_T/U_0 d}{1 + R_m v_A^2 / \langle \tilde{v}^2 \rangle} + \frac{\eta}{U_0 d}. \quad (3.28)$$

We begin with the flux expulsion regime where B_0 is sufficiently low (specifically, where $v_A^2 < U_0^2/R_m$). Here, the magnetic field (b) concentrates at the cell boundary, so that $b \sim B_0 R_m^{1/3}$ (essentially $\sim \nabla A$ increases at cell boundary) as A is expelled from the cell. In this case, we can

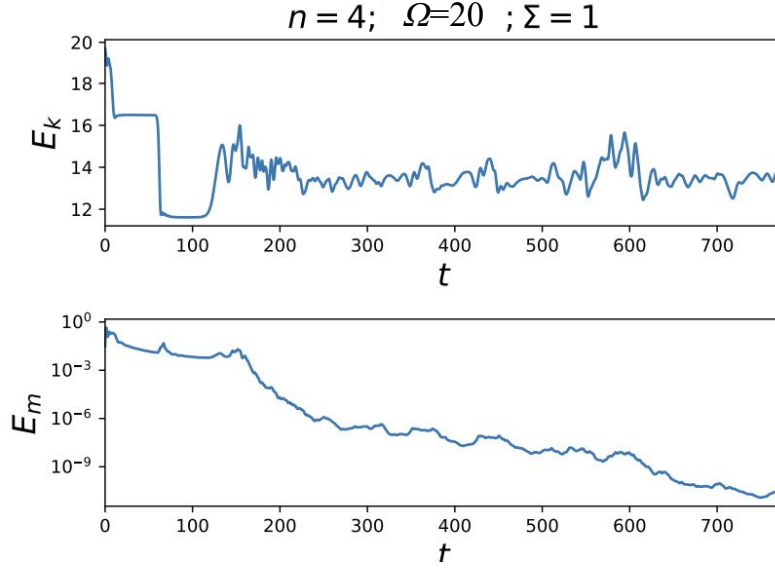


Figure 3.15. Energy time trace showcasing the suppression stage, governing fluctuations in the flow's kinetic energy. As the magnetic energy dissipates sufficiently, the kinetic energy evolution transitions from an initially stable pattern to a chaotic and random one, with the magnetic field's influence becoming negligible.

approximate the ratio of scattering length to cell size scales as:

$$\left(\frac{\delta_A}{d}\right)^2 \approx \frac{D_T/U_0 d}{1 + R_m^{5/3} v_A^2 / \langle \tilde{v}^2 \rangle} + \frac{\eta}{DPe}. \quad (3.29)$$

If $D_T \neq 0$, the denominator of the first term on the left-hand-side is usually far in excess of unity, so turbulent broadening of the layer is weaker for the magnetic/active case as compared to the passive scalar case (Eq. 3.2). The boundary layer is stronger since $\mathbf{B}$ reduces transport through it, maintaining layering. In other words, $\mathbf{B}$ reinforces staircase structures. In the vortex disruption regime $v_A^2 \geq U_0^2/R_m$, the first term in Eq. 3.29 becomes

$$\left(\frac{\delta_A}{d}\right)^2 \xrightarrow{v_A \gg \langle \tilde{v} \rangle} \frac{D_T/U_0 d}{R_m^{5/3} v_A^2 / \langle \tilde{v}^2 \rangle}. \quad (3.30)$$

Here, $\mathbf{B}$ again reduces the scattering length at the boundary layer, quenching inter-cell turbulent transport. The magnetic field effect on η_T reduces the rate of flux transport through the boundary

layer, thus increasing the scalar's dwell time in the boundary in comparison to the flux expulsion case. The main point here is that the active scalar *reinforces* the staircase structure in *both* the flux expulsion and vortex disruption regimes. This result for the cell disruption regime is suprising, because one might expect that transport can no longer be limited once cell boundaries are no longer intact. Cell boundaries must be maintained for staircase structures to form. Previous studies on single-cell systems show that vortices do not survive in $\Sigma \geq 1$ scenarios. However, in a multiple cell system, residual cells may remain, implying that layering still can occur. Thus, the active scalar field can produce a more robust staircase structure (in both MHD regimes) than its passive scalar counterpart, regardless of the value of Σ .

For the active scalar model, we conduct direct numerical simulations (DNS) of the NS and active-scalar equations (Eqs. 3.17 and 3.19), utilizing the Dedalus framework [100]. Our simulations adopt a pseudo-spectral approach with a 3/2 de-aliasing rule and a fourth-order Runge-Kutta method for time stepping ¹. Note that numerical methods are consistent with those outlined in prior research [99, 60, 61]. The computational domain is a doubly periodic box with length L and a spatial resolution of $N = 128$. The initial conditions for the vorticity and active scalar are given by

$$\psi(t = 0) = \frac{U_0}{n} (\cos nx + \cos ny), \quad (3.31)$$

$$A(t = 0) = A_0 \cos x, \quad (3.32)$$

with A_0 set to one.

Similar to Sec. 3.2.1, the dynamics are driven by the force F_ω , exploring the stages of the fluctuating cellular array by increasing the value of Ω . To compare with previous passive scalar studies, our study spans effective Reynolds numbers from 1 to 500, with a focus on examination of spatiotemporal chaos within the cellular lattice. Again, this model facilitates the exploration of

¹Time in the simulation is advanced as a physical variable. For $U_0 = 1$ and $L = \pi/3$, one cell turnover time is $\tau = \pi/3$, so $t = 50$ corresponds to roughly 52 turnovers. The timestep Δt is adaptively controlled by the Dedalus framework for stability.

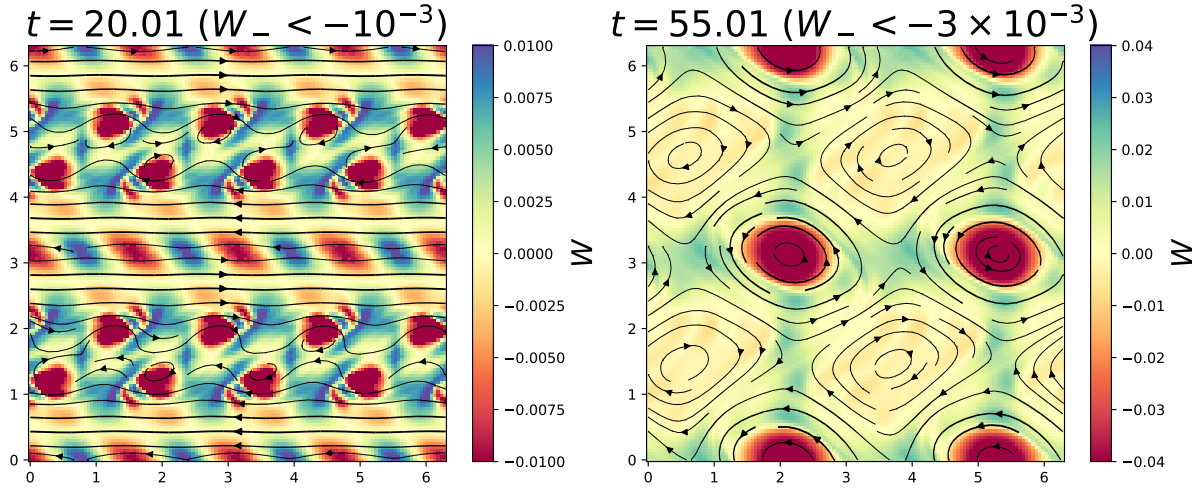


Figure 3.16. Contours of the Okubo-Weiss field for $\Omega = 500(n = 4, \Sigma = 10)$ at various times demonstrate that even within the regime of vortex disruption, the array with $n = 4$ still maintains closed streamlines. This observation illuminates the persistence of scalar layered structure depicted on the right-side of Fig. 3.13. [Reproduced with permission from Phys. Rev. E 111, 055107. Copyright 2025 American Physical Society.]

a broad range of excitations typical of the modest levels of turbulence, where effective Reynolds numbers range from 10 to 100.

In order to understand the behavior of the active scalar feedback on the fluctuating cellular flow for different Reynolds numbers, we present Figs. 3.13 and 3.14, which show contours and profiles of the magnetic potential and magnetic field in both the flux expulsion regime ($\Sigma \leq 1$) and vortex disruption regime ($\Sigma > 1$). In the first case, the magnetic back reaction on the flow is weak. In Fig. 3.13, for the case of $\Omega = 1$, we demonstrate the increase of the concentration of magnetic field at the cell boundaries and the homogenization of the active scalar within the vortices during the suppression stage, for $n = 2$ and $n = 4$. To confirm the presence of a layered structure, we analyze the scalar concentration averaged in x , as a function of y ,

$$\langle A(y) \rangle_x = \frac{1}{L} \int_0^L A(x, y) dx. \quad (3.33)$$

Our analysis of Fig. 3.14 confirms that magnetic layering is a distinct feature of the flux expulsion and frozen cellular array regime for both $n = 2$ and $n = 4$.

To investigate the longevity and stability of the vortex array and staircase structure in the flux expulsion regime, we study scenarios where $\Omega \geq 1$. Initially, flux expulsion sustains the array structure. However, as the magnetic field weakens, the field wrapped around cell boundaries disintegrates, and the system loses its cellular structure. In Fig. 3.15, we demonstrate how magnetic fields constrain kinetic energy fluctuations up to $t = 200$ for the case of $\Omega = 20$ in an $n = 4$ array. Once the magnetic field sufficiently dissipates, the flow transitions to a turbulent state with no apparent structure. The decay of the magnetic potential is consistent with Zeldovich's theorem. As a result, for Ω values where the array isn't frozen (i.e., fixed), the magnetic field maintains the cellular array structure only for a limited time.

Now, we shift our attention to the vortex disruption regime ($\Sigma > 1$). Here, magnetic back reaction on the flow is significant. To characterize and quantify the effects of Ω and M on the vortex array, we measure vortex array disruption by utilizing the Okubo-Weiss field (W) [62, 63], defined as

$$W = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 - \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)^2 \quad (3.34)$$

$$= W_+ - W_- . \quad (3.35)$$

The terms W_+ and W_- represent the normal and shear components of the rate of strain, and the vorticity, respectively. The Okubo-Weiss field measures the relative strength of shear strain to vorticity, characterizing the local fluid flow topology. Specifically, regions of $W < 0$ indicate vortex-dominated zones, where rotation dominates over strain, while $W > 0$ corresponds to regions of strain-dominated deformation.

The relevance of the Okubo-Weiss field to this study lies in its ability to identify changes in the vortex structure and quantify the extent of vortex disruption. In the vortex disruption regime, the magnetic back reaction introduces additional forces that distort the vortex array. This feedback alters the local flow topology, with regions of high strain disrupting the coherence of vortices. By analyzing W , we can evaluate how Ω (the Reynolds number) and M (the magnetic

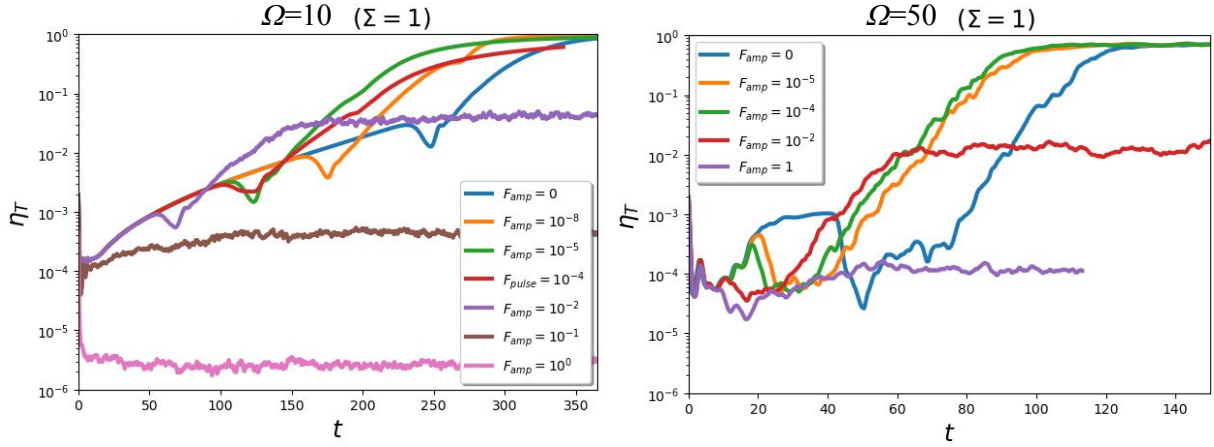


Figure 3.17. When forcing is present, turbulent quenching can be sustained or even improved indefinitely, as long as the forcing is sustained. [Reproduced with permission from Phys. Rev. E 111, 055107. Copyright 2025 American Physical Society.]

field strength) influence the spatial distribution of strain and vorticity, shedding light on the mechanisms that drive vortex mergers and the breakdown of the staircase structure. In this context, the Okubo-Weiss field serves as a bridge between local flow topology and macroscopic transport properties. It provides a means to track the transition from a rotationally dominated flow, which supports staircase formation, to a strain-dominated flow, which undermines layering. The interplay between W , Ω , and M thus encapsulates the critical dynamics that govern the resilience of the staircase structure in the presence of active scalar feedback and increasing flow complexity.

To understand the persistence of layering in regimes where $\Sigma > 1$, we examine the Okubo-Weiss field (constructed using streamlines) in Fig. 3.16 at the same time as shown in Fig. 3.13. Here, staircase structures are similar, but flow structure differs. The Okubo-Weiss parameter helps identify vortical regions, and the streamlines indicate whether cell streamlines are closed, which is essential for layering. We use a measure commonly used to identify vortical regions.

Vortical regions are determined by $W < -0.2\sigma$, where σ represents the standard deviation of W [103, 104]. In Fig. 3.16, the original form of the cellular array is no longer apparent. However, during the initial disruption, some small closed streamlines remain intact. As time progresses and the magnetic field weakens, viscous forces become more pronounced, leading to cell mergers, observable at $t = 55$. Our study shows that in a multiple vortex system, closed streamlines can still persist in the dynamic regime. Counter-intuitively, layering persists in the vortex disruption regime. Thus, layering emerges as a feature common to both the flux expulsion and vortex disruption regimes.

As a partial summary, we observe that layered structures emerge in both the flux expulsion and vortex disruption regimes during the magnetic diffusion suppression stage (see Fig. 3.14). In the flux expulsion regime, magnetic field lines are expelled to the cell boundaries, which stabilizes the cellular array structure and enables the layered structure to persist. Surprisingly, in the vortex disruption regime, not all closed streamlines are fully destroyed. This partial retention of vortex structures enables layering of A to persist even in a disrupted array. Thus, the formation and persistence of layering in A appear to be characteristic features of fluctuating cellular arrays across both 2D MHD regimes, for Σ values ranging from 1 to 500. However, as Ω increases, the lifetime of these layered structures decreases. This is due to the excitation of higher modes at larger Ω , which perturbs the cellular array, generates electric currents, and accelerates magnetic field decay. To address the magnetic field decay and sustain the layered structure (i.e., suppress turbulent scattering) during the suppression stage, we next investigate staircase formation within the fluctuating cellular array model under random external forcing of A . This approach aims to prolong the suppression stage and enhance the resilience of the layered structures. The results are compared to cases with $F_A = 0$.

With external forcing of A

We have demonstrated that staircase structures without external magnetic forcing can exist and be sustained on a slow τ_{η_T} timescale through a feedback process between the flow

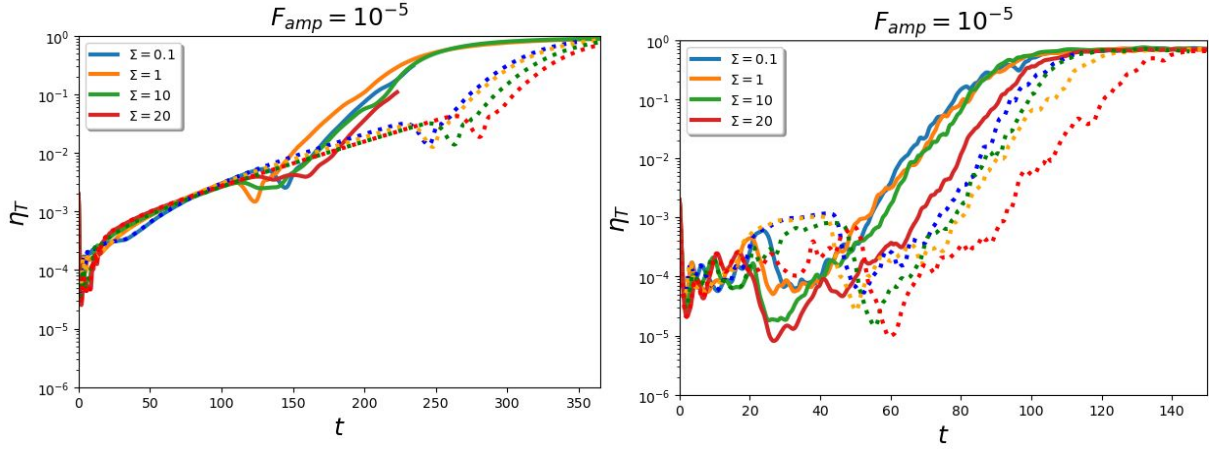


Figure 3.18. Simulations with a forcing amplitude of $F_{\text{amp}} = 10^{-5}$ for $\Omega = 10$ (left) and $\Omega = 50$ (right) for different Σ values, spanning both the flux expulsion and vortex disruption regimes. Consistent with Fig. 3.17, we observe that an amplitude of $F_{\text{amp}} = 10^{-5}$ temporarily reduces the value of η_T for a brief period. [Reproduced with permission from Phys. Rev. E 111, 055107. Copyright 2025 American Physical Society.]

and the magnetic field. However, the inherently limited lifetime of the magnetic field in 2D is a recurring theme in this paper. In this subsection, we investigate the impact of external magnetic potential forcing on sustaining the model's longevity and transport mechanisms.

To this end, we introduce an additional term, $F_A = F_{\text{amp}} \hat{F}_A$, to the right side of Eq. 3.17, representing global stochastic white noise magnetic potential forcing. We apply a Gaussian filter to the white noise to attenuate high frequency components that could result in unrealistic outcomes while preserving the overall shape of the noise. Through various simulations, we examine the influence on flow suppression for different magnitudes (F_{amp}) of the stochastic forcing, specifically applied to different Σ values across a range of Ω values.

First, we present an approximate calculation to establish expectations for η_T from

simulations with external forcing. Considering Eq. 3.17 with the initial conditions

$$A = B_0 y + \tilde{A}, \quad (3.36)$$

$$\mathbf{B} = B_0 \hat{x} + \tilde{\mathbf{B}}, \quad (3.37)$$

we express the two-dimensional MHD equation for the magnetic potential fluctuation as:

$$\frac{\partial \tilde{A}}{\partial t} + \nabla \psi \times \hat{\mathbf{z}} \cdot \nabla \tilde{A} = -\tilde{u}_y \langle B \rangle + \eta \nabla^2 \tilde{A} + \tilde{F}_a, \quad (3.38)$$

where we define $\langle A \rangle$ as $\langle A(y) \rangle$. Here, $\tilde{F}_a$ represents the general expression for the (random) magnetic stirring force, with $d\tilde{A}/dt = \tilde{F}_a$, where $\tilde{A}$ denotes the random perturbation of A . Multiplying by $\tilde{A}$, assuming the fluctuation correlation length is smaller than the scale of $\langle B \rangle$ variation, and integrating over space ($\langle \rangle$), we obtain:

$$\frac{\partial \langle \tilde{A}^2 \rangle}{\partial t} + \langle \nabla \cdot \mathbf{u} \tilde{A}^2 \rangle = 2 \left[-\langle \tilde{u}_y \tilde{A} \rangle \langle B \rangle + \eta \langle \tilde{A} \nabla^2 \tilde{A} \rangle + \langle \tilde{A} \tilde{F}_a \rangle \right]. \quad (3.39)$$

For stationary systems with periodic boundary conditions, the left-hand side of Eq. 3.39 vanishes.

Integrating by parts on the right-hand side gives:

$$\langle \tilde{B}^2 \rangle = \frac{-\langle \tilde{u}_y \tilde{A} \rangle \langle B \rangle + \langle \tilde{A} \tilde{F}_a \rangle}{\eta}. \quad (3.40)$$

We can write $\langle \tilde{u}_y \tilde{A} \rangle$ in the form of a Fick's law ($\langle \tilde{u}_y \tilde{A} \rangle = -\eta_T \partial \langle A \rangle / \partial y = -\eta_T \langle B \rangle$), yielding:

$$\langle \tilde{B}^2 \rangle = \frac{\eta_T \langle B \rangle^2 + \langle \tilde{F}_a^2 \rangle \tau_a}{\eta}. \quad (3.41)$$

Here, τ_a is the auto-correlation time of the (random) magnetic stirring force $\tilde{F}_a$. We can insert this into the following representation of η_T , derived from a quasilinear expression for the turbulent

$$\text{flux } (\Gamma = -\langle \delta A \psi_y \rangle + \langle A_y \delta \psi \rangle),$$

$$\eta_T = \left(\langle \tilde{u}^2 \rangle - \langle \tilde{B}^2 \rangle \right) \tau, \quad (3.42)$$

where τ represents the average correlation time of the turbulence. By inserting Eq. 3.41 into Eq. 3.42, we arrive at the following approximation for η_T with random external forcing

$$\eta_T \sim \frac{\left(\langle \tilde{u}^2 \rangle - \langle \tilde{F}_a^2 \rangle \tau_a \right) \tau}{1 + R_m \frac{v_A^2}{\langle u^2 \rangle}}. \quad (3.43)$$

Here, the negative sign in front of $\langle \tilde{F}_a \rangle$ indicates that η_T decreases when external forcing of A is applied to Eq. 3.17. In the remainder of this subsection, we explore several scenarios illustrating how turbulent transport is suppressed ($\eta_T \ll \eta_k$) by the application of external magnetic potential forcing.

We investigate the impact of stochastic magnetic forcing on the suppression stage lifetime. Figure 3.17 presents results from selected simulations, where η_T is measured directly from flow velocity and magnetic field time traces. The figure shows the evolution of η_T for various F_{amp} values, consistently applied at $\Sigma = 1$ and across different Ω values. For $0 < F_{amp} < 10^{-2}$, η_T initially decreases early in the simulation but eventually returns to η_k , similar to the zero-forcing case. To sustain η_T , we examine the effect of pulsed external forcing on A . By applying a pulsed amplitude of 10^{-4} every $t = 10$, we extend the suppression stage duration, achieving a result comparable to the $F_{amp} = 10^{-5}$ case without pulsing. This demonstrates that pulsed forcing at small amplitudes effectively prolongs the suppression of turbulent scattering.

We also conduct simulations with an amplitude of $F_{amp} = 10^{-5}$ for similar Ω values and different Σ values, exploring both the flux expulsion and the vortex disruption regimes (see Fig. 3.18). Consistent with the observations in Fig. 3.17, we find that an amplitude of $F_{amp} = 10^{-5}$ reduces the value of η_T for a brief period (specifically $100 < t < 150$ for $\Omega = 10$ and $20 < t < 50$ for $\Omega = 50$). *It is important to note that, as in cases discussed earlier with $F_{amp} < 10^{-2}$, we do*

not observe layering of A once $\eta_T/\eta_k \rightarrow 1$.

While the small forcing amplitudes studied thus far decrease η_T , they do not indefinitely extend the lifetime of inhomogeneous mixing (i.e., suppression). However, for $F_{amp} \geq 10^{-2}$, the lifetime of layered structures is extended. We first examine the case with $\Omega = 10$ in Fig. 3.17. At $F_{amp} = 10^{-2}$, η_T does not fully revert to η_k but remains steady at around $\sim 10^{-2}$. This pattern persists as F_{amp} increases. Both $F_{amp} \sim 10^{-1}$ and $F_{amp} \sim 1$ exhibit consistently smaller values of turbulent resistivity compared to η_k and the corresponding zero forcing case. Simulation results qualitatively agree with the estimate in Eq. 3.43, which predicts that greater forcing amplitude can further decrease the value of η_T . These results indicate that external random magnetic potential forcing can enhance layering of A by maintaining a smaller η_T relative to η_k .

Partial summary: In this subsection, we explore a how stochastic external forcing of magnetic potential affects suppression lifetime and provide an approximation to η_T in such scenarios (see Eq. 3.43). Our simulations show that forcing amplitudes of $0 \leq F_{amp} \leq 1$ reduce turbulent transport during the suppression stage, aligning with theoretical predictions. However, for forcing amplitudes less than 10^{-2} , turbulent resistivity eventually evolves to η_k over time. Enhancing confinement for $F_{amp} < 10^{-2}$ is feasible through pulsing or short-term external forcing. For forcing amplitudes greater than 10^{-2} , $\eta_T \ll \eta_k$ is maintained. Thus, sufficiently high forcing amplitudes or pulsing of A can extend the lifetime of active scalar staircase structures in a fluctuating cellular array.

3.3 Relevance to magnetic confinement

At this point, the reader may be puzzling over the question of why this paper is included in a collection of works from a plasma physics conference. To this end, it should be said that

- staircases have been observed in magnetic confinement experiments [33].
- many purported models of $\mathbf{E} \times \mathbf{B}$ staircases have been proposed [55, 41, 34, 43, 44].
- most models involve $\mathbf{E} \times \mathbf{B}$ shear as the mechanism of micro-barrier formation [55, 25].

Thus, the conventional wisdom on staircases on magnetized plasma is that of a lattice of $\mathbf{E} \times \mathbf{B}$ shear micro-barriers, at which profiles steepen, and which are separated by mixing regions where profiles flatten [1, 3, 35].

This work is complementary to the conventional approach in several ways. It offers an *alternative* micro-barrier formation mechanism, derived from the $Pe \gg 1$ condition, which is relevant to an array of cells which hover near marginal stability. This is indeed the natural state of turbulence in confinement devices, on account of the universally observed phenomenon of ‘profile stiffness’ [58]. Thus, cells are likely weakly overlapping or tangent – so that some fluctuation-induced transport occurs – but not strongly overlapping, in which case the transport would be larger than that which is observed. This paper examines the physics of layer formation in a fluctuating cellular flow, which is a simple model of such “turbulence near marginal stability”. It shows how a synergy of cellular flow and collisional scattering between cells, can generate a staircase. Here the cells and collisional diffusion are analogous to an array of modes and neoclassical transport, respectively in the confinement problem. This mechanism is applicable to systems near marginality.

However, the model goes further by proposing a semi-quantitative answer to the eternal question of “how near is near?”. Recall, we estimated that the ratio of the boundary layer width to the cell width is

$$\frac{\delta}{d} = \left(\frac{D_T}{dU_0} + \frac{1}{Pe} \right)^{1/2}. \quad (3.44)$$

This suggests, then a level of turbulent diffusion $D_T \sim dU_0/Pe$ or $D_T \gtrsim dU_0/Pe$ may qualify as “near marginal” but $D_T \gg dU_0/Pe$ will not, since δ/d will no longer be small. Of course, D_T can – and should – be linked to the Reynolds number Ω and the forcing strength. Results here indicate that the criteria $Pe \gg 1$ and $\kappa > 1.5$ are satisfied well into regimes of moderate Reynolds number. Thus, we see the bounds on ‘quasi-marginality’ are not tight. Further study of this “How near is near?” question is a high priority task.

In a related vein, the study of systems of tangential cells highlights the important role of cell boundary layers and the interplay of turbulent and collisional transport in such systems. To this end, note that $\delta^2/d^2 \sim 1/\text{Pe}$ implies that the effective diffusivity of the cellular array is $D_{\text{eff}} \sim (U_0 d D)^{1/2}$ – the geometric mean of the naive turbulent diffusivity $U_0 d$ and the collisional diffusion D [46]. This result is in marked contrast to the conventional procedure of adding $D_{\text{turb}} + D_{\text{Neo}}$. Instead, the geometric mean controls the global transport in the system, since it sets the rate of scattering through boundary layers. We note here that a similar theme appears in the Chalker-Coddington model of electron transport associated with the Quantum Hall Effect [108]. In that case, quantum fluctuations allow transport through junctions between neighboring cells. The effect of such junctions on avalanching is an interesting open question.

The model presented here is intentionally minimalist—focusing on the essential physics of staircase formation without relying on more complex feedback mechanisms involving zonal flows, turbulence, and modulational instability, as discussed in Sec. 3.1. Our aim is to extract as much physical insight as possible and relate key quantities in this simplified model to characteristics of magnetically confined plasmas. While idealized, many elements of the marginally overlapping cellular structure modeled here correspond to features in magnetic confinement devices. For example, convection cells resemble $\mathbf{E} \times \mathbf{B}$ vortices, zonal flows play the role of cell boundaries, collisional diffusion sets the baseline transport, and the scalar concentration may be interpreted as the spatial distribution of turbulence intensity.

Furthermore, the range of effective Reynolds numbers explored in this study is relevant to drift-wave turbulence. In such systems, the ratio of the inverse correlation time to the Landau damping rate—i.e., the ratio of nonlinear to linear time scales—can serve as a surrogate Reynolds number. This is particularly meaningful in high-Kubo-number regimes, where transport is percolative and stochasticity arises from structural features of the medium itself. Owing to these parallels, the model offers valuable heuristic insight and serves as a useful complement to more elaborate staircase models.

Most significantly, this work contributes detailed understanding of staircase formation

in the near-marginal regime—where cells are only weakly overlapping and the turbulent heat flux approaches the neoclassical level. This scenario is highly relevant for magnetic confinement, where understanding the transition from marginal stability to full turbulence is critical for improving transport models. While the model includes many ingredients relevant to fusion plasmas, it is not intended as a complete or exact representation. Accordingly, we now pose several open questions that reflect real challenges in magnetic fusion science—questions that extend beyond what idealized models alone can resolve and must be addressed through experiment and data-driven analysis.

The questions listed below extend the analysis beyond idealized systems and toward unresolved challenges in fusion experiments. Bridging simplified cellular models and the complex turbulence of magnetized plasmas is essential for understanding how staircases form, evolve, and interact with perturbations in real devices. These are high-priority tasks for developing validated, predictive models of transport regulation near marginality. Among the most compelling open issues are:

1. Do temperature perturbations or heat pulses generated by modulated electron cyclotron heating (MECH) trigger or amplify preexisting staircases? Is there a threshold in δT for staircase onset? Do staircases form during heat pulse wave train steepening? How do heat pulses or avalanches interact with a staircase? Can the staircase be characterized as an array of semi-permeable barriers?
2. Do staircases form in both temperature and density? Do density perturbations induced by modulated gas feed, supersonic molecular beam injection (SMBI) [109], or small pellets, trigger staircase formation?
3. Do pulses evolve into jams and then staircases? Cold pulse simulations suggest that jams and barriers form during inward propagation [110] of the cold pulses, but experimental tests of this process do not exist. How are staircases related to ‘non-locality’ phenomena?

4. What is the critical parameter for staircase formation? For the onset of turbulence in pipeflow via directed percolation [111] – a process akin to avalanching – the Reynolds number is the relevant parameter. By analogy, one might argue that Q/ν_{eff} , i.e., the ratio of the heat flux to collisionality, is relevant to staircase formation. Indeed, there already is some indication that is the case [52]. However, Q/ν manifestly is not dimensionless, so other physics must be in play.
5. Do staircases re-assemble, after being destroyed by a large perturbation? What are the time scales and the dynamics? What limits structure size?
6. Can the emergent scale invoked by two scale mixing models be identified? Is it a variant of the Rhines scale?
7. How might staircases collapse into macroscopic transport barriers, especially ITB's? This evolution has been addressed by reduced models [110], but again, there is no direct experimental test of this process.

These questions illustrate that much remains to be learned about layering and staircase formation in magnetized plasmas, particularly regarding their formation thresholds, response to perturbations, and relevance to transport bifurcations in fusion devices.

3.4 Conclusion

This work develops a unifying heuristic model for understanding staircase formation in both passive and active scalar systems, building on and expanding beyond prior purely numerical studies. In contrast to earlier publications, which focused on presenting computational results in isolation, the present study identifies common physical principles and resilience criteria governing staircase dynamics in both systems. A central result of this paper is that a single model—grounded in basic transport principles and supported by reduced equations—can explain staircase formation across both scalar types. The fluctuating vortex array, whether coupled to

a passive tracer or an evolving magnetic potential, provides a minimal yet powerful model for capturing the physics of layer formation, barrier persistence, and staircase breakdown.

In this study, we have explored the key physics governing transport barrier formation and maintenance in both passive and active scalar models, focusing on the dynamics of inter-cellular layers. In the introduction, we provide a back-of-envelope derivation based on a random walk model, predicting the scattering of scalar concentration across boundary layers in a fluctuating cellular array. This analysis identifies parameters where turbulence or intrinsic noise impacts the boundary layer, a crucial element for staircase formation. Building on this, we examine the active scalar case, where the fluctuating cellular array is coupled to a mean magnetic field threading the array. Due to flux expulsion, the magnetic field accumulates in the inter-cell boundary layer, reinforcing it and suppressing active scalar transport between cells. As a result, active scalar dynamics inherently strengthen and stabilize inter-cell boundary layers. These theoretical derivations underpin the numerical results presented in this work.

In the case of passive scalar transport in a fluctuating cellular array, we find that staircase structures form and persist across a broad range of Reynolds numbers ($0 < \Omega < 40$), relevant to effective Reynolds numbers observed in magnetic fusion confinement. In addition, we establish criteria to define the resiliency of staircase structures. The key quantities identified are the Peclet number (Pe) and profile curvature (κ). Our results show that the Peclet number is sustained at $Pe > 1$, a necessary condition for staircase formation based on previous studies of fixed cellular arrays. The second criterion, derived from our analysis, is that a staircase structure must maintain a sufficiently high curvature. We find that $\kappa \geq 1.5$ is adequate for staircase formation. The Peclet number serves as a microscopic measure of transport properties, while profile curvature provides a macroscopic measure of staircase integrity. It is important to note that κ is an extensive quantity and thus depends on the system size. Although we did not investigate an appropriately normalized value for κ , this will be addressed in future research.

For the active scalar in the fluctuating cellular array model, we observe that staircase structures emerge and persist during the suppression period in both the flux expulsion and,

surprisingly, the vortex disruption regimes. However, the suppression period is brief, ending when the magnetic field decays to a level insufficient to resist cell jittering. In the flux expulsion regime, magnetic field lines are expelled to the cell boundaries, stabilizing the cellular array structure and supporting the formation of staircases. In the vortex disruption regime, we find that not all closed streamlines—essential for staircase formation—are fully destroyed. This partial preservation of vortex structures enables the staircase structure of A to persist, even under conditions of significant vortex disruption.

Recognizing that the suppression lifetime is limited, we introduce stochastic external forcing of the magnetic potential to investigate its impact on suppression duration. Our findings reveal that such forcing reduces turbulent transport during the suppression stage, aligning with theoretical predictions. For forcing amplitudes below 10^{-2} , however, the suppression lifetime cannot be sustained indefinitely. Nonetheless, it is feasible to extend the lifetime through pulsing or short-term external forcing. Therefore, sufficiently high forcing amplitudes or pulsing methods can effectively prolong the persistence of a magnetic potential staircase structure.

These experiments, though simple, illuminate the much more complex physics observed in magnetic confinement devices. We first present a basic model without feedback (passive scalar) and then extend the study to include a model with novel feedback mechanisms (active scalar). This progression complements existing models in magnetic confinement devices and helps motivate deeper investigation into the phenomenon of layering observed near marginal stability—a theme we expand upon in the discussion below.

Crucially, this study extends beyond basic modeling and contributes to ongoing questions in magnetic confinement fusion. We show that the simplified vortex array model captures features relevant to plasma turbulence near marginal stability, including layering, transport suppression, and the emergence of semi-permeable barriers. While idealized, the parallels between $\mathbf{E} \times \mathbf{B}$ convection, zonal shear regions, and collisional diffusion make this a useful interpretive tool for understanding turbulence in fusion-relevant regimes. By focusing on the marginally overlapping regime, this study offers insight into transport regulation in near-threshold systems—an essential

topic in fusion science.

Several extensions of the present work suggest themselves. First, studies with a larger system size, run for a longer time, would enable determining the pdf of scalar flux as a function of vortex array Reynolds number. Second, it would be interesting to add a β -effect to the underlying vortex array model. This would introduce zonal flow shearing effects, which could enhance barriers [2]. Third, one should explore a wider range of noise excitations. In particular, it would be interesting to explore crystal breather modes, with the only irreversibility being that in the inter-cell boundary layers. What then would be the fate of the staircase? In the active scalar model, a study of flux driven dynamics of the scalar staircase (i.e., here impose an external flux of A) would be of considerable interest. Would such a system form magnetic barriers which self-regulate the scalar transport. Clearly, many interesting directions are possible, and many exciting results await discovery.

Chapter 4, in full, is currently being prepared for submission for publication of the material. F.R. Ramirez and P.H. Diamond (2026). The dissertation author was the primary investigator and author of this material.

Chapter 4

Near marginal layered transport states in a flux-driven turbulence model

4.1 Introduction

Spatially inhomogeneous transport structures are widely observed in driven nonequilibrium systems across a broad range of physical contexts [49, 75, 51]. These structures are characterized by the organization of a mean scalar into relatively well-mixed regions separated by narrow interfaces, leading to strongly localized transport and enhanced gradient stiffness [46, 48, 47, 83]. Staircase-like layering has been reported in stratified geophysical turbulence and the oceanic thermocline [37, 70, 39], planetary atmospheres exhibiting zonal banding [37], cryospheric and melt-layer systems [79], double-diffusive convection [76, 77, 36, 78], and active transport media [112]. Similar forms of mesoscale organization also arise in magnetically confined plasmas [35, 1, 53, 54, 41, 113, 42, 114, 40], where turbulent transport organizes into alternating regions of enhanced mixing and localized transport suppression. Despite the diversity of these systems, a common feature is the emergence of coherent mesoscale structure through nonlinear feedback between transport, gradients, forcing, and redistribution processes.

In magnetic fusion energy (MFE) plasmas, transport is inherently flux-driven. Power supplied internally through fusion reactions or externally through auxiliary heating is transported radially outward by turbulent fluctuations and collisional processes before ultimately being lost at the plasma boundary. The confinement properties of the system therefore depend sensitively

on how turbulence redistributes heat and particles across the plasma column. In particular, near-marginal operating regimes are often associated with strong transport stiffness [115], avalanche-like transport events [27, 116, 117], and the spontaneous formation of localized transport barriers [2, 110, 55, 56, 59]. Recent gyrokinetic simulations [3, 57, 118] and experimental observations [4, 119, 35] suggest that layered transport structures, often referred to as $E \times B$ staircases, play an important role in regulating turbulent spreading, suppressing radial transport, and improving confinement through mesoscale organization of transport activity.

Despite their apparent importance, the physical mechanisms governing staircase formation and persistence remain not fully understood. Full gyrokinetic and flux-driven turbulence simulations contain a wide range of coupled spatial and temporal scales, nonlinear interactions, and geometric effects, making it difficult to isolate the minimal ingredients responsible for layered transport self-organization. In addition, the computational cost of such simulations limits systematic exploration of parameter sensitivity, metastability, and long-time transport evolution. Reduced transport models therefore remain an essential theoretical tool for identifying dominant physical mechanisms, constructing interpretable macroscopic diagnostics, and developing conceptual understanding of transport regulation near marginality [55, 54, 57, 24, 120].

One influential reduced description of layered transport formation was developed by Balmforth, Llewellyn Smith, and Young (BLY) for stratified turbulent fluids [38]. In the BLY model, the evolution of a mean scalar gradient and turbulent kinetic energy are coupled through a mixing-length closure, leading to a nonlinear flux-gradient relation capable of producing layer-forming instabilities. A key feature of the BLY model is the emergence of a nonmonotone equilibrium flux-gradient relation, shown schematically in Fig. 4.1. In this regime, there exists a range of gradients for which increasing the local gradient reduces the turbulent flux, producing a negative effective diffusivity and a subsequent layering instability. The model successfully reproduces several characteristic features of staircase dynamics, including interface formation, coarsening, metastability, and edge-layer propagation.

The present work is motivated by the question of whether layered transport states can

emerge and persist in flux-driven turbulent systems even when the local constitutive transport relation remains monotone (see Appendix 6.8). In the reduced transport model studied here, turbulent transport evolves self-consistently through coupling between a mean temperature profile and turbulent kinetic energy. Unlike the BLY model, however, the local equilibrium flux-gradient relation remains monotone,

$$Q(G) = \chi_0 G + CG^3, \quad C > 0, \quad (4.1)$$

as illustrated schematically in Fig. 4.1. Consequently, staircase formation in the present system does not arise from a local negative-diffusivity instability. Instead, the emergence and persistence of layered transport states are associated with nonlocal transport redistribution, spatially inhomogeneous turbulent mixing, and near-marginal transport balance. For a detailed comparison of the two models see Table 4.1.

This distinction motivates several central questions addressed throughout this work:

1. For what conditions do layered transport states emerge in a flux-driven turbulent system with a monotone constitutive transport law?
2. What determines the persistence, coarsening, and eventual collapse of staircase structures?
3. How does transport redistribution regulate confinement and energy storage near marginality?
4. Can nonlocal turbulent spreading produce memory, hysteresis, and bistability-like behavior even in the absence of a locally nonmonotone flux-gradient relation?

To address these questions, we study a reduced temperature–kinetic energy (T – K) transport model describing turbulent transport evolution in a one-dimensional flux-driven system. The model combines local turbulent production, nonlinear damping, and nonlocal turbulent spreading within a self-consistent transport closure. Beginning from initially inhomogeneous transport

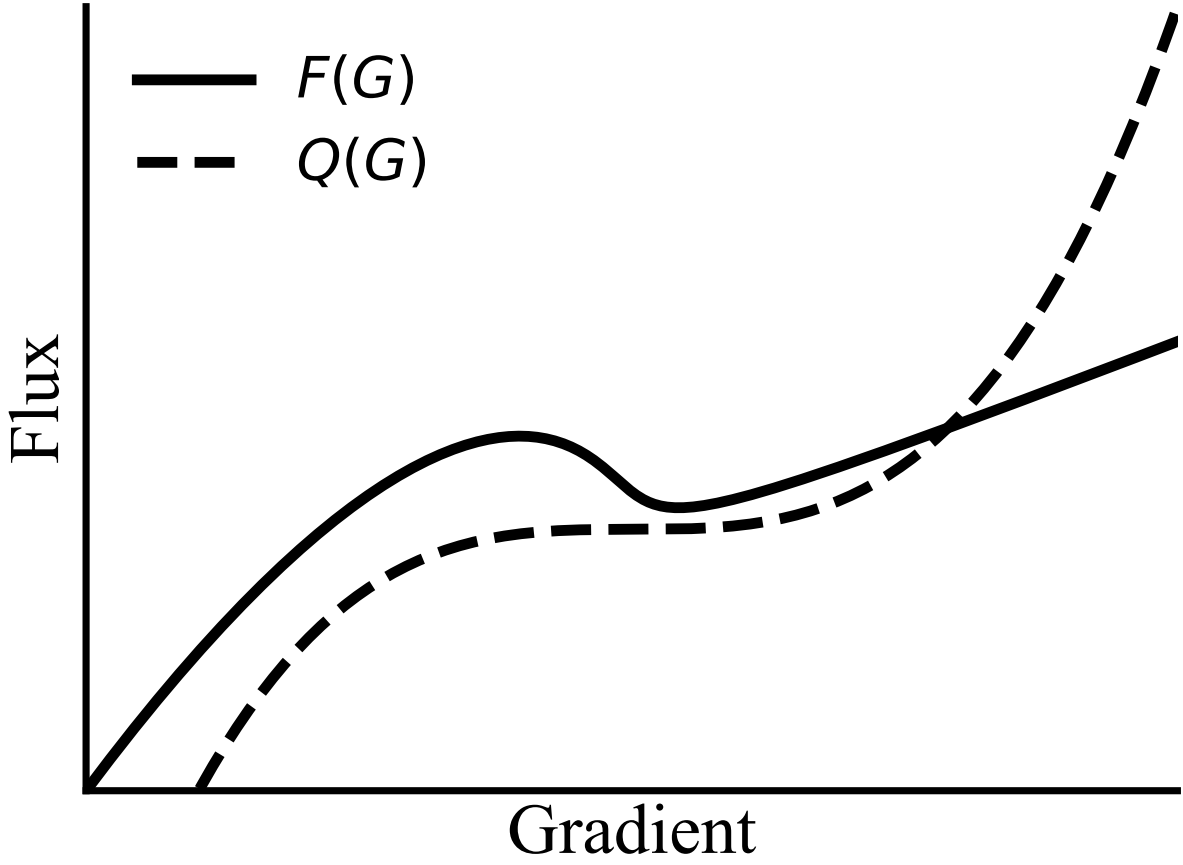


Figure 4.1. Comparison of the local equilibrium flux–gradient relations for the BLY (solid) and T – K (dashed) models. The BLY model exhibits a nonmonotone flux–gradient relation associated with local layer-forming instability, whereas the T – K model remains monotone. Derivations of both relations are provided in Appendix 6.8.

fields, the system spontaneously develops layered transport states characterized by alternating regions of enhanced transport and localized transport suppression. We introduce several macroscopic diagnostics to quantify staircase formation, transport contrast, redistribution dynamics, and memory effects, including measures based on profile curvature, transport inhomogeneity, and hysteresis within near-marginal temperature excursions.

Our results show that long-lived layered transport states occur most readily near marginality, where turbulent redistribution and nonlinear damping become quasi-balanced. In this regime, the system exhibits enhanced transport contrast, prolonged staircase persistence, and improved confinement properties. Increasing turbulent spreading modifies both the lifetime and memory

Table 4.1. Comparison between the BLY model and the present flux-driven T – K model.

Feature	BLY	Present work
Primary variables	Buoyancy gradient (g) and turbulent kinetic energy (e)	Temperature (T) and turbulent kinetic energy (K)
Transport closure	Mixing-length diffusivity $\chi \sim l e^{1/2}$	Turbulent diffusivity $\chi_T = \tau_{ac} K$
Mixing length	Dynamic, $l = e^{1/2}(e + g)^{-1/2}$	Fixed autocorrelation time τ_{ac}
Energy source	External stirring	Mean-gradient free energy
Boundary conditions	No-flux boundaries	Flux-driven temperature gradient
Local flux–gradient relation	Nonmonotone	Monotone
Equilibrium flux law	$F(g) = \frac{E(g)g}{\sqrt{E(g) + g}}$	$Q(G) = \chi_0 G + CG^3$
Layering mechanism	Local instability with $F'(g) < 0$	Dynamic interplay between damping and spreading timescales

of staircase states, producing path-dependent transport evolution and hysteresis-like behavior despite the absence of a locally bistable constitutive law. These findings suggest that metastable layered transport states and transport memory may arise generically from nonlocal turbulent redistribution processes in flux-driven systems operating near marginal stability.

The paper is structured as follows. Sec. 4.2 introduces the reduced temperature–kinetic energy (T – K) transport model, describes the physical interpretation of the governing equations and transport closure, and outlines the numerical implementation, staircase diagnostics, and relation to the BLY model. Sec. 4.3 establishes a baseline staircase state and introduces the layered transport structures that emerge from the interaction between turbulent production, nonlinear damping, and turbulent redistribution. Sec. 4.4 investigates staircase persistence, transport contrast, and redistribution dynamics, identifies weak-, intermediate-, and strong-drive transport regimes, and examines the connection between transport redistribution, enhanced confinement, and near-marginal metastable states. Sec. 4.5 explores the role of nonlocal turbulent spreading in transport memory, hysteresis, and path-dependent staircase evolution through core-temperature sweep experiments. Sec. 4.6 extends the model to include Reynolds-stress-driven

mean flows and examines how shear generation modifies transport redistribution and broadens the stability window of persistent staircase states. Finally, Sec. 4.7 summarizes the main results and discusses their implications for layered transport organization, confinement regulation, and staircase dynamics in flux-driven turbulent systems.

4.2 Reduced transport model

We consider a reduced one-dimensional transport model for the coupled evolution of a mean temperature profile $T(x, t)$ and a turbulent kinetic energy field $K(x, t)$. The model is intended to isolate the minimal ingredients required for layered transport organization in a flux-driven, near-marginal system. Rather than resolving the full turbulent dynamics, the turbulence is represented by a spatially dependent intensity field K , which determines the local turbulent diffusivity.

The physical picture is motivated by transport in weakly turbulent cellular systems. In this interpretation, localized regions of finite K represent turbulent transport cells or fluctuations (see Fig. 4.2). These cells are initialized as a finite array of localized Gaussian structures. Their weak overlap allows transport to spread across the domain while retaining strong spatial inhomogeneity in the turbulent diffusivity. This provides a minimal setting in which localized mixing regions, transport barriers, and staircase-like profiles can emerge self-consistently.

The model is assumed to operate in a weakly turbulent regime with Kubo number $K_u \lesssim 1$, so that large-scale transport can be approximated by a diffusive closure. The turbulent diffusivity is therefore related to the turbulence intensity through a characteristic autocorrelation time. The system is driven by an imposed temperature gradient at the boundaries, which produces a prescribed input heat flux through the background diffusivity. In this sense, the model is intrinsically flux-driven.

4.2.1 Governing equations

The governing equations are

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(\chi_T \frac{\partial T}{\partial x} \right) + \chi_0 \frac{\partial^2 T}{\partial x^2}, \quad (4.2)$$

$$\frac{\partial K}{\partial t} = \frac{\partial}{\partial x} \left(\beta \chi_T \frac{\partial K}{\partial x} \right) + g \chi_T \frac{\partial T}{\partial x} - K_c K^{3/2} - \alpha \Gamma, \quad (4.3)$$

where χ_0 is a small background diffusivity, χ_T is the turbulent diffusivity, β controls turbulent spreading, g controls gradient-driven turbulent production, K_c controls nonlinear turbulent damping, and $\alpha \Gamma$ represents turbulent kinetic energy sink associated with Reynolds-stress-driven mean flows.

Throughout most of this work we focus on the limit

$$\alpha = 0, \quad (4.4)$$

so that staircase formation arises solely from the interaction between turbulent production, nonlinear damping, and nonlocal turbulent spreading. This allows us to isolate the transport-driven mechanisms responsible for layered state formation. The effects of self-consistent mean-flow generation are considered separately in Sec. 4.6, where we extend the model to include mean-flow evolution and associated Reynolds-stress feedback.

Equations (4.2) describes the evolution of the mean temperature subject to both turbulent and collisional diffusion. Equations (4.3) describes the evolution of turbulence intensity through three competing processes. The first term on the right-hand side redistributes turbulent intensity through nonlocal turbulent spreading. The second term represents production of turbulent fluctuations by the mean temperature gradient. The final term provides nonlinear damping and saturation of K .

The turbulent diffusivity is closed by

$$\chi_T(x, t) = \tau_{ac} K(x, t), \quad (4.5)$$

where τ_{ac} is the turbulent autocorrelation time. This closure is the analogue of a mixing-length estimate in which the turbulent diffusivity is proportional to the fluctuation intensity multiplied by a correlation time. The resulting feedback loop is

$$\partial_x T \rightarrow K \rightarrow \chi_T \rightarrow \partial_x T, \quad (4.6)$$

which provides the basic mechanism for transport self-organization.

4.2.2 Initial and boundary conditions

The initial temperature profile is taken to be linear,

$$T(x, 0) = \frac{T_0}{L} x, \quad (4.7)$$

so that the initial temperature gradient is spatially uniform. The initial turbulent kinetic energy is prescribed as a series of localized Gaussian turbulent intensity seeds,

$$K(x, 0) = K_0 \sum_{n=1}^N \exp \left[- \left(\frac{x - x_n}{w} \right)^2 \right], \quad (4.8)$$

where K_0 is the initial turbulence amplitude, w is the turbulent intensity seed width, and x_n is the center of the n th Gaussian. Unless stated otherwise, we use $N = 6$ Gaussians located at

$$x_n = \frac{n}{7}, \quad n = 1, \dots, 6. \quad (4.9)$$

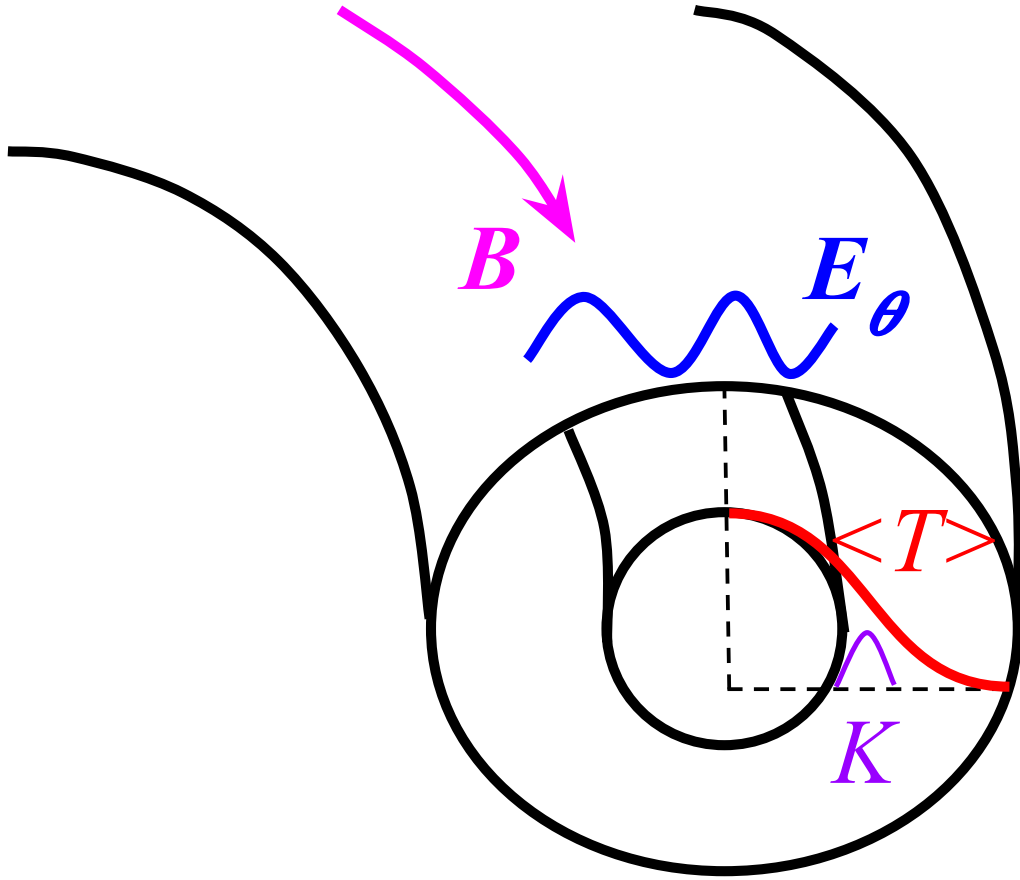


Figure 4.2. Illustration of the physical motivation for the T - K model. Flux-driven boundary conditions, a mean temperature gradient, and turbulent fluctuations interact to produce transport organization and staircase formation in a simplified representation of core-to-edge plasma transport.

We impose Neumann boundary conditions on the kinetic energy,

$$\partial_x K|_{x=0,L} = 0, \quad (4.10)$$

while the temperature uses fixed-flux boundary conditions,

$$\partial_x T|_{x=0,L} = \frac{T_0}{L}. \quad (4.11)$$

Since the turbulent transport vanishes at the boundaries, the imposed temperature gradient

produces the input heat flux

$$Q_{\text{in}} = \chi_0 \frac{T_0}{L}. \quad (4.12)$$

Thus, T_0 controls the external drive, while K_0 controls the initial turbulent intensity.

Equations (4.2)–(4.3) are solved on a one-dimensional domain $x \in [0, L]$ with $L = 1$. The simulations are performed using the open-source plasma fluid framework BOUT++ [121]. The simulations use a uniform spatial grid with 512 points and adaptive time stepping. Resolution tests were performed to ensure that the layered structures and transport contrasts discussed below are well resolved.

4.2.3 Staircase diagnostics

A central objective of this work is to quantify the emergence and evolution of layered transport states. To accomplish this, we introduce two macroscopic diagnostics based solely on the mean temperature profile: a curvature metric that quantifies interface sharpness and a staircase strength metric that quantifies the degree of profile segmentation. Because both quantities are constructed directly from the mean profile, they may be applied equally to numerical simulations or experimental measurements.

The first diagnostic is a normalized profile curvature. We begin by computing the curvature of the temperature profile,

$$\kappa(x, t) = \frac{\partial_x^2 T}{\left[1 + (\partial_x T)^2\right]^{3/2}}. \quad (4.13)$$

The staircase curvature metric is then defined as

$$\kappa_{\text{stair}} = \frac{\langle |\kappa| \rangle}{\langle |\partial_x T| \rangle / L}, \quad (4.14)$$

where the averages are taken over the analysis region and L is the corresponding spatial extent. This normalization removes the dependence on the overall profile gradient and provides a dimensionless measure of interface sharpness. Large values of κ_{stair} indicate the presence of narrow jumps separating relatively flat step regions.

In addition, to characterize the degree of profile segmentation, we introduce a staircase strength diagnostic based on the spatial distribution of temperature gradients. We first define the normalized gradient distribution

$$p_i = \frac{|\partial_x T|_i}{\sum_j |\partial_x T|_j}. \quad (4.15)$$

The associated normalized gradient entropy is

$$Z_{\text{norm}} = \frac{-\sum_i p_i \ln p_i}{\ln N}, \quad (4.16)$$

where N is the number of spatial points within the analysis region. We then define an entropy-based staircase measure,

$$S_{\text{ent}} = 1 - Z_{\text{norm}}. \quad (4.17)$$

This quantity increases as gradients become concentrated into a small number of localized interfaces.

To further distinguish strongly segmented profiles from smoothly varying profiles, we introduce a jump-strength measure,

$$S_{\text{jump}} = 1 - \frac{\langle |\partial_x T| \rangle}{\max(|\partial_x T|)}. \quad (4.18)$$

This quantity approaches zero when gradients are distributed uniformly throughout the domain and approaches unity when the profile contains isolated sharp jumps.

The final staircase strength diagnostic combines both measures,

$$S_{\text{stair}} = \sqrt{S_{\text{ent}} S_{\text{jump}}}. \quad (4.19)$$

By construction,

$$0 \leq S_{\text{stair}} \leq 1. \quad (4.20)$$

Small values correspond to smooth or weakly structured profiles, whereas large values indicate well-developed staircase organization with strongly localized interfaces. Throughout this work, κ_{stair} is used as a measure of interface sharpness, while S_{stair} serves as the primary global diagnostic for staircase formation and persistence.

4.2.4 Relation to the BLY model

The present model is closely related in spirit to the stratified turbulence model of BLY, but differs in its physical setting and in the mechanism responsible for staircase formation (see Table 4.1). Both models couple a mean scalar field to a turbulent energy variable, use a mixing-length-type transport closure, and allow turbulent intensity to redistribute spatially. Both also produce long-lived layered states and slow coarsening dynamics.

The main distinction is that BLY obtains layering from a nonmonotone local equilibrium flux–gradient relation, whereas the present model is flux-driven and has a monotone local constitutive relation. Therefore, the staircase states studied here do not originate from a local negative-diffusivity interval. Instead, they arise from the coupled evolution of turbulent production, nonlinear damping, and nonlocal redistribution of turbulent transport. A detailed derivation of the two flux–gradient relations is given in Appendix 6.8, and the main modeling differences are summarized in Table 4.1.

4.3 Staircase formation

We begin by establishing a baseline staircase state that will serve as the reference configuration throughout the remainder of this work. Unless otherwise stated, simulations use

$$\begin{aligned}\tau_{ac} &= 10, \quad g = 10^{-1}, \quad K_c = 10^{-1}, \\ K_0 &= 1, \quad T_0 = 1, \quad \beta = 0,\end{aligned}$$

with initial turbulent intensity seed width $w = 0.02$.

The choice $K_0 = 1$ ensures that turbulent transport initially dominates over the background collisional diffusivity, $\chi_T \gg \chi_0$, allowing a visible staircase structure to develop. The choice $\beta = 0$ isolates the staircase-forming dynamics in the absence of turbulent spreading and is consistent with previous studies showing that weaker turbulent spreading promotes sharper staircase formation and stronger interface development [34].

Figure 4.3 shows the resulting staircase state. The initially localized turbulent intensity seeds organize the mean temperature profile into alternating regions of weak and strong temperature gradient, producing a sequence of relatively flat steps separated by narrow jumps. This reference case lies within a long-lived regime and will serve as the baseline for all subsequent parameter scans.

4.4 Transport redistribution and staircase persistence

The central question addressed in this section is what determines staircase persistence and collapse. To answer this question, we first examine how the competition between turbulent production, nonlocal spreading and nonlinear damping influences the evolution of the turbulent diffusivity field.

Although parameter scans were performed in g , K_c , and β , we focus initially on scans of the production parameter g . Unlike the nonlinear damping coefficient K_c , which primarily

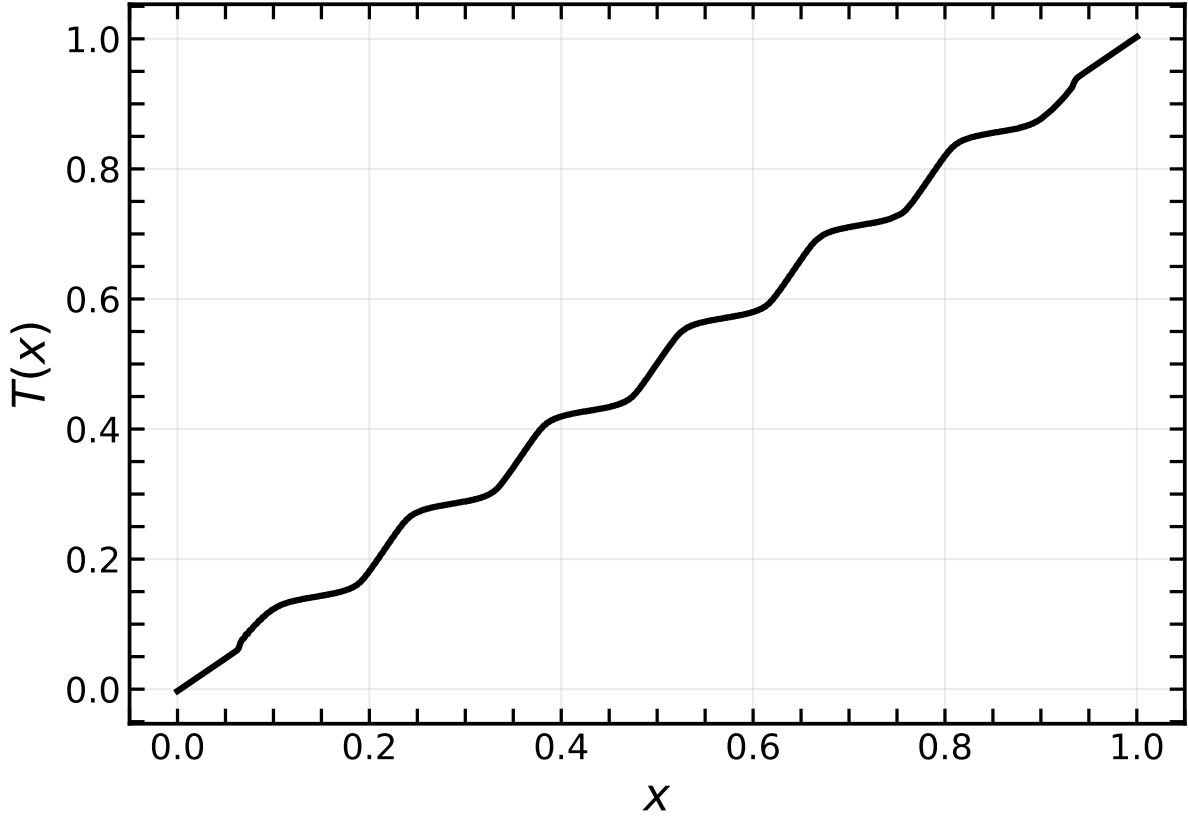


Figure 4.3. Baseline staircase state for $T_0 = K_0 = 1$, $\tau_{ac} = 10$, $K_c = g = 10^{-1}$, and $\beta = 0$. The initial linear temperature profile evolves into a staircase profile consisting of relatively flat steps separated by narrow jumps. This case serves as the reference configuration throughout the remainder of this work.

regulates turbulent saturation and numerical stability, the production parameter is directly related to the free-energy drive available to the turbulence and is therefore more closely connected to experimentally accessible control parameters such as the core temperature gradient. Production scans provide a convenient way to identify the dynamical regimes of the model before performing the more refined near-marginal scans in T_0 and β considered later.

For completeness, we briefly summarize the role of the damping coefficient. Large values of K_c suppress turbulence, causing transport to revert toward the collisional state and preventing sustained staircase formation. Conversely, as $K_c \rightarrow 0$, staircase interfaces become increasingly sharp and the transport field becomes progressively more rigid, eventually leading to numerical

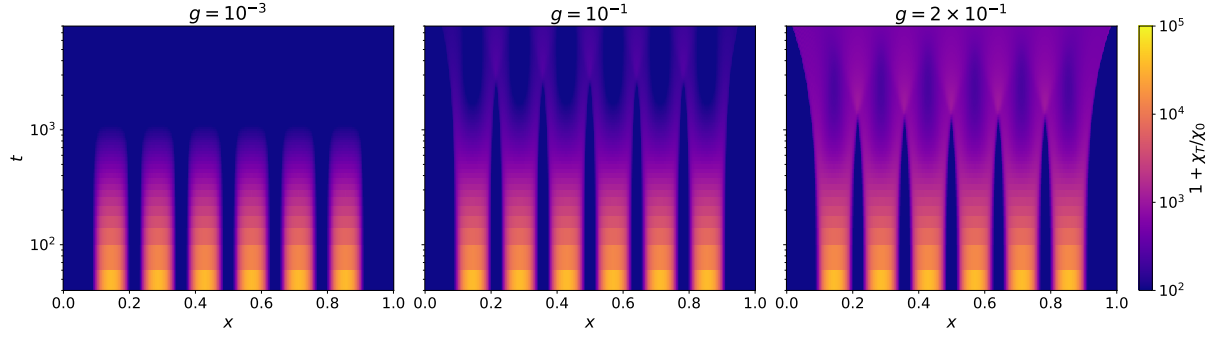


Figure 4.4. Space–time evolution of the turbulent diffusivity, χ_T , for weak, intermediate, and strong drive regimes (left to right). In the weak-drive regime, the turbulent diffusivity remains localized near its initially prescribed seed locations and gradually decays with time, causing transport to become increasingly dominated by the background collisional diffusivity. In the strong-drive regime, turbulent transport becomes progressively homogenized across the domain, reducing the transport contrast required for staircase maintenance. The intermediate-drive regime exhibits a redistribution of turbulent diffusivity from the initial seed locations toward the staircase jump regions. Here, sufficient diffusivity contrast is maintained, allowing the layered transport structure to persist for extended times.

instability. Throughout the remainder of this work we therefore adopt $K_c = 10^{-1}$, which provides a balance between numerical stability and long-lived staircase formation.

4.4.1 Three transport regimes

Production parameter scans reveal three distinct transport regimes: weak drive, intermediate drive, and strong drive. Figure 4.4 shows representative examples of the space–time evolution of the turbulent diffusivity in each regime. In the weak-drive regime, turbulent transport remains localized near its initial seed locations but gradually decays with time. As the turbulence weakens, transport becomes increasingly dominated by the background collisional diffusivity, $\chi_T \ll \chi_0$, and the staircase eventually relaxes toward a nearly linear profile. In this regime, layered transport structures form transiently, but cannot be sustained.

The strong-drive regime is characterized by excessive turbulent production. Here, turbulent transport becomes progressively homogenized throughout the domain, producing $\chi_T \gg \chi_0$, but with little spatial variation. As the transport contrast decreases, neighboring staircase steps merge and the layered structure coarsens toward a nearly uniform transport state (i.e., staircase

steps coalesce). Thus, staircase collapse may occur not only through turbulent decay, but also through excessive transport homogenization.

Between these limits lies the intermediate-drive regime, which contains the baseline staircase state shown in Fig. 4.3. In this regime, turbulent transport is redistributed away from the initial seed locations and accumulates near the staircase jump regions. Although turbulent intensity is expelled from the step interiors, sufficient transport contrast is maintained to preserve the distinction between steps and jumps. Consequently, staircase structures persist for times significantly longer than those observed in either the weak- or strong-drive limits.

To illustrate this behavior, Fig. 4.5 shows the evolution of the turbulent diffusivity profile for a representative intermediate-drive case. Each curve corresponds to a temporal snapshot of χ_T , with colors progressing from early (red) to late (blue) times. As the system evolves, turbulent transport redistributes from the initially prescribed seed locations and develops a persistent spatial contrast between regions of strong and weak transport. The dashed lines indicate the late-time values of $\chi_{T,\max}$ and $\chi_{T,\min}$. Unlike the weak- and strong-drive regimes, the intermediate-drive state approaches a quasi-steady configuration in which

$$\Delta\chi_T = \chi_{T,\max} - \chi_{T,\min} > 0. \quad (4.21)$$

The maintenance of this transport inhomogeneity preserves the distinction between staircase steps and jumps and is therefore a key ingredient for staircase persistence.

These observations suggest that staircase longevity is controlled by the degree of transport inhomogeneity. To quantify this effect, we define the transport contrast,

$$\Delta\chi_T = \chi_{T,\max} - \chi_{T,\min}, \quad (4.22)$$

which serves as a measure of inhomogeneous mixing within the system. Figure 4.6 compares the transport contrast with the staircase strength for all parameter scans. Each marker corresponds to

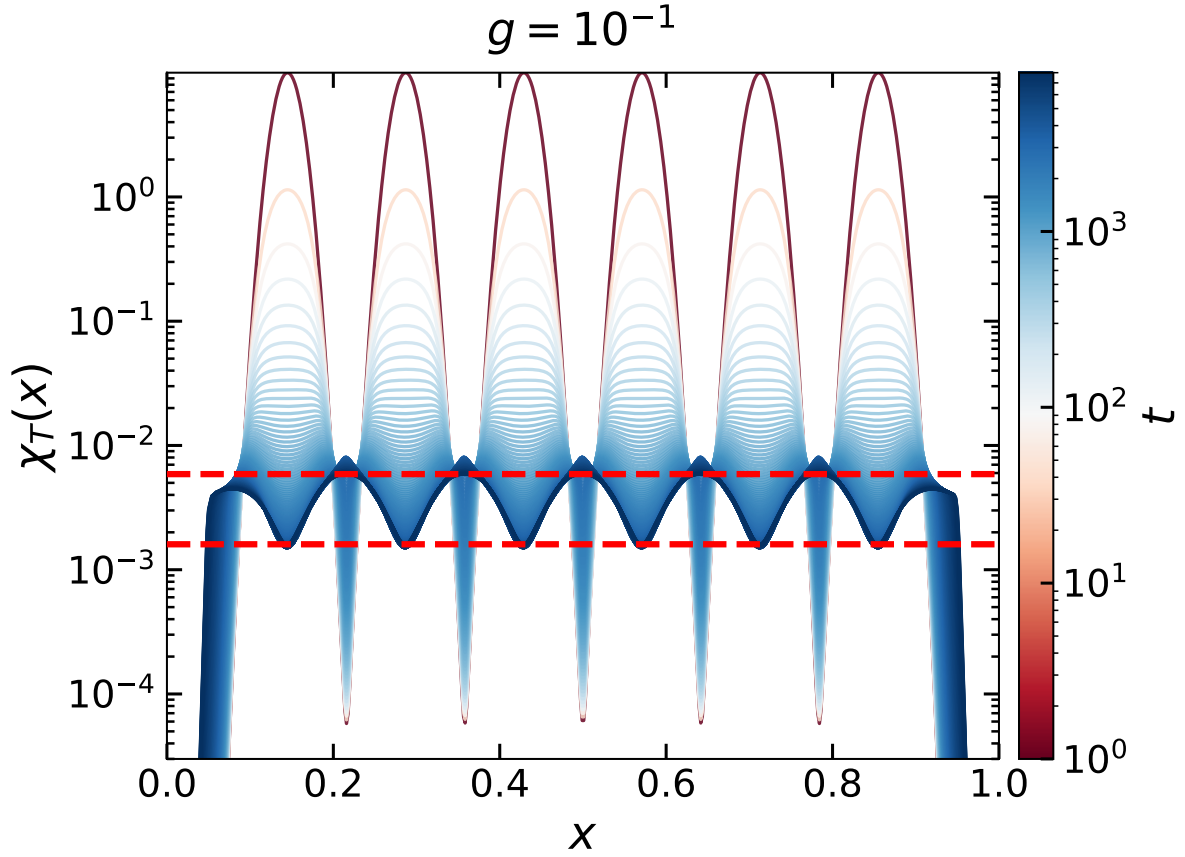


Figure 4.5. Evolution of the turbulent diffusivity profile for the baseline staircase state. The dashed red lines denote the late-time values of the maximum and minimum turbulent diffusivities, $\chi_{T,\max}$ and $\chi_{T,\min}$, respectively. The persistence of a finite transport contrast maintains the distinction between staircase steps and jumps and is therefore a key ingredient for staircase longevity.

a different parameter family, while only the indicated parameter is varied within a given scan and all remaining parameters are held fixed at their baseline values.

A clear correlation is observed between transport contrast and staircase organization. Visible staircase states (determined qualitatively), defined by

$$S_{\text{stair}} \geq 10^{-2}, \quad (4.23)$$

are consistently associated with sustained transport inhomogeneity. Across the parameter space

explored here, staircase states generally maintain

$$\frac{\Delta\chi_T}{\chi_0} \gtrsim 10^2. \quad (4.24)$$

These results indicate that maintaining sufficient transport contrast is a necessary condition for staircase persistence and provide a quantitative criterion for distinguishing long-lived layered states configurations.

4.4.2 Redistribution rate and staircase lifetime

The preceding analysis suggests that staircase persistence requires the maintenance of a sufficient transport contrast. We now seek a quantity that characterizes how rapidly this contrast evolves and redistributes throughout the domain.

To quantify the degree of transport inhomogeneity, we consider the coefficient of variation of the effective diffusivity,

$$\chi = \chi_T + \chi_0, \quad (4.25)$$

defined as

$$\text{CV}_\chi(t) = \frac{\sigma_\chi(t)}{\langle\chi\rangle(t)}, \quad (4.26)$$

where σ_χ and $\langle\chi\rangle$ denote the spatial standard deviation and mean of the effective diffusivity, respectively. Large values of CV_χ correspond to strongly inhomogeneous transport, whereas $\text{CV}_\chi \rightarrow 0$, indicates a spatially uniform transport state.

This behavior is consistent with the three transport regimes identified previously. In the strong-drive regime, turbulent transport becomes homogenized across the domain, causing CV_χ to collapse toward zero. In the weak-drive regime, turbulent transport decays until $\chi_T \ll \chi_0$, leaving the uniform collisional diffusivity as the dominant transport mechanism and again

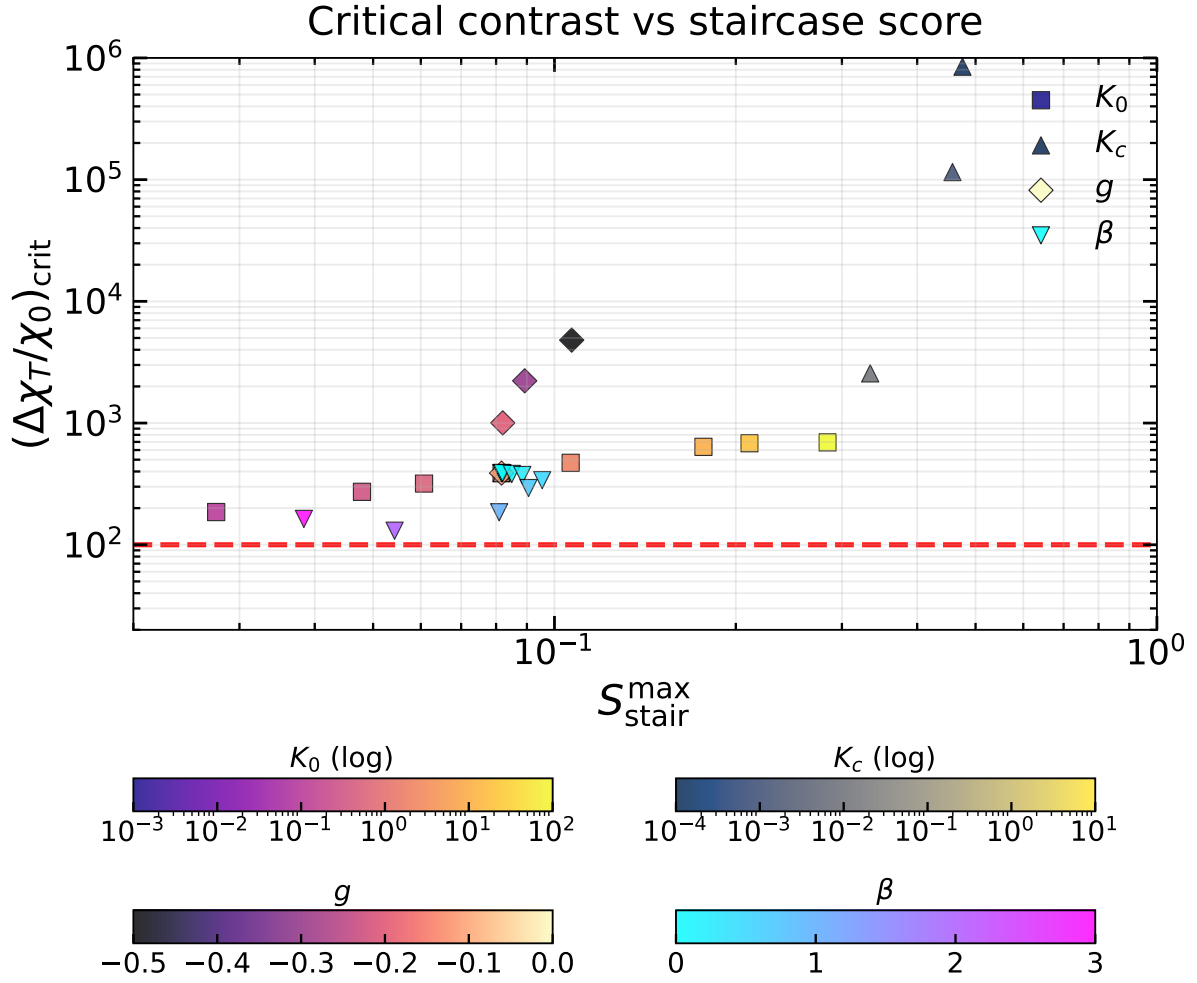


Figure 4.6. Transport contrast versus staircase strength for all parameter scans. Each marker corresponds to a different parameter family. Within a given scan, only the indicated parameter is varied, while all remaining parameters are held fixed at their baseline values. Across the parameter space explored here, persistent staircase states generally maintain a transport contrast of $\Delta\chi_T/\chi_0 \gtrsim 10^2$, indicating that sustained transport inhomogeneity is a necessary condition for staircase persistence.

producing $\text{CV}_\chi \rightarrow 0$. Only the intermediate-drive regime maintains a finite coefficient of variation for extended times, reflecting the persistence of transport contrast required for staircase maintenance.

Figure 4.7 shows the time evolution of CV_χ for representative production-parameter scans. Strongly driven cases exhibit an initial decrease, followed by a temporary increase and eventual collapse toward zero. As the drive is reduced, this collapse is progressively delayed

and the transport contrast is maintained for longer times. Weakly driven cases also decay toward zero, but do so through turbulent decay rather than transport homogenization.

The initial minimum observed in CV_χ marks a transition in the transport dynamics. Inspection of the turbulent diffusivity evolution reveals that this point corresponds to the redistribution of turbulent transport from the initial Gaussian seed locations toward the staircase jump regions (see Fig. 4.5). Here, turbulent transport is expelled from the step interiors and accumulates near the interfaces that ultimately sustain the layered state. We refer to the associated timescale as the expulsion time (similar dynamics to that of Refs. [97, 98, 96, 122]).

Although the expulsion time provides a useful characterization of transport reorganization, its identification becomes increasingly sensitive in long-lived staircase states where CV_χ remains finite for extended times. To better characterize the redistribution dynamics across all regimes, we instead define the redistribution rate

$$\gamma_{\text{redist}} = \left[\frac{d}{dt} \ln(CV_\chi) \right]_{\text{max}}, \quad (4.27)$$

which measures the maximum rate of change of the transport contrast. The redistribution rate serves as a dynamical measure of how rapidly turbulent transport reorganizes throughout the system. Because γ_{redist} remains well defined even for long-lived staircase states, it provides a more robust measure of staircase dynamics than a direct estimate of the expulsion time. In the following section, we use γ_{redist} to examine the relationship between transport redistribution, energy storage, and near-marginal confinement.

4.4.3 Near-marginal confinement enhancement

The redistribution rate provides a dynamical measure of staircase persistence. We now examine how staircase persistence relates to confinement by introducing a simple measure of

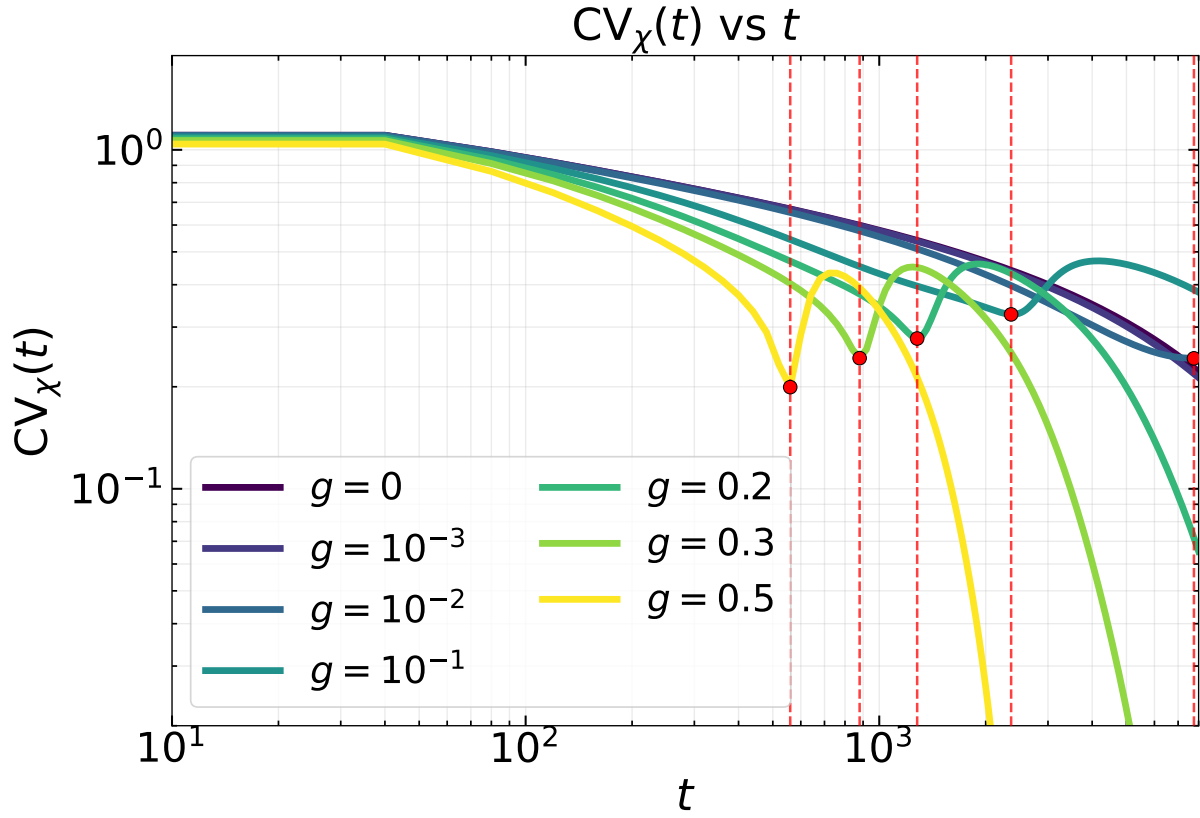


Figure 4.7. Time evolution of the coefficient of variation of the effective diffusivity, CV_χ . The coefficient of variation approaches zero when turbulent transport either becomes spatially uniform or decays below the background diffusivity, $\chi_T \ll \chi_0$. Red markers denote the redistribution event, where turbulent transport migrates from the initial Gaussian seed locations toward the staircase jump regions. In the intermediate-drive regime, CV_χ remains finite, indicating the sustained transport contrast required for staircase persistence.

energy storage within the system. The total stored energy is defined as

$$E(t) = \int_0^L T(x, t) dx, \quad (4.28)$$

and the excess stored energy relative to the initial linear profile is

$$E_{\text{store}}(t) = E(t) - E(0). \quad (4.29)$$

This quantity measures the increase in energy storage associated with staircase formation and transport suppression.

For each simulation, we define the maximum stored energy as the value of E_{store} at the time of strongest staircase organization, identified by the maximum staircase curvature κ_{stair} . Figure 4.8 shows the maximum stored energy as a function of redistribution rate for scans in both the core temperature T_0 and turbulent spreading parameter β . The central question is how close the system must operate to marginality in order to access an improved confinement regime. Since the imposed heat flux scales with T_0 , increasing T_0 corresponds to driving the system further above the collisional transport level. The figure shows that for weak turbulent spreading ($\beta \lesssim 0.5$), the condition $T_0 \simeq 1$, corresponding to $Q_{\text{in}} \approx Q_{\text{coll}}$, lies near the onset of enhanced energy storage. For stronger turbulent spreading, the same drive level places the system directly within the improved storage regime.

An important feature of the figure is the role of turbulent spreading. Although staircase formation is strongest in the $\beta \rightarrow 0$ limit, moderate turbulent spreading ($\beta \sim 0.1$) produces larger stored energies while maintaining comparable redistribution rates. This suggests that a finite level of turbulent spreading can enhance confinement by efficiently redistributing turbulent transport from the staircase step interiors toward the jump regions, thereby reinforcing transport barriers without immediately destroying transport contrast.

For cases with $\beta > 0.1$, the transition to enhanced energy storage is accessed at input fluxes below the collisional transport flux. These same cases exhibit redistribution rates that are larger by approximately an order of magnitude. This trend suggests that stronger turbulent spreading promotes more efficient transport redistribution, allowing the system to access the enhanced-storage regime at lower levels of external drive.

The figure also reveals a tradeoff between energy storage and staircase persistence. Increasing the drive generally increases the maximum stored energy, but also increases the redistribution rate. Since larger redistribution rates correspond to more rapid transport reorganization, the resulting staircase states tend to be shorter lived. Conversely, long-lived staircase

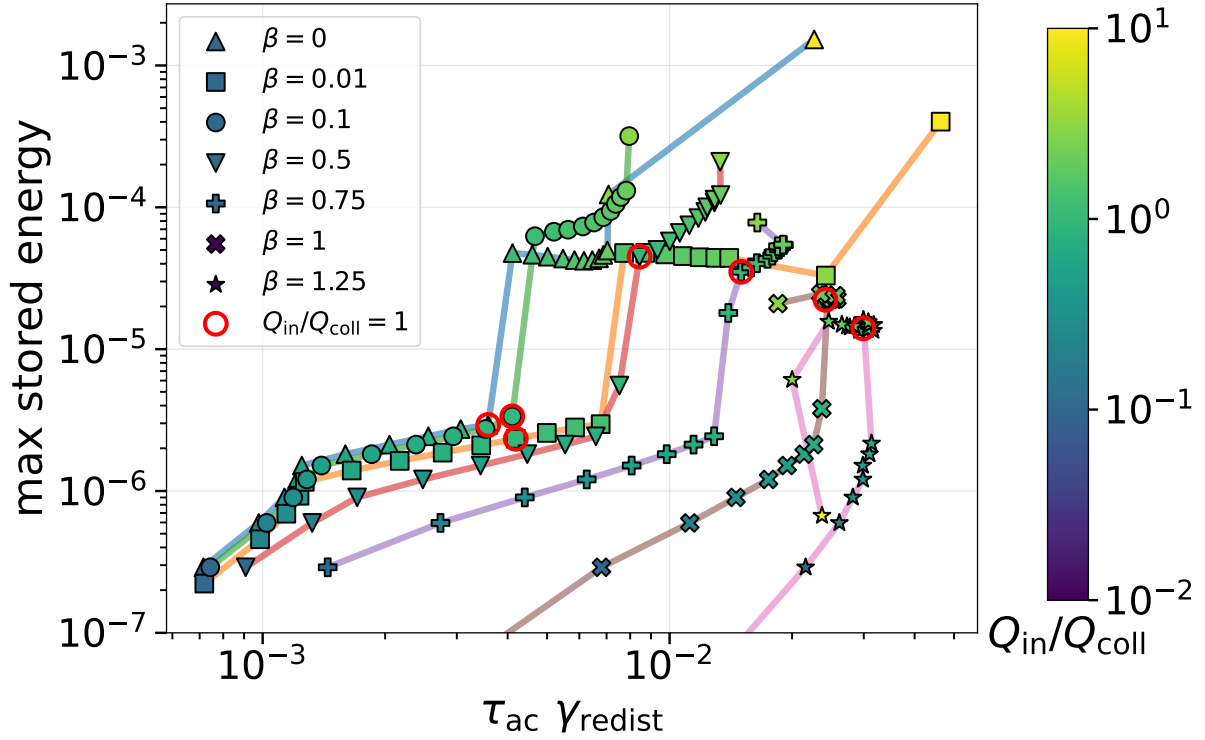


Figure 4.8. Stored energy versus normalized redistribution rate. Marker style denotes the turbulent spreading parameter β , while marker color indicates $Q_{\text{in}}/Q_{\text{coll}}$. The transition to enhanced energy storage occurs near $Q_{\text{in}} \approx Q_{\text{coll}}$. Increased stored energy is accompanied by larger redistribution rates, indicating a tradeoff between confinement enhancement and staircase lifetime.

structures are associated with slower redistribution but typically exhibit reduced energy storage. Thus, the regime of practical interest lies near marginality, where the system balances enhanced confinement against staircase longevity.

The preceding analysis suggests that staircase persistence is closely tied to the operating point of the system relative to marginality. In particular, Fig. 4.8 indicates that the transition to enhanced energy storage occurs near

$$Q_{\text{in}} \approx Q_{\text{coll}}, \quad (4.30)$$

where the externally imposed heat flux approximately matches the collisional transport level. This result provides a quantitative answer to the question of how near the system must be to

marginality in order to access an improved confinement regime.

The same figure also suggests the existence of a metastable operating window. When the drive is substantially below the collisional level, turbulent transport decays and the staircase eventually relaxes toward a nearly linear profile. Conversely, when the drive is significantly above the collisional level, turbulent transport becomes increasingly homogenized and staircase steps coalesce. Between these limits lies a narrow region near $Q_{\text{in}}/Q_{\text{coll}} \sim 1$, where staircase structures remain long-lived while simultaneously maintaining enhanced energy storage.

This observation suggests that staircase persistence is governed by a competition between transport redistribution and turbulent relaxation processes. The existence of long-lived staircase states near marginality therefore points to a dynamically selected metastable regime rather than a conventional equilibrium state. To investigate this mechanism, we next examine the relationship between the redistribution rate and the nonlinear damping timescale.

4.4.4 Competing timescales and the metastable regime

The preceding analysis identified a near-marginal operating window in which staircase structures remain long-lived while simultaneously exhibiting enhanced energy storage. We now investigate whether this behavior can be understood in terms of competing transport timescales.

The nonlinear damping term (relaxation timescale) in Eq. (4.3) implies the characteristic timescale

$$\gamma_{\text{damp}} = K_c K^{1/2}. \quad (4.31)$$

If we normalize using the autocorrelation timescale $\tau_{\text{ac}} = 10$, we find that $\tau_{\text{ac}} \gamma_{\text{damp}} \propto \sqrt{K}$. This then indicates that the evolved turbulent kinetic energy provides a measure of nonlinear damping.

Figure 4.9 compares the normalized damping and redistribution rates. Here, both rates are normalized by the autocorrelation time. Each curve corresponds to a different value of the turbulent spreading parameter β , and the marker colors indicate the measured ratio $Q_{\text{in}}/Q_{\text{coll}}$.

Several notable features emerge. Simulations operating near $Q_{\text{in}}/Q_{\text{coll}} \approx 1$, cluster within a plateau-like region of the parameter space. Within this region, the damping rate varies only weakly over a range of redistribution rates, indicating a quasi-balanced state in which neither turbulent decay nor turbulent homogenization dominates the dynamics.

The extent of this plateau depends sensitively on the turbulent spreading parameter. The clearest plateau structure occurs for

$$0 \leq \beta \lesssim 0.5, \quad (4.32)$$

with the $\beta = 0$ and $\beta = 0.1$ cases exhibiting nearly identical behavior. The $\beta = 10^{-2}$ case maintains approximately constant damping rate over the largest range of redistribution rates, suggesting particularly robust staircase persistence. As β is increased further, the plateau gradually weakens and the curves become increasingly sensitive to changes in redistribution rate. Thus, strong turbulent spreading appears to destabilize the quasi-balanced state.

The structure of Fig. 4.9 also provides a dynamical interpretation of the three transport regimes identified previously. The lower-left branch corresponds to weakly driven systems in which turbulent transport decays and collisional transport ultimately dominates. The upper-right branch corresponds to strongly driven systems in which turbulent transport becomes homogenized and the staircase eventually collapses through loss of transport contrast. Between these limits lies the quasi-balanced regime, where redistribution, spreading, and damping remain sufficiently balanced to maintain the transport inhomogeneity required for staircase persistence.

These observations indicate that staircase longevity is governed by a competition between transport redistribution and turbulent relaxation processes. Near $Q_{\text{in}}/Q_{\text{coll}} \sim 1$, the system enters a dynamically selected metastable state in which transport contrast is maintained for extended times despite the absence of a static transport bifurcation. Across the turbulent-spreading scans considered here ($0 \leq \beta \leq 1.25$), persistent staircase states occupy a quasi-balanced regime

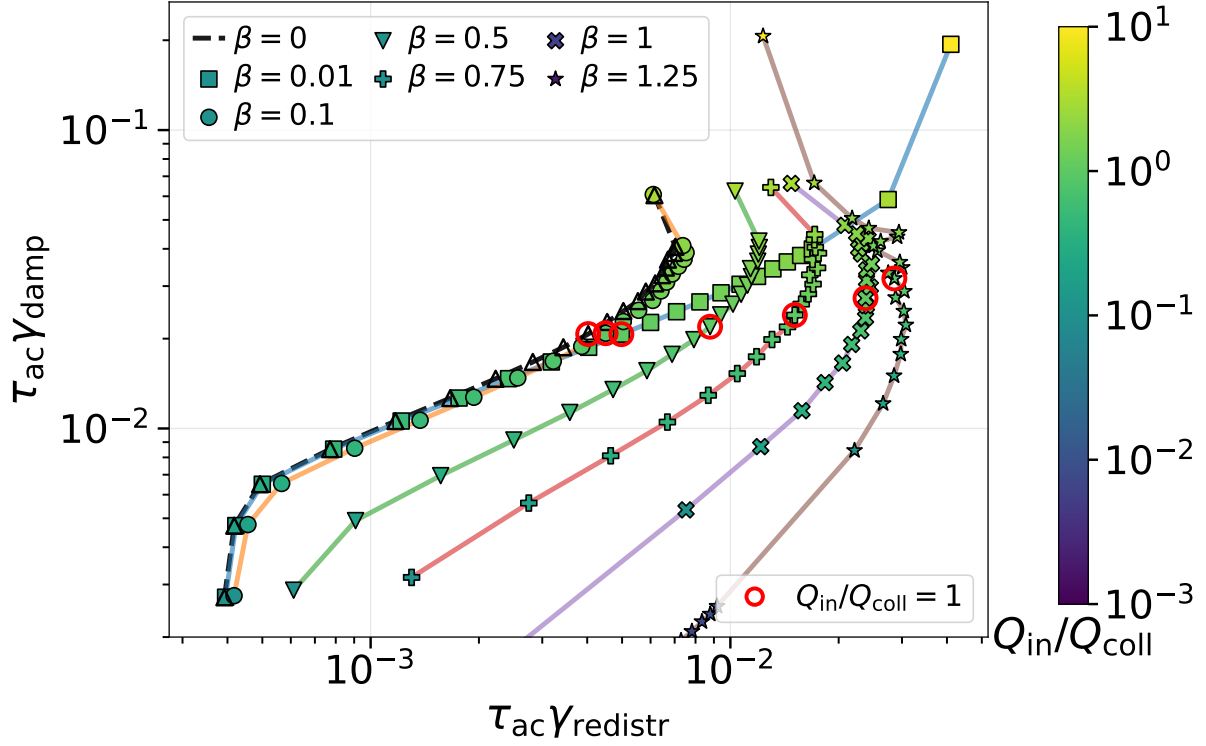


Figure 4.9. Plot of damping and redistribution rates. The competition between damping and redistribution separates the parameter space into weakly driven, quasi-balanced, and strongly driven regimes. Weakly driven states occupy the lower-left branch and decay toward the collisional state, whereas strongly driven states occupy the upper-right branch and lose transport contrast through staircase steps coalescence. Between these limits lies a quasi-balanced regime in which long-lived staircase structures persist. Most persistent staircase states satisfy $1 \lesssim \gamma_{\text{damp}}/\gamma_{\text{redist}} \lesssim 10$, suggesting a near-marginal metastable window.

satisfying

$$1 \lesssim \frac{\gamma_{\text{damp}}}{\gamma_{\text{redist}}} \lesssim 10, \quad (4.33)$$

where damping and redistribution remain comparable and neither process strongly dominates the other.

Unlike the BLY model, where layering emerges from a nonmonotone flux–gradient relation, the local constitutive relation of the present model remains monotone. Consequently, the metastable behavior observed here does not arise from a local transport rollover or constitutive

bistability. Instead, it emerges dynamically through the competition between turbulent intensity production, transport redistribution, and nonlinear decay. The resulting staircase states therefore represent a form of dynamically selected multistate behavior in a flux-driven transport system.

In Sec. 4.6, we revisit the role of competing timescales in the presence of self-consistent mean flows. There, Reynolds-stress-driven mean-flow generation introduces an additional transport-regulation mechanism that modifies the balance between turbulent growth, redistribution, and relaxation. Before considering these effects, however, we first examine how nonlocal transport influences memory and path-dependent staircase evolution.

4.5 Nonlocal transport and memory

The preceding sections identified a near-marginal operating window in which staircase structures remain long-lived while simultaneously exhibiting enhanced energy storage. The results of Figs. 4.8 and 4.9 further showed that this regime is sensitive to deviations from marginality. Changes in the imposed heat flux can drive the system toward either a weak-drive state, where turbulent transport decays and the profile relaxes toward a linear state, or a strong-drive state, where transport homogenization leads to staircase step coalescences.

This sensitivity naturally raises the question of whether the staircase response depends on the path by which the system is driven. To investigate this possibility, we perform a series of core-temperature sweeps about the near-marginal operating point for different β parameters and examine the resulting hysteresis behavior.

4.5.1 Core-temperature sweeps

Figure 4.10 shows a sequence of core-temperature sweeps spanning $0.6 \leq Q_{\text{in}}/Q_{\text{coll}} \leq 1.4$, starting from the near-marginal state $Q_{\text{in}} = Q_{\text{coll}}$. The input flux is varied through successive temperature increments $\delta T_0 = 0.05$, applied every $\delta t = 1.2 \times 10^3$, allowing the staircase sufficient time to adjust between perturbations.

The resulting trajectories in Fig. 4.10 are plotted as a function of the staircase curvature,

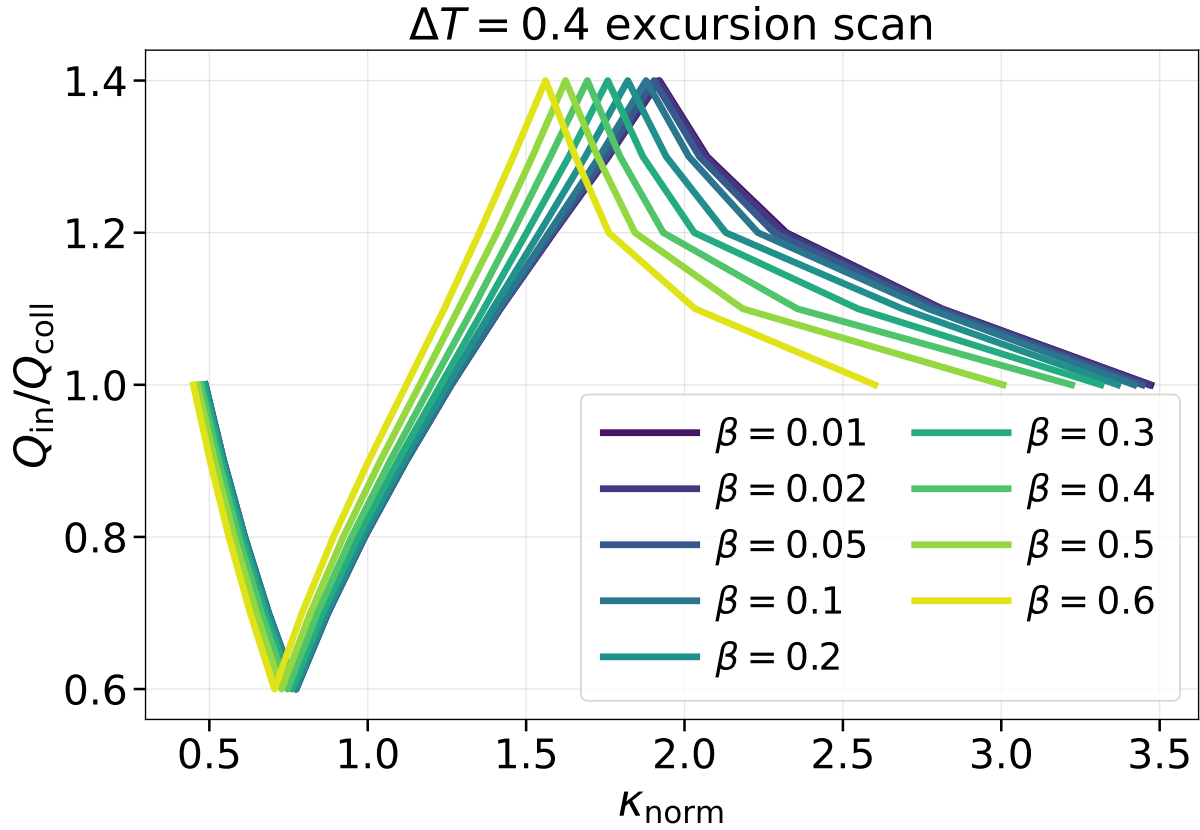


Figure 4.10. Core-temperature sweeps spanning $0.6 \leq Q_{\text{in}}/Q_{\text{coll}} \leq 1.4$, plotted as a function of staircase curvature. The system exhibits clear hysteresis, with increasing and decreasing flux sweeps following distinct paths. The response is directionally asymmetric, as increasing flux accesses a broader range of staircase curvatures than decreasing flux, indicating path-dependent staircase evolution and finite memory.

κ_{stair} , which serves as a macroscopic measure of interface strength. If the staircase response were uniquely determined by the instantaneous drive, the trajectories would collapse onto a single curve. Instead, the system exhibits a clear path dependence: increasing and decreasing flux sweeps follow distinct branches. Furthermore, the response is directionally asymmetric, with increasing flux accessing a broader range of staircase curvatures than decreasing flux. These observations indicate that the staircase retains memory of its previous state and suggest the existence of metastable transport configurations.

4.5.2 Hysteresis and memory strength

To quantify the memory effects observed in Fig. 4.10, we introduce a normalized hysteresis measure,

$$H_{\text{area}} = \int |Y_{\text{up}}(X) - Y_{\text{down}}(X)| dX, \quad (4.34)$$

$$H_{\text{area}}^{\text{norm}} = \frac{H_{\text{area}}}{(X_{\text{max}} - X_{\text{min}})(Y_{\text{max}} - Y_{\text{min}})}, \quad (4.35)$$

where $H_{\text{area}}^{\text{norm}}$ measures the area enclosed by the hysteresis loop after removing its overall scale.

To quantify directional asymmetry, we define the hysteresis skewness

$$S_{\text{skew}} = \frac{A_{\text{low}} - A_{\text{high}}}{A_{\text{low}} + A_{\text{high}}}, \quad (4.36)$$

where A_{low} and A_{high} denote the lower- and higher-drive portions of the hysteresis loop, respectively. Negative values indicate a response weighted toward the higher-drive branch, while positive values indicate weighting toward the lower-drive branch.

Figure 4.11 summarizes the dependence of both quantities on the turbulent spreading parameter β . Several trends emerge. First, the normalized hysteresis area increases with β , indicating that stronger turbulent spreading enhances the memory of the staircase state. Second, the magnitude of the skewness also increases with β , demonstrating that turbulent spreading amplifies the directional asymmetry of the response.

Although the increase in hysteresis area is modest, it is remarkably robust across the range of perturbation amplitudes considered. This behavior suggests that the memory properties of the staircase are controlled primarily by the redistribution dynamics rather than by the magnitude of the perturbation itself. In other words, nonlocal turbulent spreading governs the geometry of the hysteresis loop and therefore the degree of memory retained by the system.

Taken together, these results indicate that the near-marginal staircase state is not only metastable but also exhibits a finite memory of its previous transport state. Increasing turbulent

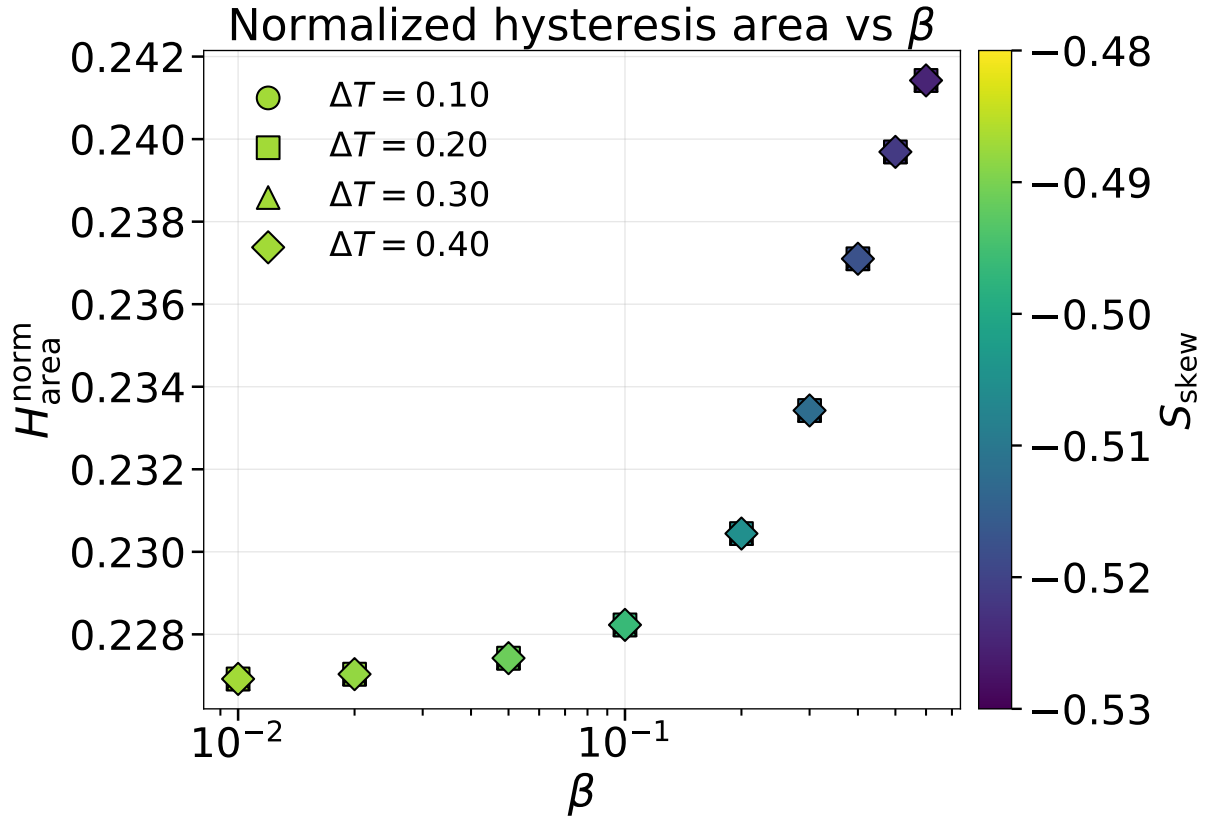


Figure 4.11. Normalized hysteresis area and skewness as functions of the turbulent spreading parameter β . Both quantities increase with β , indicating that stronger turbulent spreading enhances the memory of the staircase state while simultaneously increasing the directional asymmetry of its response to the input flux.

spreading (or redistribution rate) strengthens both the memory and the directional asymmetry of the response, highlighting the important role of nonlocal transport in staircase evolution.

The sensitivity of the staircase to both transport redistribution and nonlinear damping raises a natural question. Can an additional transport-regulation mechanism stabilize the staircase and widen the near-marginal operating window? In the present model, turbulent transport is regulated solely through nonlinear damping and redistribution. However, many turbulent systems also generate mean flows that extract energy from the turbulence and regulate transport through shear (see Fig. 4.12). Motivated by this possibility, we next extend the model to include a simple self-consistent mean-flow generation through Reynolds-stress feedback and examine whether flow generation can enhance staircase stability.

4.6 Mean-flow regulation of staircase persistence

The staircase states discussed thus far arise from the interaction between turbulent production, nonlinear damping, and turbulent spreading. In this section, we extend the reduced transport model to include self-consistent mean-flow generation through Reynolds-stress feedback. The temperature equation, turbulent diffusivity closure, and initial turbulence-intensity seed configuration remain unchanged from Sec. 4.2. Our focus here is on the coupled evolution of the turbulence intensity, mean flow, and fluctuating wavevector.

The mean-flow extension introduces two additional fields: a mean flow $U(x, t)$ and a radial wavenumber $k_x(x, t)$. The mean flow is generated entirely by the turbulence and is initialized as

$$U(x, 0) = 0, \quad (4.37)$$

so that all subsequent flow evolution arises self-consistently through Reynolds-stress forcing. As in the base model, the turbulence intensity determines the turbulent diffusivity through $\chi_T = \tau_{ac}K$.

The turbulent kinetic energy equation becomes

$$\frac{\partial K}{\partial t} = \frac{\partial}{\partial x} \left(\beta \chi_T \frac{\partial K}{\partial x} \right) + g \chi_T \frac{\partial T}{\partial x} - K_c K^{3/2} - R_{xy} S, \quad (4.38)$$

where

$$S = \frac{\partial U}{\partial x} \quad (4.39)$$

is the mean-flow shear. Relative to Eq. (4.3), the additional term $R_{xy}S$ represents the transfer of turbulent kinetic energy into the mean flow. Consequently, the mean flow acts as an additional sink of turbulence intensity.

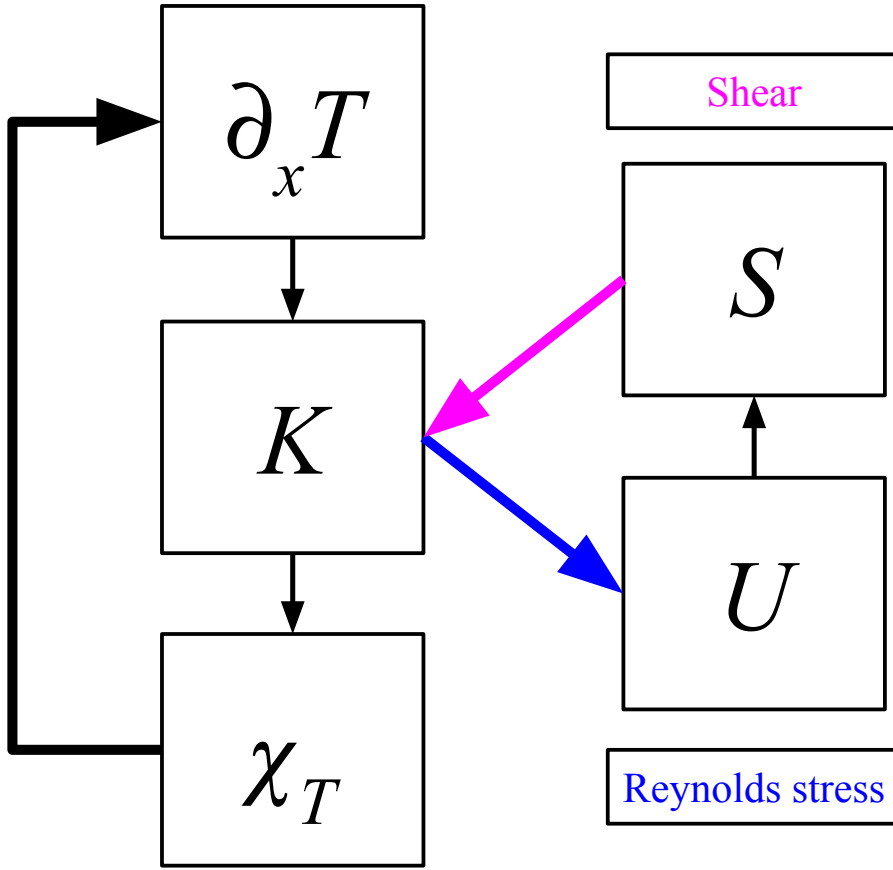


Figure 4.12. Schematic of the feedback loops in the T – K – U model. Turbulent intensity drives mean-flow generation through Reynolds-stress transfer. The resulting mean-flow shears turbulent structures, modifies the Reynolds stress, and regulates turbulence intensity near staircase jump regions, establishing a turbulence–flow feedback loop.

The Reynolds stress is modeled using the closure

$$R_{xy} = C_R \left(\frac{k_x k_y}{k_x^2 + k_y^2} \right) K, \quad (4.40)$$

where k_y is held fixed at unity and C_R controls the coupling strength between turbulent fluctuations and mean-flow generation. Larger values of C_R increase the efficiency with which turbulence drives mean flows. Including C_R directly in Eq. (4.40) ensures that changes in coupling strength affect both the turbulent kinetic energy and mean-flow equations simultaneously, thereby preserving the energetic consistency of the Reynolds-stress transfer mechanism.

The mean flow evolves according to

$$\frac{\partial U}{\partial t} = -\frac{\partial R_{xy}}{\partial x} - \mu_U U + \nu_U \frac{\partial^2 U}{\partial x^2}, \quad (4.41)$$

where the Reynolds-stress divergence drives the flow, μ_U provides linear damping, and ν_U is a small viscosity. Throughout this section we use $\mu_U = 0.1$, $\nu_U = 10^{-5}$.

To account for the distortion of turbulent structures by the shear flow, we evolve the radial wavenumber using the eikonal relation

$$\frac{\partial k_x}{\partial t} = -k_y S. \quad (4.42)$$

This equation describes the shearing of turbulent fluctuations by the mean flow. Regions of finite shear modify the radial wavenumber, which in turn alters the Reynolds stress through Eq. (4.40). The resulting feedback provides a simple model for turbulence–flow interaction [123, 24, 89].

The mean flow and radial wavenumber satisfy homogeneous Neumann boundary conditions,

$$\partial_x U|_{x=0,L} = 0, \quad \partial_x k_x|_{x=0,L} = 0, \quad (4.43)$$

while the initial condition for the radial wavenumber is taken to be spatially uniform ($k_{x0} = 1$),

$$k_x(x, 0) = k_{x0}. \quad (4.44)$$

Throughout this section, we investigate the effects of turbulent spreading and turbulence–flow coupling by varying the parameters β and C_R . The parameter β controls the redistribution of turbulence intensity, while C_R controls the efficiency of Reynolds-stress-driven mean-flow generation. Together, these parameters determine the balance between transport redistribution, shear generation, and turbulent dissipation.

The principal question addressed in this section is whether Reynolds-stress-driven mean flows can widen the stability window of the persistent staircase state. The T – K model exhibited a metastable near-marginal regime in which staircase structures remained long-lived but occupied a relatively narrow region of parameter space. The introduction of a self-consistent mean flow provides an additional transport-regulation mechanism that may stabilize transport contrast against rapid redistribution.

As in the transport-only model, the local steady-state flux–gradient relation remains monotone (see Appendix 6.9). Consequently, any enhancement of staircase persistence cannot be attributed to a local transport rollover or Phillips-type negative diffusivity. Instead, persistence must arise dynamically through the competition between turbulent redistribution, nonlinear damping, and Reynolds-stress-driven shear generation.

To characterize the influence of the mean flow, we define the shear rate

$$\gamma_{\text{shear}} = \max |\partial_x U|. \quad (4.45)$$

Figure 4.13 compares the shear rate with the redistribution rate for weak turbulence–flow coupling strengths. A visible nonmonotone relationship emerges. For small redistribution rates, mean-flow generation is weak and the resulting shear is insufficient to significantly regulate transport. In this regime, nonlinear damping remains the dominant relaxation mechanism and staircase structures decay. As the redistribution rate increases, Reynolds-stress transfer becomes more effective and the shear rate increases. Beyond a finite interval, however, redistribution becomes sufficiently strong that staircase steps begin to coalesce and transport contrast is lost. The result is a finite window in which redistribution and shear generation remain balanced, allowing persistent staircase states to be sustained.

The mean flow also modifies the relationship between redistribution and damping. As the turbulence–flow coupling strength C_R is increased, the redistribution-rate curves broaden and become less sensitive to variations in the damping rate. This behavior suggests that Reynolds-

stress-driven shear provides an additional stabilization channel that supplements the nonlinear damping already present in the T – K model.

To quantify the competition between redistribution and transport regulation, we define

$$\mathcal{M} = \frac{\gamma_{\text{redist}}}{\gamma_{\text{shear}} + \gamma_{\text{damp}}}. \quad (4.46)$$

This quantity compares the rate at which transport contrast is redistributed against the combined rates that act to regulate that redistribution. Across the parameter scans considered here, long-lived staircase states cluster near $\mathcal{M} \sim 10^{-1}$. This result suggests that persistent staircase structures require the combined shear and damping rates to exceed the redistribution rate by approximately an order of magnitude. When redistribution dominates, staircase steps coalesce and transport contrast is lost. Conversely, when shear generation and damping sufficiently regulate redistribution, persistent staircase states can be maintained.

To quantify the extent of the stable staircase regime, we define the logarithmic width

$$W_{\log} = \log \left(\frac{\gamma_{\text{redist}}^{\max}}{\gamma_{\text{redist}}^{\min}} \right). \quad (4.47)$$

Figure 4.14 shows $W_{\log}$ as a function of turbulent spreading strength for several values of the turbulence–flow coupling parameter C_R . For weak coupling strengths ($C_R \ll 1$), the stability-window widths remain similar. However, increasing turbulent spreading over, $10^{-2} \lesssim \beta \lesssim 1$, produces a substantial increase in the stability-window width, from approximately $W_{\log} \sim 0.35$ to $W_{\log} \sim 0.55$.

This increase indicates that moderate turbulent spreading broadens the range of redistribution rates over which persistent staircase states can exist. Physically, stronger redistribution promotes shear generation, which feeds back on the turbulence through Reynolds-stress transfer and enlarges the parameter region supporting long-lived transport contrast. Thus, turbulent spreading does not simply redistribute turbulence; when coupled to mean-flow generation, it can

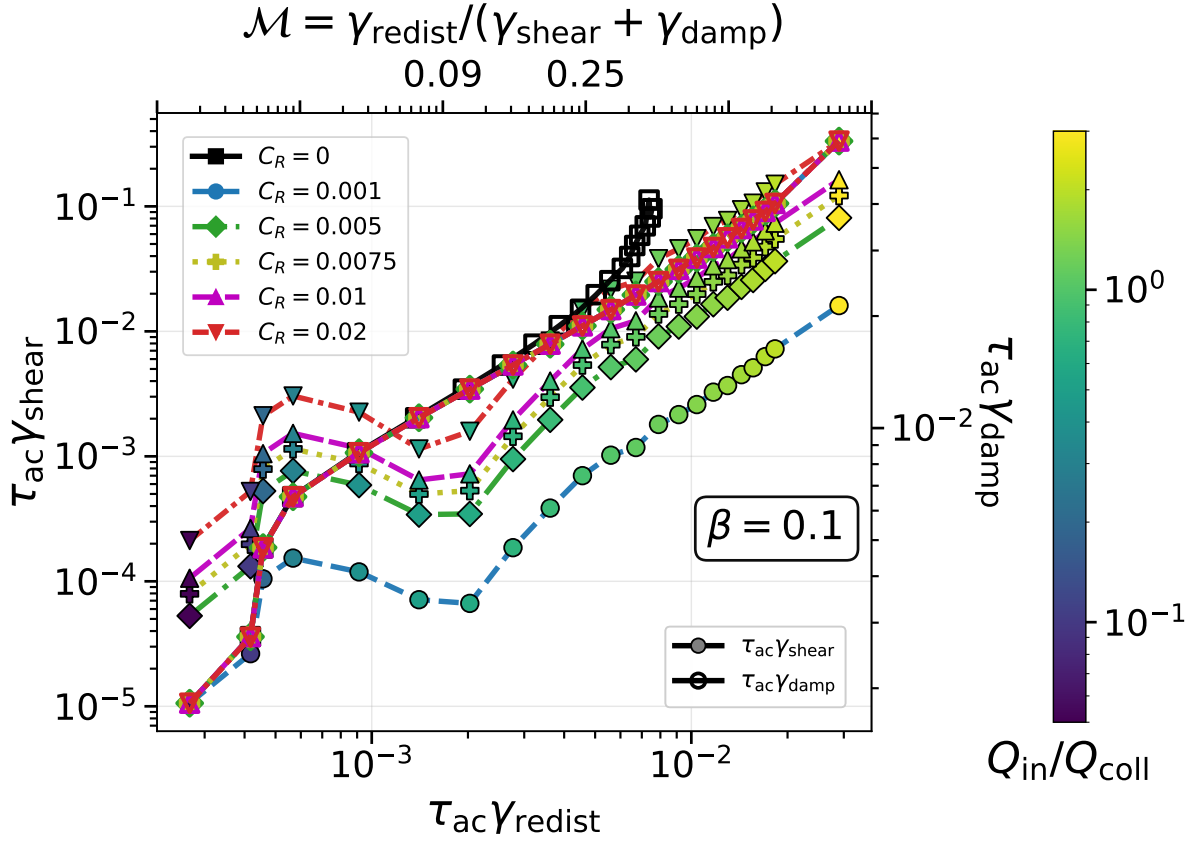


Figure 4.13. Comparison of the redistribution, damping, and shear rates for $\beta = 0.1$. A nonmonotone relationship is observed between the shear and redistribution rates, producing a finite stability window in which persistent staircase structures are sustained. Within this window, long-lived staircase states cluster near $\mathcal{M} \sim 0.1$, indicating a balance between transport redistribution and the mechanisms that regulate it. Increasing the turbulence–flow coupling strength broadens the metastable region and reduces the sensitivity of the redistribution rate to damping.

enhance the effectiveness of shear regulation and increase staircase robustness.

Taken together, these results reinforce a central conclusion of this work. Both the T – K and T – K – U models possess monotone local flux–gradient relations and therefore do not admit a Phillips-type negative-diffusivity instability. Staircase persistence should instead be interpreted as the complex interaction between timescales.

In the T – K model, the turbulent spreading competes with nonlinear damping, producing a narrow metastable near-marginal window. The introduction of Reynolds-stress-driven mean flows adds an additional regulation channel through shear generation. Even weak turbulence–flow

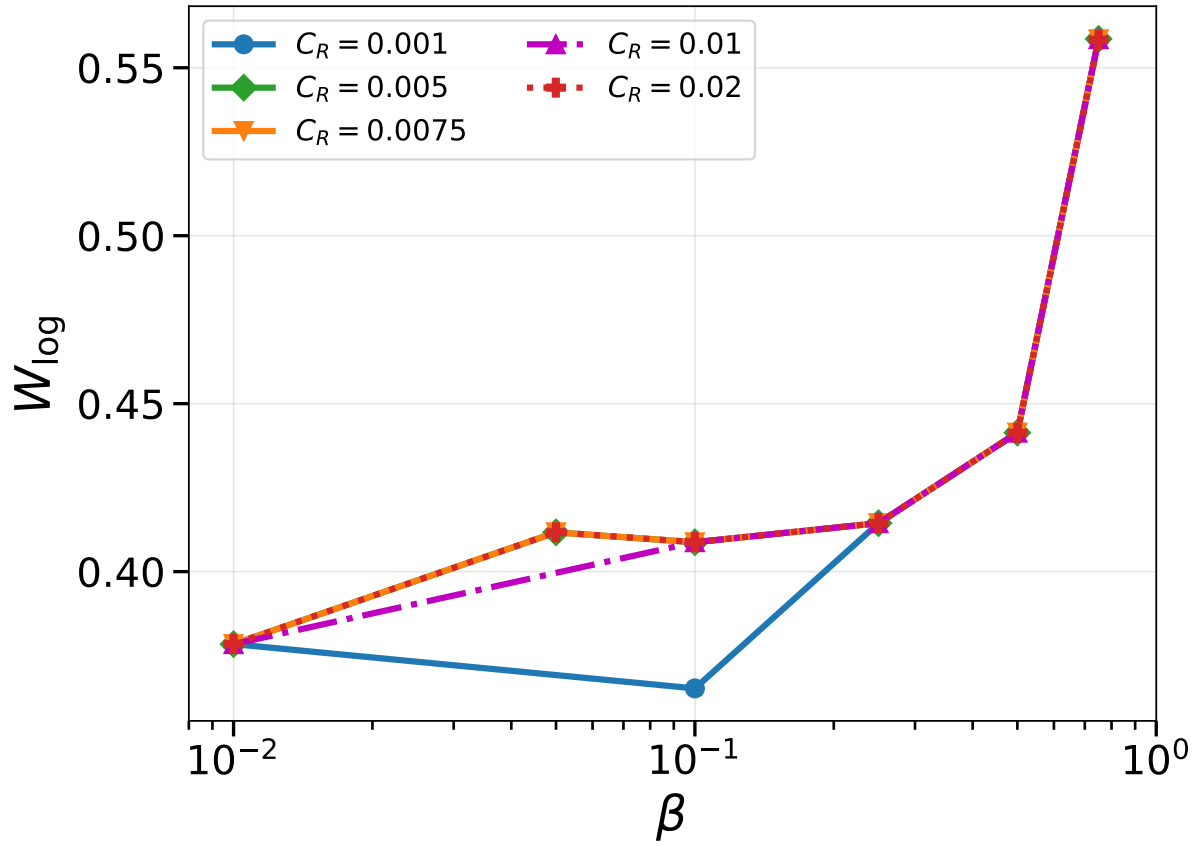


Figure 4.14. Logarithmic width of the redistribution-rate stability window, $W_{\log}$, versus the turbulent spreading parameter β . Curves correspond to different turbulence–flow coupling strengths, C_R . While the dependence on C_R is weak, increasing turbulent spreading significantly broadens the stability window, increasing $W_{\log}$ from approximately 0.35 to 0.55. This behavior suggests that stronger redistribution enhances shear generation and expands the parameter range supporting persistent transport contrast and staircase structures.

coupling substantially broadens the region of persistent staircase states by stabilizing transport contrast against rapid redistribution.

The resulting staircase structures are therefore dynamically maintained rather than constitutively selected. Persistence emerges from the balance between redistribution, damping, and shear regulation, despite the absence of a local transport bifurcation.

4.7 Conclusion

In this work, we investigated staircase formation, persistence, and regulation in reduced flux-driven transport models intended to capture key aspects of core–edge transport dynamics in magnetically confined plasmas. Two models were considered: a transport-only T – K model and a mean-flow extension, the T – K – U model. Both models possess a monotone local flux–gradient relation and therefore differ fundamentally from the BLY model, in which layering originates from a nonmonotone equilibrium flux–gradient relation and an associated local layer-forming instability. Instead, the staircase states studied here arise through the interplay of turbulent spreading, nonlinear damping, and – in the extended model – Reynolds-stress-driven shear generation.

For the T – K model, parameter scans revealed three transport regimes: weak drive, intermediate drive, and strong drive. Weakly driven states relax toward the collisional transport limit, while strongly driven states undergo transport homogenization and staircase steps coalesce. Between these limits lies an intermediate-drive regime in which staircase structures persist for extended times. Across the parameter space explored here, persistent staircase states generally maintain a transport contrast of $\Delta\chi_T/\chi_0 \gtrsim 10^2$, indicating that sustained transport inhomogeneity is a necessary condition for staircase persistence.

To quantify staircase longevity, we introduced the coefficient of variation of the effective diffusivity and used its evolution to define a redistribution rate, γ_{redist} , which characterizes the rate at which turbulent transport reorganizes throughout the domain. This quantity provides a useful dynamical proxy for staircase persistence and reveals a clear tradeoff between confinement enhancement and staircase lifetime. Near $Q_{\text{in}} \approx Q_{\text{coll}}$, the system transitions to a regime of enhanced stored energy. However, increasing stored energy is accompanied by more rapid transport redistribution, ultimately leading to staircase coalescence and the loss of transport contrast.

The near-marginal staircase state was found to be metastable and governed by a com-

petition between redistribution and damping. Most persistent staircase states satisfy $1 \lesssim \gamma_{\text{damp}}/\gamma_{\text{redist}} \lesssim 10$, indicating that long-lived staircase structures require damping and redistribution timescales of comparable magnitude. This quasi-balanced regime defines a narrow window in parameter space where enhanced confinement and sustained transport contrast coexist.

To examine the effects of nonlocal transport, we performed core-temperature sweeps across the near-marginal regime. These experiments revealed finite hysteresis and path-dependent staircase evolution. Both the hysteresis area and directional asymmetry increase with the turbulent spreading parameter β , indicating that stronger turbulent spreading enhances memory of the transport state. The resulting hysteresis behavior further supports the interpretation of the sustained staircase state as a metastable transport configuration that is highly sensitive to deviations from marginality.

We then extended the model to include self-consistent mean-flow generation through Reynolds-stress feedback. The resulting T – K – U model introduces an additional regulation pathway in which turbulent kinetic energy drives mean flows that subsequently shear and regulate the turbulence. A nonmonotone relationship emerges between the shear and redistribution rates, producing a finite stability window in which persistent staircase states are maintained. Long-lived staircase states cluster near $\mathcal{M} \sim 10^{-1}$, suggesting that staircase persistence requires the combined damping and shear rates to exceed the redistribution rate by approximately an order of magnitude. Furthermore, Reynolds-stress-driven shear broadens the stability window and reduces the sensitivity of staircase persistence to variations in damping, thereby improving staircase robustness.

Taken together, these results indicate that staircase persistence in both models is dynamically maintained rather than constitutively selected. Unlike the BLY model, where layering is associated with a local transport instability, the staircase structures studied here emerge from the competition of nonlocal transport processes and the balance of emergent transport timescales. In this sense, the persistent staircase state should be viewed as a metastable self-organized transport configuration rather than the consequence of a local flux–gradient rollover.

Beyond their theoretical implications, the results suggest several diagnostics that may be useful in experiments. The staircase curvature and staircase strength diagnostics introduced here provide macroscopic measures of interface sharpness and profile segmentation that can be obtained directly from mean temperature, density, or pressure profiles. Likewise, measurements of transport contrast, shear rate, damping rate, and transport redistribution may provide experimentally accessible indicators of staircase persistence and proximity to near-marginal operating conditions.

Future work should extend the present model to two dimensions in order to investigate the coupled evolution of transport redistribution, Reynolds stresses, and self-generated shear flows in interacting turbulent eddies. A natural next step is to replace the turbulence-intensity description with a dynamically evolving fluctuation field, allowing the study of spatiotemporal chaos, intermittent transport events [60, 82, 83, 122, 124, 64], and finite-Kubo-number effects on staircase formation and persistence [58]. Such extensions may provide a closer connection between the reduced transport framework developed here and experimentally relevant turbulent transport regimes in magnetically confined plasmas.

Chapter 5

Summary and future research

In this dissertation, we investigated staircase formation, persistence, and transport regulation using a hierarchy of reduced models motivated by transport phenomena in magnetically confined plasmas. The central objective was to identify the minimal physical mechanisms required for layered transport structures to emerge and persist, and to understand how these mechanisms evolve as increasingly realistic forms of feedback and self-organization are introduced. Beginning from passive scalar transport in cellular flows and progressing to active scalar dynamics and reduced flux-driven turbulence models, a common physical picture emerged.

A central result of this work is that staircase formation is fundamentally a consequence of transport inhomogeneity. Across all models considered, layered structures emerge when transport organizes into regions of relatively strong mixing and regions of comparatively weak transport. In the passive scalar system, these regions are separated by the boundaries between convective cells. In the active scalar system, magnetic flux expulsion reinforces these boundaries and further suppresses inter-cell transport. In the flux-driven transport models, transport barriers arise from sustained transport contrast and persist through a balance between transport redistribution, nonlinear damping, and mean-flow shear. Although the underlying mechanisms differ, all cases exhibit the same essential feature: the generation and maintenance of persistent transport inhomogeneity.

The passive scalar studies demonstrated that feedback is not required for staircase

formation. Inhomogeneous mixing alone is sufficient to produce layered structures provided the diffusive transport time across cell boundaries remains long compared to the cell turnover time. Extending the fixed cellular array to a fluctuating vortex array revealed that staircase structures remain resilient over a broad range of modest effective Reynolds numbers. While fluctuations lead to cell mergers and less regular layering, staircase formation persists provided the cellular Peclet number remains large and profile curvature remains sufficiently high. These studies also showed that transport barriers form before homogenization within the cells, suggesting that staircase formation should be viewed primarily as a barrier-formation process rather than as a consequence of homogenization.

The active scalar studies extended this picture by introducing feedback between the transported quantity and the underlying flow. Here, the transported scalar was identified with magnetic potential in a reduced magnetohydrodynamic system. Magnetic flux expulsion concentrates magnetic field at cell boundaries, suppresses transport across interfaces, and strengthens the staircase structure. A notable result is that staircase structures persist not only in the flux-expulsion regime but also in the vortex-disruption regime, despite significant modification of the underlying flow. This persistence arises because vortex disruption does not completely eliminate the cellular transport network. While some cells are strongly distorted or destroyed, a sufficient population of residual cells remains to maintain transport barriers and preserve the layered structure. This highlights an important distinction from previous vortex-disruption studies focused on isolated vortices [96], as staircase persistence in a cellular array depends on the collective transport properties of the remaining cell population rather than the survival of any individual vortex. More generally, these studies established that feedback is not necessary for staircase formation but can substantially enhance staircase resiliency by increasing the disparity between local mixing and inter-cell transport.

The passive and active scalar results were subsequently unified within a common transport framework. Random-walk models of multi-scatterer transport were shown to capture the essential physics governing barrier crossing and staircase persistence. Within this framework, the cellular

Peclet number emerges as a microscopic measure of transport suppression, while profile curvature provides a macroscopic measure of staircase integrity. Together, these quantities provide a practical set of figures of merit for characterizing layered transport states. The unified framework further clarifies how fluctuations, stochastic forcing, and feedback modify transport barriers and regulate staircase morphology across a broad class of systems.

The final stage of this work revealed a distinct route to staircase persistence in turbulent transport systems. Reduced flux-driven models for the coupled evolution of temperature, turbulent kinetic energy, and mean flow were shown to support long-lived staircase states despite possessing a monotone local flux–gradient relation. Unlike classical staircase theories [38], these systems do not admit a local layer-forming instability. Instead, staircase persistence emerges through self-organization near marginality and is maintained by the competition between transport redistribution, nonlinear damping, and Reynolds-stress-driven shear generation. The analysis identified a metastable regime in which these competing processes remain comparable and sustained transport contrast is preserved. The introduction of mean-flow feedback broadens this regime, improves staircase resilience, and generates a finite stability window for long-lived transport barriers. These results demonstrate that staircase persistence need not originate from a local transport instability, but can instead emerge dynamically through the balance of nonlocal transport processes and emergent transport timescales.

Taken together, the studies presented here suggest two complementary routes to staircase persistence. In passive and active scalar systems, persistence follows from transport inhomogeneity generated by the separation between rapid local mixing and slow barrier-crossing transport. In near-marginal turbulent transport systems, persistence emerges through a balance between redistribution, damping, and shear regulation. Despite their differences, both mechanisms generate long-lived transport barriers and sustained transport contrast. Viewed from this perspective, staircase structures are not tied to a particular instability or transport model, but instead represent a broader class of self-organized transport states maintained by dynamical constraints on mixing and transport.

Beyond their relevance to magnetic confinement fusion, the results presented here connect to a broader class of layered transport phenomena observed in geophysical fluids, stratified turbulence, and other systems exhibiting self-organization. The reduced models developed throughout this dissertation provide a physically transparent foundation for understanding how transport barriers form, evolve, and persist. They also suggest experimentally accessible diagnostics, including profile curvature, transport contrast, and characteristic transport timescales, that may be used to identify and quantify layered transport states in more complex systems.

Several directions for future research naturally follow from this work. A particularly important extension is the generalization of the reduced flux-driven transport models to two dimensions. Such an approach would allow the simultaneous evolution of transport redistribution, Reynolds stresses, self-generated shear flows, and fluctuating eddy structures within a single framework. By incorporating dynamically evolving vortex arrays similar to those studied in the passive and active scalar systems, it may be possible to investigate how finite Kubo number effects, turbulent spreading, and coherent transport events influence staircase formation and persistence near marginality. Incorporating dynamically evolving fluctuation fields would also provide a closer connection to intermittent transport events, avalanches, and spatiotemporal chaos. Finally, direct comparisons with experimental measurements and gyrokinetic simulations may help establish whether the timescale balances identified here provide a useful framework for interpreting staircase formation and transport regulation in magnetically confined plasmas.

More broadly, the results of this dissertation suggest that staircase formation and persistence can be understood within a unified theoretical foundation based on transport inhomogeneity and the dynamical processes that sustain it, including feedback and timescale competition. By isolating these mechanisms in a hierarchy of reduced physical models, this work provides a foundation for interpreting layered transport structures in fusion plasmas and offers a pathway toward understanding how self-organized transport barriers may contribute to improved confinement in future magnetic fusion devices.

Chapter 6

Appendices

6.1

Magnetic Confinement Motivation

A current subject of interest in magnetic confinement is the $\mathbf{E} \times \mathbf{B}$ staircase [33, 1, 3]. Non-equilibrium, near marginal plasmas can sometimes naturally evolve towards a globally organized critical state of spatially segregated micro barriers, interspersed between sectors of strong, avalanche-like transport [110, 125, 27, 116]. The regime of “near marginality” is a relevant operating regime for present and future magnetic confinement devices. Here ‘marginality’ refers to the normalized temperature gradient for which the turbulent heat flux exceeds neoclassical heat flux [2].

Evidence of the $\mathbf{E} \times \mathbf{B}$ staircase has been observed in current fusion devices [119, 55, 4, 113] and simulations [114, 54, 118]. In simulations, the pressure profile plot shows the coexistence of profile flat spots and strong shear layers, which act as mini-barriers. The shear layers correspond to strongly localized temperature gradients. Avalanches occur in flat spots between shear layers and zonal flows (ZF) occur in the areas of mini-barriers. Note that ZF formation is the result of nonlocal interactions (in k -space) between unstable drift-wave modes. ZF are understood to regulate turbulence by “eddy shearing”.

Two ideas from self-organization that can explain the $\mathbf{E} \times \mathbf{B}$ staircase are $\mathbf{E} \times \mathbf{B}$ shear feedback and jams. The shear feedback process involves the zonal shears, turbulence, and their

mutual interaction by modulation processes. It is useful to view drift-wave turbulence and ZF as separate populations which interact via a “predator-prey” feedback loop: the drift waves (the prey) grow due to the gradient (instability) drive, while the ZF (the predator) “feed” upon the drift wave population by Reynolds stresses. For the case of jams, it is useful to draw inspiration from traffic flow theory, where time delay is the key element [55, 126, 56]. In traffic flow, a driver’s prompt (i.e., short) reaction maintains smooth traffic flow, while longer driver reaction times trigger jams [106]. In the case of flux-driven turbulence, heat flux jams occur when there is sufficient time delay between temperature modulations and local heat flux. This leads to the growth of patterns that resemble shock trains, from which develop the staircase ‘jumps’. Jumps here are simply the outcome of steepening the temperature gradient.

6.2

Derivation of Reynolds number and Vortex Equation

The forced incompressible Navier-Stokes equation in 2D is of the form,

$$\frac{d\mathbf{u}'}{dt'} + \frac{\nabla' p'}{\rho} = \nu \nabla'^2 \mathbf{u}' - \alpha' \mathbf{u}' + F_{\text{amp}} (\sin ky' \hat{x} - \sin kx' \hat{y}), \quad (6.1)$$

where α' is the Ekman-friction and ν the kinematic viscosity. Here the density ρ is set to 1. This system has the solution,

$$\mathbf{u}' = \frac{F_{\text{amp}}}{k^2 \nu} (\sin ky' \hat{x} - \sin kx' \hat{y}). \quad (6.2)$$

To derive the Reynolds number for the forced NS equation, we use the maximum speed of Eq. 6.2,

$$U_0 = \frac{F_{\text{amp}}}{k^2 \nu}. \quad (6.3)$$

For computational purposes, we chose the following normalization for position and time

$$\mathbf{x} = \frac{k}{n} \mathbf{x}', \quad (6.4)$$

$$t = \frac{F_{\text{amp}}}{nk\nu} t', \quad (6.5)$$

where n is the spatial frequency of the forcing. We can then normalize and take the curl of Eq. 6.1 to derive the vorticity equation,

$$\frac{d}{dt} \omega = \frac{1}{\Omega} \nabla^2 \omega - \alpha \omega + F_\omega. \quad (6.6)$$

Here Ω represents the Reynolds number and α the non-dimensional Ekman friction. These two control parameters are defined by the following quantities:

$$\Omega = \frac{nF_{\text{amp}}}{k^3 \nu^2}, \quad (6.7)$$

$$\alpha = \frac{n\nu k \alpha'}{F_{\text{amp}}}. \quad (6.8)$$

The forcing function F_ω is defined as

$$F_\omega = -\frac{n^3}{\Omega} (\cos nx + \cos ny). \quad (6.9)$$

A detailed analysis on Eq. 6.1 can be found in [99].

6.3

Effective Diffusivity (D^*) Calculation

We compute the value of D^* using

$$\langle \Gamma \rangle = -D^* \nabla \langle n \rangle_y. \quad (6.10)$$

The left-hand side (LHS) represents the flux of scalar concentration through the region between $x_2 = (3/4)L$ and $x_1 = (1/4)L$ at different points in time, as illustrated in Fig. 1.31. Thus, we have the following equation

$$\langle \Gamma \rangle = \frac{\Delta x}{\Delta t} (n_{\text{tot}, t_f}(x = [x_1, x_2]) - n_{\text{tot}, t_i}(x = [x_1, x_2])), \quad (6.11)$$

where Δx represents the difference between x_2 and x_1 . Note that Δt represents 8.3 diffusive cycles. Here, we define the diffusive time scale (τ_D) with length scale $d = \pi/3$. For the right-hand side (RHS) of Eq. (6.10), we use Eq. (1.10) to evaluate the value of $\langle n \rangle_y$ at the two points x_2 and x_1 at time t_f ,

$$\nabla \langle n \rangle_y = \frac{\langle n_{t_f}(x_2) \rangle_y - \langle n_{t_f}(x_1) \rangle_y}{\Delta x}. \quad (6.12)$$

By performing these calculations, we can compute a numerical value for D^* . Note that for the oblique staircase profiles, we replace the length L with L' .

6.4 Active Scalar Problem

For the active scalar problem, we convert the passive scalar (n) to an active scalar (A),

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) A = \eta \nabla^2 A, \quad (6.13)$$

which will result in a system isomorphic to 2D MHD. Such systems are used to study flux expulsion [97, 98] and vortex bursting [96]. Here, the active scalar is coupled to the vorticity

equation, thus creating feedback. The governing vorticity equation will be of the form,

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \omega = \frac{1}{\Omega} \nabla^2 \omega + \boxed{\mathbf{B} \cdot \nabla \nabla^2 A} + F_\omega - \alpha \omega, \quad (6.14)$$

$$\mathbf{B} = \hat{\mathbf{z}} \times \nabla A. \quad (6.15)$$

We box the term, which couples the vorticity equation and Eq. (6.13). For the active scalar, we consider an initial profile $A = B_0 x$, which results in a vertical magnetic field $B = B_0 \hat{\mathbf{y}}$. Here, the value of B_0 plays a crucial role in how the active scalar impacts the dynamics of the flow. Depending on the magnitude of B_0 , the flow can be in a kinematic or dynamic regime. The regime is determined by

$$M = \left(\frac{v_A}{U_0}\right)^2 R_m. \quad (6.16)$$

The terms v_A and U_0 represent the Alfvén speed and the characteristic flow speed, respectively. Here R_m represents the magnetic Reynolds number. Note that v_A is proportional to B_0 , thus the strength of B_0 will determine the dynamics of the flow. First, we will discuss $M < 1$, the flux expulsion regime. For sufficiently small values of B_0 (i.e., $v_A^2 < U_0^2/R_m$), the vortex will expel magnetic field lines to the vortex boundary. Due to magnetic field expulsion, the boundary layer of a cell will be reinforced, thus maintaining the cellular structure. In our cellular array model, we expect that the magnetic field will support the cell array structure when fluctuations are present (i.e., B_0 will elasticize the cell array). Since magnetic field lines are expelled to the boundaries, the active scalar essentially becomes homogenized ($\nabla A = 0$) inside the vortices, thus resulting in a staircase structure. Of course, unless the magnetic field is forced, the structure is only temporary since the magnetic field eventually decays in 2D.

The other case is the vortex bursting regime ($M > 1$). If B_0 is too large (i.e., $v_A^2 > U_0^2/R_m$), vortices will be severely disrupted. In this scenario, Alfvénization takes place. Alfvénization is the conversion of hydrodynamic turbulence to Alfvén wave turbulence. Thus, we expect that the

formation of a staircase structure will not be possible. Until now, the problem of flux expulsion and vortex bursting has been addressed only in the context of a single eddy. To expand on this problem, we can use the setup of the vortex array, but now the vortices will be threaded by a magnetic field. In our upcoming study, we will address (through scans of B_0) what occurs to staircase dynamics. As magnetic fields get threaded around vortices, will this result in a more robust staircase? For what values of B_0 will the staircase structure no longer form? These are questions we plan to answer in the next phase of research.

6.5 Staircases in magnetic confinement physics

In magnetically confined plasmas, the study of mesoscopic phenomena in drift-wave turbulence has transitioned, into a study of pattern formation problems. The focus is on understanding the interaction between zonal flows [2] and avalanches [25]. This interaction creates the $\mathbf{E} \times \mathbf{B}$ staircase [33], characterized by a quasiperiodic pattern of mean field and fluctuation intensity. Both simulations and experimental data confirm the existence of this staircase structure in pressure and temperature profiles [1, 35]. Corrugations, marked by strong localized temperature gradients, form a persistent layer of localized shear, acting as a permeable transport barrier. Thus, $\mathbf{E} \times \mathbf{B}$ shear layer feedback regulates avalanches, which can promote mixing [3].

In the plasma community, several explanations have been proposed to elucidate the phenomenon of layering in magnetic confinement. One common approach posits that the formation process arises from scale selection. In this context, inhomogeneous mixing is attributed to the interplay of mixing on two length scales [38]. Staircase formation hinges on bi-stability, of the flux, resulting from the interaction between an excitation scale and an emergent scale [39, 40], which depends on the gradient and intensity of the transported quantity. Notably, the emergent scale for drift-wave turbulence is related to the Rhines scale [42], defined by the cross-over between the wave interaction mismatch frequency and the decorrelation rate characteristic of the inverse cascade. The interaction between these mixing scales triggers the self-sharpening of

modulations, leading to staircase structure formation. Another approach suggests that staircase structures emerge due to a time delay between temperature modulations and local heat flux, which lead to flux ‘jams’ [55]. In this model, heat flux jams form and foster the growth of gradient steepening layers that shape the staircase “jumps”. For an in-depth exploration of these two approaches, refer to the corresponding literature [41, 34, 57, 126, 106].

Many (but not all) properties of marginally overlapping cells can be found or correlated with features in magnetic confinement devices. For example, convection cells in the array are similar to $\mathbf{E} \times \mathbf{B}$ convection cells induced by density and potential gradients, while zonal flows represent the boundary layer between cells. Background diffusion within the array mirrors ambient collisional diffusion, and the scalar field is as turbulence intensity. The effective Reynolds numbers explored in this paper can be related to those in drift-wave turbulence. The ratio of inverse correlation time to the Landau damping rate (i.e., nonlinear rate to the linear rate) may be used as a surrogate for the Reynolds number. Given the numerous characteristics in common between these systems, investigating passive and active scalar dynamics in a setting where layering naturally emerges due to differing mixing timescales is a study of interest.

6.6 2D MHD basics

In 2D MHD, a background magnetic field can lead to either the flux expulsion (kinematic) or vortex disruption (dynamic) regime. These regimes are determined by Σ as discussed in Sec. 2.1. With fixed characteristic flow speed (U_0) and magnetic Reynolds number fixed (R_m), the strength of the background magnetic field (B_0) determines the regime. In scenarios where B_0 is sufficiently low (specifically, where $v_A^2 < U_0^2/R_m$), vortices expel magnetic field lines towards their periphery on a timescale of:

$$\tau_f = \tau_\eta^{1/3} \tau_H^{2/3}. \quad (6.17)$$

Here, τ_η and τ_H represent the magnetic diffusivity timescale and cell circulation timescale, respectively. Note that Eq. 6.17, requires that R_m to be large in order for there to be a distinction between the flux expulsion and cell turnover timescale. During this expulsion process, the active scalar (magnetic potential) tends towards homogenization (i.e., $\nabla A = 0$) within the vortices, forming a magnetic potential staircase step. In the flux expulsion regime, the ratio of boundary layer width (“jump”) to cell length (“step”) satisfies,

$$\frac{\delta}{d} \sim R_m^{-1/3}. \quad (6.18)$$

For a detailed derivation of the resulting $R_m^{1/3}$ scaling, see [97, 98, 70]. Utilizing magnetic flux conservation, we can alternatively express the scaling as the ratio of the magnetic field strength at the boundary to the background magnetic field as:

$$\frac{B_0}{b} \sim R_m^{-1/3}. \quad (6.19)$$

Given that $b > B_0$ for large magnetic Reynolds numbers, as the field is expelled from the cells, the gradient of the active scalar at the cell boundary sharpens. The dynamics of active scalar staircase formation in this regime, initialized as $A = B_0 x$, are shown in Fig. 6.1.

The other case of interest is the cellular disruption regime, where large B_0 values ($v_A^2 > U_0^2/R_m$) lead to significant disruption of vortices [96]. This disruption triggers Alfvénization, transforming hydrodynamic cells into Alfvén wave turbulence. The disruption of vortices can affect inhomogeneous mixing, as intact cell boundaries are essential for this and for forming the staircase structure, as found in studies of passive scalar layering in a FVA. We will see however, that some closed streamlines survive for $\Sigma > 1$.

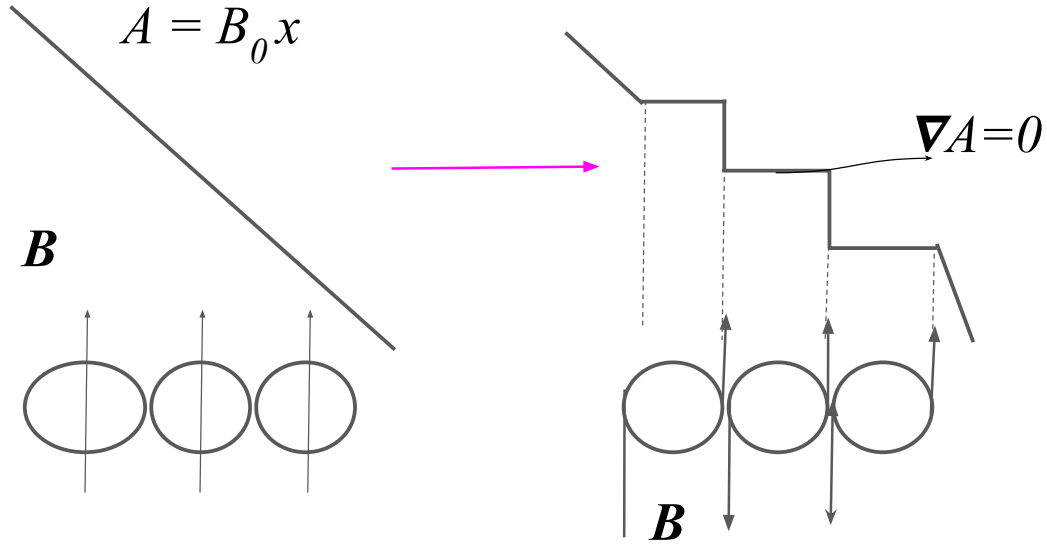


Figure 6.1. Consider an initial magnetic potential profile which is linear. A vortex array will homogenize A within cells, thus expelling magnetic field to the inter-cell boundaries, and so also producing a staircase in A .

6.7 Boundary layer scales and transport

To derive a scaling of scattering length to cell size as depicted in Fig. 2.4, consider the following inequality

$$\langle \delta x^2 \rangle < d^2, \quad (6.20)$$

where $\langle \delta x^2 \rangle$ represents the mean square extent of the boundary layer. The extent of the layer can also be expressed as

$$\delta x = \delta x_T + \delta x_M, \quad (6.21)$$

where δx_T and δx_M represent the turbulent scattering and molecular diffusion scattering step, respectively. We assume no cross-correlation between the two scattering steps – caused by

turbulence and collisions, respectively (i.e., $\langle \delta x_T \delta x_M \rangle = 0$).

For a fixed cellular array, we can derive a scaling by breaking down the mean square displacements into integrals involving velocity correlations, which simplify into diffusion coefficients

$$\langle \delta x^2 \rangle = \langle \delta x_T^2 \rangle + \langle \delta x_M^2 \rangle \quad (6.22)$$

$$= \int^t dt' \int^t dt'' \langle v_T(t') v_{-T}(t'') \rangle + \int^t dt' \int^t dt'' \langle v_c(t') v_c(t'') \rangle \quad (6.23)$$

$$= \int^t dt' \tau_{c_T} \langle \tilde{v}_k^2 \rangle + \int^t dt' D \quad (6.24)$$

$$= (D_T + D)t, \quad (6.25)$$

Here, $t = d/U_0$ and τ_{c_T} represent the transit time through the layer and the exponential decay characteristic time of $\langle v_T(t') v_{-T}(t'') \rangle$. The two diffusive terms, D_T and D , represent the turbulent diffusivity and the collisional diffusivity. Therefore, the scaling is

$$\frac{\langle \delta x^2 \rangle}{d^2} = \frac{D_T}{U_0 d} + \frac{1}{\text{Pe}}, \quad (6.26)$$

where $\text{Pe} = U_0 d/D$. In the case where $D_T = 0$, the scaling is determined by Pe alone [46, 48].

We can approximate when turbulence will “matter” by taking $\langle \delta x^2 \rangle = d^2$. For $\text{Pe} \gg 1$, turbulence matters when:

$$D_T \gtrsim \left(1 - \frac{1}{\text{Pe}}\right) U_0 d. \quad (6.27)$$

Strong turbulence requires that $D_T \gg U_0 d$. This result indicates that to maintain the staircase structure, we need $D_T < U_0 d$ when turbulent fluctuations are present.

Now, consider how a magnetic field influences the mean square scattering length ($\langle \delta x^2 \rangle_A$)

when threaded through a vortex array. This situation changes the scaling of the scattering length:

$$\frac{d}{dt}\langle\delta x^2\rangle_A \approx -\frac{\langle\tilde{v}_r\tilde{A}\rangle}{\nabla\langle A\rangle} + \eta \quad (6.28)$$

$$= \frac{\langle\tilde{v}_r^2\rangle\tau_c}{1 + R_m v_A^2/\langle\tilde{v}^2\rangle} + \eta \quad (6.29)$$

$$= \frac{D_T}{1 + R_m v_A^2/\langle\tilde{v}^2\rangle} + \eta. \quad (6.30)$$

Integrating with respect to time, dividing by d^2 , and using the transit time through the layer as the timescale, we get

$$\frac{\langle\delta x^2\rangle_A}{d^2} = \frac{D_T/U_0 d}{1 + R_m v_A^2/\langle\tilde{v}^2\rangle} + \frac{\eta}{DPe}. \quad (6.31)$$

Since $1 + R_m v_A^2/\langle\tilde{v}^2\rangle > 1$, turbulent broadening is weaker for the magnetic/active case, so cellular interfaces are sharper. For flux expulsion, where $b \rightarrow R_m^{1/3} B_0$ in the boundary layer, the scaling can be written as

$$\frac{\langle\delta x^2\rangle}{d^2} \sim \frac{D_T/U_0 d}{1 + R_m^{5/3} v_A^2/\langle\tilde{v}^2\rangle}. \quad (6.32)$$

Here, **B** weakens transport through the boundary layer, thus maintains layering. In the vortex disruption regime (i.e., $R_m v_A^2/\langle\tilde{v}^2\rangle > 1$), we have:

$$\frac{\langle\delta x^2\rangle}{d^2} \sim \frac{D_T/U_0 d}{R_m^{5/3} v_A^2/\langle\tilde{v}^2\rangle}, \quad (6.33)$$

Again, **B** weakens transport through the boundary layer. As a figure of merit, $\eta_T \sim D_T/1 + R_m v_A^2/\langle\tilde{v}^2\rangle \sim D_T/(v_A^2 \tau_c/\eta) \sim \eta \langle\tilde{v}^2\rangle/v_A^2$, for $R_m v_A^2/\langle\tilde{v}^2\rangle > 1$. This allows increased $\langle\tilde{v}^2\rangle$ for the same η_T and $\langle\delta x^2\rangle_A/d^2$, provided v_A^2 is also larger. Thus, B_0 reinforces staircase structure to array fluctuations. Alternatively, increased B_0 reduces $\langle\delta x^2\rangle_A/d^2$ and so sharpens inter-cell transport barriers.

6.8 Equilibrium flux–gradient relations

We derive the local equilibrium flux–gradient relations for the BLY model and for the reduced flux-driven T – K model studied in this work. This comparison clarifies an important distinction between the two models: in BLY, staircase formation is associated with a nonmonotone equilibrium flux–gradient relation, whereas in the present model the local flux–gradient relation is monotone. Thus, staircase formation in the present system must arise from nonlocal redistribution and spatially inhomogeneous turbulent transport, rather than from a local negative-diffusivity instability.

6.8.1 BLY equilibrium flux–gradient relation

Consider the nondimensional BLY equipartition model for the mean buoyancy gradient

$$g = \partial_z b \tag{6.34}$$

and turbulent kinetic energy density e :

$$\partial_t g = \partial_z^2 \left(l e^{1/2} g \right), \tag{6.35}$$

$$\partial_t e = \beta \partial_z \left(l e^{1/2} \partial_z e \right) - l e^{1/2} g + \varepsilon l^{-1} (1 - e) e^{1/2}, \tag{6.36}$$

where the mixing length is

$$l = \frac{e^{1/2}}{(e + g)^{1/2}}. \tag{6.37}$$

For a spatially uniform steady state,

$$g = \text{const.}, \quad e = \text{const.}, \tag{6.38}$$

the diffusive spreading term in the e equation vanishes. The steady local energy balance is therefore

$$0 = -le^{1/2}g + \varepsilon l^{-1}(1-e)e^{1/2}. \quad (6.39)$$

Substituting the mixing length gives

$$0 = -\frac{eg}{(e+g)^{1/2}} + \varepsilon(e+g)^{1/2}(1-e). \quad (6.40)$$

Thus,

$$eg = \varepsilon(e+g)(1-e). \quad (6.41)$$

Defining

$$r = \frac{1}{\varepsilon}, \quad (6.42)$$

this becomes

$$reg = (e+g)(1-e). \quad (6.43)$$

Solving this quadratic equation for the physical positive branch of e gives

$$E(g) = \frac{1}{2} \left[1 - g(1+r) + \sqrt{(1-g(1+r))^2 + 4g} \right]. \quad (6.44)$$

The equilibrium buoyancy flux is

$$F(g) = le^{1/2}g. \quad (6.45)$$

Using the same expression for l , the BLY equilibrium flux–gradient relation becomes

$$F(g) = \frac{E(g)g}{\sqrt{E(g)+g}}. \quad (6.46)$$

For sufficiently large r , specifically

$$r > 7 + 4\sqrt{3}, \quad (6.47)$$

the function $F(g)$ becomes nonmonotone. Hence, there exists a finite interval of buoyancy gradients for which

$$F'(g) < 0. \quad (6.48)$$

In this range, a local increase in the mean gradient reduces the turbulent flux, producing an effective negative diffusivity. This is the local layer-forming instability in BLY.

6.8.2 Local flux–gradient relation for the present model

We now derive the corresponding local flux–gradient relation for the reduced flux-driven T – K model used in the present work. Let

$$G = |\partial_x T| \quad (6.49)$$

denote the magnitude of the mean temperature gradient. The turbulent diffusivity is taken to be

$$\chi_T = \tau_{ac} K, \quad (6.50)$$

where K is the turbulent kinetic energy density and τ_{ac} is the turbulent autocorrelation time.

To isolate the local equilibrium flux–gradient relation, we neglect turbulent spreading

and consider the local steady balance between turbulent production and nonlinear damping in the K equation:

$$0 = g_0 \chi_T G - K_c K^{3/2}. \quad (6.51)$$

Using $\chi_T = \tau_{ac} K$, this becomes

$$g_0 \tau_{ac} K G = K_c K^{3/2}. \quad (6.52)$$

For the nontrivial branch $K > 0$, division by K gives

$$K^{1/2} = \frac{g_0 \tau_{ac}}{K_c} G. \quad (6.53)$$

Therefore,

$$K = \left(\frac{g_0 \tau_{ac}}{K_c} \right)^2 G^2. \quad (6.54)$$

The turbulent diffusivity is then

$$\chi_T = \tau_{ac} K = \tau_{ac} \left(\frac{g_0 \tau_{ac}}{K_c} \right)^2 G^2. \quad (6.55)$$

The total heat flux magnitude is

$$Q(G) = (\chi_0 + \chi_T) G, \quad (6.56)$$

where χ_0 is the background collisional diffusivity. Substituting the expression for χ_T gives

$$Q(G) = \chi_0 G + \tau_{ac} \left(\frac{g_0 \tau_{ac}}{K_c} \right)^2 G^3. \quad (6.57)$$

Equivalently,

$$Q(G) = \chi_0 G + CG^3, \quad (6.58)$$

$$C = \tau_{ac} \left(\frac{g_0 \tau_{ac}}{K_c} \right)^2 > 0. \quad (6.59)$$

Thus,

$$\frac{dQ}{dG} = \chi_0 + 3CG^2 > 0. \quad (6.60)$$

The local equilibrium flux–gradient relation in the present model is therefore monotone for all G . Consequently, the staircase states studied here do not arise from a local negative-diffusivity interval analogous to the BLY mechanism. Instead, they result from the coupled evolution of turbulent transport, flux-driven profile adjustment, and nonlocal redistribution of turbulent intensity.

6.9 T – K – U equilibrium flux-gradient relation

We consider the local homogeneous limit of the T – K – U model. In the numerical model, the turbulent diffusivity is

$$\chi_T = \tau_{ac} K, \quad \chi = \chi_0 + \chi_T. \quad (6.61)$$

The temperature equation is

$$\partial_t T = \partial_x (\chi \partial_x T). \quad (6.62)$$

Since the kinetic-energy production term in the code is

$$-g\chi_T\partial_x T, \quad (6.63)$$

we define the positive imposed gradient as

$$G \equiv -\partial_x T. \quad (6.64)$$

Then the heat flux magnitude is

$$Q = \chi G = (\chi_0 + \tau_{ac}K) G. \quad (6.65)$$

The Reynolds stress is defined in the code as

$$R_{xy} = C_R \left(\frac{k_x k_y}{k_x^2 + k_y^2} \right) K, \quad (6.66)$$

where k_y is constant and k_x evolves according to the eikonal equation. The mean shear is

$$S = \partial_x U. \quad (6.67)$$

The Reynolds-stress energy transfer is

$$\Gamma = R_{xy}S. \quad (6.68)$$

Thus the local kinetic-energy balance is

$$0 = g\tau_{ac}KG - K_c K^{3/2} - R_{xy}S. \quad (6.69)$$

The mean-flow equation is

$$\partial_t U = -\partial_x R_{xy} - \mu_U U + \nu_U \partial_x^2 U. \quad (6.70)$$

In an exactly spatially homogeneous state, $\partial_x R_{xy} = 0$, so no mean shear is generated. Therefore, to estimate the effect of the mean-flow feedback on the local flux-gradient relation, we use a finite- q local amplitude estimate,

$$\partial_x U \sim qU, \quad \partial_x^2 U \sim -q^2 U, \quad \partial_x R_{xy} \sim qR_{xy}. \quad (6.71)$$

The steady mean-flow balance gives

$$(\mu_U + \nu_U q^2) U \sim -qR_{xy}, \quad (6.72)$$

and therefore

$$S = \partial_x U \sim -\frac{q^2}{\mu_U + \nu_U q^2} R_{xy}. \quad (6.73)$$

The sign depends on the phase relation between R_{xy} and U . For the energy-transfer estimate, we take the case in which the Reynolds stress transfers energy from turbulence into the mean flow, so that

$$R_{xy} S \sim \lambda_U K^2, \quad (6.74)$$

with

$$\lambda_U = C_R^2 \frac{q^2}{\mu_U + \nu_U q^2} \left(\frac{k_x k_y}{k_x^2 + k_y^2} \right)^2. \quad (6.75)$$

For mean-flow generation acting as a sink of turbulent kinetic energy, $\lambda_U > 0$.

Substituting this estimate into (6.69) gives

$$0 = g\tau_{ac}KG - K_cK^{3/2} - \lambda_UK^2. \quad (6.76)$$

For the nontrivial branch $K > 0$, division by K gives

$$g\tau_{ac}G = K_cK^{1/2} + \lambda_UK. \quad (6.77)$$

Hence the homogeneous fixed-flux-gradient relation can be written parametrically as

$$K(G) = \left[\frac{-K_c + \sqrt{K_c^2 + 4\lambda_Ug\tau_{ac}G}}{2\lambda_U} \right]^2, \quad (6.78)$$

where the positive root is chosen because $K^{1/2} \geq 0$. The resulting flux-gradient relation is

$$Q(G) = \chi_0G + \tau_{ac}G \left[\frac{-K_c + \sqrt{K_c^2 + 4\lambda_Ug\tau_{ac}G}}{2\lambda_U} \right]^2. \quad (6.79)$$

For fixed positive λ_U , the relation is monotone

$$\frac{dQ}{dG} > 0. \quad (6.80)$$

Therefore, the homogeneous T – K – U flux-gradient relation remains monotone when the Reynolds stress transfer acts as a turbulent-energy sink.

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