Turbulent Equipartition Pinch of Toroidal Momentum in Spherical Torus

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Abstract

We present a new analytic expression for turbulent equipartition (TEP) pinch of toroidal angular momentum originating from magnetic field inhomogeneity of spherical torus (ST) plasmas. Starting from a conservative modern nonlinear gyrokinetic equation [Hahm et al., 1988 Phys. Fluids 31, 2670], we derive an expression for pinch to momentum diffusivity ratio without using a usual tokamak approximation of $B \propto 1/R$ which has been previously employed for TEP momentum pinch derivation in tokamaks [Hahm et al., 2007 Phys. Plasmas 14, 072302]. Our new formula is evaluated for model equilibria of National Spherical Torus eXperiment (NSTX) [M.Ono et al., 2001 Nuclear Fusion 41, 1435] and Versatile Experiment Spherical Torus (VEST) [K.J. Chung et al., 2013 Plasma Sci. Technol. 15, 244] plasmas. Our result predicts stronger inward pinch for both cases, as compared to the prediction based on the tokamak formula.

Contents

Turbulence-driven momentum transport in fusion plasma is a subject of both scientific and practical importance. Radial flux of angular momentum can be decomposed into three parts [1].

$$\Gamma_\phi \propto -\chi_\phi \frac{\partial u_\phi}{\partial r} + V_{pinch} u_\phi + \Pi_{rs}$$

The first term corresponds to the usual diffusive flux governed by momentum diffusivity, χ_ϕ. The last term corresponds to residual stress Π_{rs}. This piece is essential in explaining intrinsic toroidal rotation in the absence of external torque input observed in many tokamak experiments [2, 3, 4, 5, 6, 7, 8, 9]. This encompasses complex and rich set of phenomena and stimulated a considerable amount of theoretical research. A review article focusing on this topic has been recently published [10]. Finally, the second term is convective flux associated with momentum pinch, V_{pinch}. This term plays a crucial role in determining radial profile of the toroidal rotation. There is accumulating evidence of momentum pinch from many tokamak and spherical torus experiments [11, 12, 13, 14, 15].

Following analytic works on the Turbulent Equipartition Pinch (TEP) [16, 17, 18] and pinch driven by fluid ion temperature gradient mode [19], there have been numerous, mostly numerical, theory publications over the years. Review articles focusing on this subject cover progress mostly in local simulations [20] and their validations against experiments [21]. In addition, it's worthwhile to note a steady progress in more challenging global nonlinear simulations of momentum pinch [22, 23, 24, 25]. In this work, we extend the TEP momentum pinch theory to high-β, low aspect ratio plasmas encountered in spherical torus.

Momentum pinch can come from various physical mechanisms including magnetic field inhomogeneity and wave-particle resonant interaction. It can be classified into two parts. The first part is the turbulent equipartition part [16]. Like TEP pinch of particle [26, 27, 28, 29], this comes from the compressibility of $E \times B$ perturbed velocity in toroidal geometry. We note that TEP pinch theory of particle has been extended to ST geometry [30]. Its underlying physics is relatively simple and robust. Its magnitude and scaling are mode-independent as long as an electrostatic turbulence is present. A simple derivation is based on the conservation of angular momentum density, but can be traced back to a formal derivation [16] from the nonlinear gyrokinetic equation [31].

The second part is the thermoelectric pinch. Unlike TEP pinch which is mode-independent, a specification of electron dynamics is necessary for calculation of thermoelectric pinch. It is difficult to make an analytic estimation of the thermoelectric (THE) pinch in general, but progress has been made for pure ITG instability assuming an adiabatic (Boltzmann) response of electrons for both fluid regime [19] and kinetic regime [32]. In the context of gyrofluid theory for ions, simple analytic expression of momentum pinch for pure ITG instability has been derived in [19], and can be decomposed into TEP part and thermoelectric (THE) part. Even with gyrofluid description for ions, a derivation of an analytic expression for trapped electron mode (TEM) can be very complicated due to an inclusion of nonadiabatic trapped electron response. At least, a procedure for that calculation and a rough estimation of THE pinch for collisionless trapped electron mode (CTEM) have been presented in [1]. For CTEM, THE pinch should be subdominant to TEP pinch if the gyrofluid ion response is used.

While validations of pinch theory against experimental results [12] were actively performed with semi-quantitative success on spherical torus such as NSTX [33, 34], most theoretical derivations [16, 19] used low-β and high aspect ratio approximations. In this work, we extend the TEP momentum pinch theory [16] to high-β and lower aspect ratio plasmas encountered in ST's. Although a complete pinch formula must include both the TEP part and thermoelectric part for detailed comparisons to experiments, that requires a labor-intensive case-by-case numerical work. Furthermore, several instabilities can be at work in ST plasmas depending on parameter regimes [34], so that it is not easy to figure out a robust trend. Therefore, it is worthwhile to develop an analytic theory for the TEP part of momentum pinch which is mode-independent, with a relatively robust common element.

The TEP momentum pinch originates from the fact that magnetic curvature can modify acceleration of ions along the magnetic field, as can be appreciated from the modern nonlinear gyrokinetic equation [31]. When the magnetic curvature, $(\mathbf{b}\cdot\nabla)\mathbf{b}$, changes its direction as one moves from the low B field (bad curvature) side to the high B field (good curvature) side, the variation of fluctuation amplitude along the magnetic field (a property of ballooning fluctuations in toroidal geometry) can yield a net acceleration. This symmetry breaking mechanism due to magnetic curvature [16], alongside the $k_{\parallel}$-symmetry breaking due to the $\mathbf{E}\times\mathbf{B}$ shear responsible for the residual stress [35], constitutes the unified "$\mathbf{B}^*$- symmetry breaking" as discussed in [16].

The nonlinear electrostatic gyrokinetic equation with proper conservation laws in general geometry is given by Eqs. (19), (21) and (22) of [31] :

$$\frac{\partial F}{\partial t}+\frac{d\mathbf{R}}{dt}\cdot\nabla F+\frac{dv_{\parallel}}{dt}\frac{\partial F}{\partial v_{\parallel}}=0 \tag{1}$$

with

$$\frac{d\mathbf{R}}{dt}=v_{\parallel}\frac{\mathbf{B}^*}{B^*}+\frac{c\mathbf{b}}{e_iB^*}\times[e_i\nabla\langle\langle\delta\phi\rangle\rangle]+m_i\mu\nabla B], \tag{2}$$

and

$$\frac{dv_\parallel}{dt} = -\frac{\mathbf{B}^*}{m_i B^*} \cdot [e_i \nabla \langle\langle \delta\phi \rangle\rangle + m_i \mu \nabla B]. \tag{3}$$

Here, the gyrokinetic Vlasov equation, Eq. (1) is written in terms of the gyro-center distribution function $F(\mathbf{R}, \mu, v_\parallel, t)$, with $\mu \equiv v_\perp^2/2B$, and $\langle\langle ... \rangle\rangle$ denotes an average over the gyrophase. $\mathbf{B}^*$ is defined by

$$\mathbf{B}^* \equiv \mathbf{B} + \frac{m_i c}{e_i} v_\parallel \nabla \times \mathbf{b}$$

We can derive the nonlinear evolution of the parallel momentum density per ion mass, $nU_\parallel \equiv 2\pi \int d\mu dv_\parallel B^* F v_\parallel$, by taking a moment of the nonlinear gyrokinetic equation, Eq. (1), or equivalently of a conservative form of the nonlinear gyrokinetic equation (Eq. (24) of [31]):

$$\frac{\partial (FB^*)}{\partial t} + \nabla \cdot \left(FB^* \frac{d\mathbf{R}}{dt} \right) + \frac{\partial}{\partial v_\parallel} \left(FB^* \frac{dv_\parallel}{dt} \right) = 0. \tag{4}$$

With the Mach number using the sound speed $M_s \equiv \frac{U_0}{C_s}$, we adopt an ordering $k_\theta \rho_s > \frac{a}{qR} M_s$, and assume $M_s < 1$ so that we can ignore $\mathbf{B} \cdot \nabla n U_\parallel^2$ in comparison to $\mathbf{B} \cdot \nabla P_\parallel$. The pressure moments are defined as usual. With these considerations, we can write a nonlinear evolution equation for the parallel momentum, by multiplying Eq. (4) by $v_\parallel$ and integrating over the velocity space, to obtain

$$\begin{aligned} \frac{\partial}{\partial t}(m_i n U_\parallel) = &-c\mathbf{b} \times \nabla \delta\phi \cdot \nabla (\frac{m_i n U_\parallel}{B}) - \frac{2 c m_i n U_\parallel}{B} \mathbf{b} \times (\mathbf{b} \cdot \nabla) \mathbf{b} \cdot \nabla \delta\phi \\ &- \frac{m_i c}{e_i} \mathbf{b} \times \nabla B \cdot \nabla (\frac{P_\perp U_\parallel}{B^2}) - 3 \frac{m_i c}{e_i} \mathbf{b} \times (\mathbf{b} \cdot \nabla) \mathbf{b} \cdot \nabla (\frac{P_\parallel U_\parallel}{B}) \\ &- n_i e_i \mathbf{b} \cdot \nabla \delta\phi - \mathbf{b} \cdot \nabla P_\parallel. \end{aligned} \tag{5}$$

The 2nd term on the RHS of Eq. (5) originates from the magnetic curvature modification of the parallel acceleration in Eq. (3). The last two terms are the origin of the $\mathbf{E} \times \mathbf{B}$ shear. This has been known to produce a nondiffusive radial flux of the parallel flow and reviewed in [1]. The physics of residual stress has been extensively discussed in [10]. Therefore, from this point, we don't keep these terms in this paper, which focuses only on the inward pinch driven by toroidal effects.

In addition, as identified in [16], the 3rd and 4th terms on the RHS of Eq. (5) are responsible for the geodesic curvature driven momentum flux which is subdominant to the standard $\mathbf{E}\times\mathbf{B}$ fluctuation induced momentum flux. It is also obvious that these terms vanish in the cold ion limit of $T_i/T_e \to 0$. An expression related to the magnetic curvature in the second term on the RHS of Eq. (5) can be expressed as

$$\mathbf{b}\times(\mathbf{b}\cdot\nabla)\mathbf{b} = (\nabla\times\mathbf{b})_\perp = \frac{\mathbf{b}}{B^2}\times\nabla\left(\frac{B^2}{2}+4\pi P\right) \tag{6}$$

using the MHD equilibrium condition $\frac{1}{c}\mathbf{J}\times\mathbf{B} = \nabla P$. Note that in our previous works on TEP momentum pinch in tokamaks [16], a low-β approximation has been used, dropping the last term proportional to ∇P. Then, Eq. (5) can be further reduced to

$$\frac{\partial}{\partial t}\delta(nU_\parallel) = -c\mathbf{b}\times\nabla\delta\phi\cdot\nabla\left(\frac{nU_\parallel}{B}\right) - 2\frac{cnU_\parallel}{B^3}\mathbf{b}\times\nabla\left(\frac{B^2}{2}+4\pi P\right)\cdot\nabla\delta\phi \tag{7}$$

The radial flux of angular momentum carried by fluctuating $\mathbf{E}\times\mathbf{B}$ flow is

$$\begin{aligned}\Pi_{Ang} &= \left\langle \sum_{\mathbf{k}} \delta(M_i n U_\parallel R)_{\mathbf{k}} \delta u^*_{\mathbf{E}_{\mathbf{k}}} \right\rangle \\ &= M_i R^2 \left\langle \sum_{\mathbf{k}} \delta(n\omega_\phi)_{\mathbf{k}} \delta u^*_{\mathbf{E}_{\mathbf{k}}} \right\rangle \end{aligned} \tag{8}$$

From now, we use the angular frequency of toroidal rotation, $\omega_\phi = u_\parallel/R$, which is usually a flux fluction, as the main variable, With this in mind, Eq. (7) can be written in the following form,

$$\frac{\partial}{\partial t}\delta(n\omega_\phi) = -\frac{c\mathbf{b}\times\nabla\delta\phi}{B}\cdot\left[\nabla(n\omega_\phi) - n\omega_\phi\left\{-\frac{B}{R}\nabla\left(\frac{R}{B}\right)+\frac{2}{B^2}\nabla\left(\frac{B^2}{2}+4\pi P\right)\right\}\right]$$
$$= -\delta\mathbf{u_E}\cdot\left[\nabla(n\omega_\phi) - n\omega_\phi\left\{\frac{R}{B^3}\nabla\left(\frac{B^3}{R}\right)+\frac{8\pi}{B^2}\nabla P\right\}\right] \qquad (9)$$

Eq. (9) shows that the fluctuations in $n\omega_\phi$ is not only driven by the radial gradient of $n\omega_\phi$, but also by the gradients of B^3/R and of P. We can further arrange it as follows:

$$\delta(n\omega_\phi) = -(-i\omega_{\mathbf{k}}+\triangle\omega_{\mathbf{k}})^{-1}\delta u_{\mathbf{r},\mathbf{k}}\left[\frac{\partial}{\partial r}(n\omega_\phi) - n\omega_\phi\left\{\frac{R}{B^3}\frac{\partial}{\partial r}\left(\frac{B^3}{R}\right)+\frac{8\pi}{B^2}\frac{\partial P}{\partial r}\right\}\right] \qquad (10)$$

From Eq. (9) and (10), we obtain the radial flux of angular momentum,

$$\Pi_{Ang,r}/M_iR^2 = \left\langle\sum_{\mathbf{k}}\delta(n\omega_\phi)\delta u_r^*\right\rangle = -\chi_{Ang}\frac{\partial}{\partial r}(n\omega_\phi)+V_\phi^{TEP}n\omega_\phi \qquad (11)$$

with the angular momentum diffusivity,

$$\chi_{Ang} = \left\langle\sum_{\mathbf{k}}(Re\tau_{\mathbf{k}})|\delta u_{\mathbf{r},\mathbf{k}}|^2\right\rangle, \qquad (12)$$

and the TEP pinch

$$V_{Ang}^{TEP} = \left\langle\sum_{\mathbf{k}}(Re\tau_{\mathbf{k}})|\delta u_{\mathbf{r},\mathbf{k}}|^2\left[\frac{R}{B^3}\frac{\partial}{\partial r}\left(\frac{B^3}{R}\right)+\frac{8\pi}{B^2}\frac{\partial P}{\partial r}\right]\right\rangle \qquad (13)$$

with $Re(\tau_{\mathbf{k}}) = Re\left(\frac{1}{-i\omega_{\mathbf{k}}+\triangle\omega_{\mathbf{k}}}\right)$.

An expression in Eq. (13) should be understood as a fluctuation intensity $\left(\propto|\ \delta\mathbf{u}_{\mathbf{r},\mathbf{k}}\ |^2\right)$ weighted flux surface average. Therefore, assuming that the fluctuation amplitude peaks strongly at the low field side, the final expression for the TEP momentum pinch satisfies the relation

$$V_{Ang}^{TEP}/\chi_{Ang} = \left(\frac{R}{B^3}\frac{\partial}{\partial r}\left(\frac{B^3}{R}\right)+\frac{8\pi}{B^2}\frac{\partial P}{\partial r}\right), \qquad (14)$$

where the expression can approximately be evaluated at the point of the low field side mid-plane for a given flux surface. Of course, the minus sign expected for typical equilibrium B and

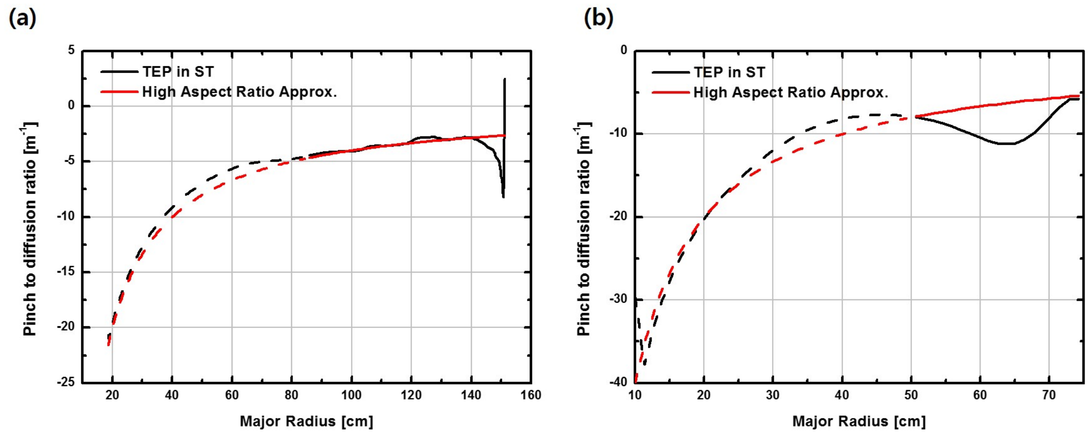

Figure 1: Pinch to diffusion ratio for typical equilibria of (a) NSTX ($\beta = 0.11, A = 1.27$)and (b) VEST ($\beta = 0.04, A = 1.25$). The black lines correspond to our new result in Eq. (14). The red lines are $-4/R$ from [16]. Here, only the solid line part corresponding to the low field side of a given flux surface should be considered.

P indicates an inward pinch. Note that Eq. (14) reduces to our previous expression of $-4/R$ in the limit of high aspect ratio and low-β [16]. Figure 1 illustrates the pinch to diffusion ratio expected from typical equilibria of NSTX and VEST spherical tori. For both examples, our new results indicate inward pinch values which are higher than those based on an earlier theory [16]. Significant difference exists for a wide range of radius for the VEST case, while more drastic localized difference is noted for the NSTX case with higher-β value. We also remind readers that the $\partial P/\partial r$ term in Eq. (14) shows up when we convert the magnetic curvature term $(\nabla \times \mathbf{b})_\perp$ contained in the $\mathbf{B}^*$ term in modern gyrokinetics [31], to $\frac{\mathbf{b}}{B^3} \times \nabla \cdot (\frac{B^2}{2} + 4\pi P)$. Therefore, we should still consider it as a part of TEP momentum pinch which originates from the magnetic field inhomogeneity. We also note that it's total P, not P_i which appears in an ITG-specific formula [32]. Since a global nonlinear gyrokinetic simulation capability for realistic ST

MHD equilibria already exists [36], we hope our theory can be compared to experiments and simulations in the near future.

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