Potential Vorticity Dynamics and Models of Zonal Flow Formation

Pei-Chun Hsu, Patrick Diamond

CASS, UCSD

TTF, April 28 – May 1, 2015
Outline

• Motivating issues
  – How to represent inhomogeneous PV mixing during relaxation processes?
  – How to calculate spatial PV flux?

• Non-perturbative analyses of the general structure of PV flux
  → structural approach
  – Minimum enstrophy model (selective decay of potential enstrophy)
    → viscous and hyper-viscous transport
  – PV-avalanche model (joint reflection symmetry)
    → K-S equation

• Perturbative analyses of transport coefficients in PV flu
  – Modulational instability, revisited
    → negative viscosity & positive hyper-viscosity
  – Parametric instability
    → Burgers’ equation

• Summary
Motivating Issues

- Real space structure of ZF is of practical interest for predictive transport modeling in quasi-2D turbulence. PV mixing in space is essential in ZF generation.

\[
\begin{align*}
\text{Taylor identity: } & \quad \left\langle \tilde{\nu}_y \nabla^2 \tilde{\phi} \right\rangle = -\partial_y \left\langle \tilde{\nu}_y \tilde{\nu}_x \right\rangle \\
& \quad \text{vorticity flux} \quad \text{Reynolds force}
\end{align*}
\]

- The relaxation dynamics is of fundamental importance in MHD and QG fluid. While Taylor's theory is successful in explaining some plasma experimental results, a relaxation model of vorticity transport is worth researching.

**Key points**

- turbulence self-organization complex — What are the general principles?
- PV flux as route to relaxed state — Is PV homogenization the case??
- zonal flow saturation — how?, especially collisionless cases?

**Generic problems**

- How to describe mean PV relaxation to a minimum enstrophy or SOC state?
**PV conservation:** \( \frac{dq}{dt} = 0 \)

<table>
<thead>
<tr>
<th>GFD: Quasi-geostrophic system</th>
<th>Plasma: Hasegawa-Wakatani system</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q = \nabla^2 \phi + \beta y )</td>
<td>( q = n - \nabla^2 \phi )</td>
</tr>
<tr>
<td>relative vorticity</td>
<td>guiding center (electron density)</td>
</tr>
<tr>
<td>planetary vorticity</td>
<td>polarization (ion density)</td>
</tr>
</tbody>
</table>

**Key:**
- How represent inhomogeneous PV mixing
- Relaxation Principles → General structure of PV flux
- Perturbation theory → Transport coefficients
Non-perturbative analyzes

i) Minimum enstrophy principle

Turbulent magnetic relaxation (J.B. Taylor, 1974)
- minimized magnetic energy subject to constant global magnetic helicity

\[
\delta \left[ \int d^3 x \frac{B^2}{8\pi} + \lambda \int d^3 x \vec{A} \cdot \vec{B} \right] = 0
\]

\[\Rightarrow \nabla \times \vec{B} = \mu \vec{B}\]

\[\Rightarrow \frac{\vec{J} \cdot \vec{B}}{B^2} = \text{const}\]

Taylor state:
- force free B field configuration
- Homogenized \(J_{||}\) profile

2D turbulence relaxation (Bretherton & Haidvogel 1976)
- minimized total enstrophy subject to constant total energy

\[
\delta \left[ \int d^2 x \frac{q^2}{2} + \lambda \int d^2 x \frac{\left(\nabla \phi\right)^2}{2} \right] = 0
\]

\[\Rightarrow q = \mu \phi\]

minimum enstrophy state:
- flow structure emergent

PV stream function
<table>
<thead>
<tr>
<th>conserved quantity (constraint)</th>
<th>total kinetic energy</th>
<th>global magnetic helicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>dissipated quantity (minimized)</td>
<td>fluctuation potential enstrophy</td>
<td>magnetic energy</td>
</tr>
<tr>
<td>final state</td>
<td>minimum enstrophy state</td>
<td>Taylor state</td>
</tr>
</tbody>
</table>

| structural approach | \( \frac{\partial}{\partial t} \Omega < 0 \Rightarrow \Gamma_E \Rightarrow \Gamma_q \) | \( \frac{\partial}{\partial t} E_M < 0 \Rightarrow \Gamma_H \) |

- Theory predict end state, but no dynamical insight
  -> flux? what can be said about dynamics?
  -> structural approach (Boozer)

\[
\frac{\partial}{\partial t} \int d^3x \frac{B^2}{8\pi} = -\int d^3x \left[ \eta J^2 - \Gamma_H \cdot \nabla \left( \frac{\langle J \rangle \cdot B}{B^2} \right) \right] < 0
\]

\[
\Rightarrow \Gamma_H = -\lambda \nabla \left( \frac{\langle J_{\parallel} \rangle}{B} \right)
\]
PV flux

→ PV conservation

mean field PV: \[
\frac{\partial \langle q \rangle}{\partial t} + \partial_y \langle \nu \cdot q \rangle = \nu_0 \partial_y^2 \langle q \rangle
\]

\( \Gamma_q \): mean field PV flux

Key Point: what form does PV flux have so that dissipate enstrophy, conserve energy

selective decay

→ energy conserved

\[
E = \int \frac{(\partial_y \langle \phi \rangle)^2}{2}
\]

\[
\frac{\partial E}{\partial t} = \int \langle \phi \rangle \partial_y \Gamma_q = -\int \partial_y \langle \phi \rangle \Gamma_q \quad \Rightarrow \Gamma_q = \frac{\partial_y \Gamma_E}{\partial_y \langle \phi \rangle}
\]

→ enstrophy minimized

\[
\Omega = \int \frac{\langle q \rangle^2}{2}
\]

\[
\frac{\partial \Omega}{\partial t} = -\int \langle q \rangle \partial_y \Gamma_q = -\int \partial_y \left( \frac{\partial_y \langle q \rangle}{\partial_y \langle \phi \rangle} \right) \Gamma_E
\]

\[
\frac{\partial \Omega}{\partial t} < 0 \Rightarrow \Gamma_E = \mu \partial_y \left( \frac{\partial_y \langle q \rangle}{\partial_y \langle \phi \rangle} \right) \quad \Rightarrow \Gamma_q = \frac{1}{\partial_y \langle \phi \rangle} \partial_y \left[ \mu \partial_y \left( \frac{\partial_y \langle q \rangle}{\partial_y \langle \phi \rangle} \right) \right]
\]
Structure of PV flux

\[
\Gamma_q = \frac{1}{\partial_y \langle \phi \rangle} \partial_y \left[ \mu \left( \partial_y \langle \phi \rangle \right) \right] = \frac{1}{\partial_y \langle \phi \rangle} \partial_y \left[ \mu \left( -\frac{\langle q \rangle}{(\partial_y \langle \phi \rangle)^2} + \frac{\partial^2_y \langle q \rangle}{\partial_y \langle \phi \rangle} \right) \right]
\]

Diffusion variable calculated by perturbation theory

Diffusion and hyper diffusion of PV

Relaxed state:

Homogenization of \( \frac{\partial_y \langle q \rangle}{\partial_y \langle \phi \rangle} \)

(Prandtl, Batchelor, Rhines, Young)

Homogenization of PV

critical scale

\[
\ell_c = \sqrt{\frac{\langle q \rangle}{\langle \phi \rangle}}
\]

\( \ell = \ell_c \) : \( \Gamma_q = 0 \) \( \rightarrow \) ZF growth rate zero

\( \ell > \ell_c \) : ZF energy growth

\( \ell < \ell_c \) : ZF energy damping
relaxed state: homogenization of \[ \frac{\partial_y \langle q \rangle}{\langle v_x \rangle} \]

→ Zonal flows track the PV gradient → PV staircase

- Highly structured profiles of the staircase is reconciled with the homogenization or mixing process required to produce it.
- Staircase may arise naturally as a consequence of minimum enstrophy relaxation.
The “minimum enstrophy”

- The relaxation rate can be derived by linear perturbation theory about the minimum enstrophy state

\[
\begin{align*}
\langle q \rangle &= q_m(y) + \delta q(y,t) \\
\langle \phi \rangle &= \phi_m(y) + \delta \phi(y,t) \\
\partial_y q_m &= \lambda \partial_y \phi_m \\
\delta q(y,t) &= \delta q_0 \exp(-\gamma_{rel} t - i\omega t + iky)
\end{align*}
\]

\[
\begin{align*}
\gamma_{rel} &= \mu \left( \frac{k^4 + 4\lambda k^2 + 3\lambda^2}{\langle v_x \rangle^2} - \frac{8q_m^2(k^2 + \lambda)}{\langle v_x \rangle^4} \right) \\
\omega &= \mu \left( -\frac{4q_m k^3 + 10q_m k \lambda}{\langle v_x \rangle^3} + \frac{8q_m^3 k}{\langle v_x \rangle^5} \right).
\end{align*}
\]

- The condition of relaxation (modes are damped):

\[
\gamma_{rel} > 0 \Rightarrow k^2 > \frac{8q_m^2}{\langle v_x \rangle^2} - 3\lambda,
\]

\[
k^2 > 0 \Rightarrow \frac{8q_m^2}{\langle v_x \rangle^2} > 3\lambda. \quad \Rightarrow q_m^2: \text{the ‘minimum enstrophy’ of relaxation}
\]

- A critical residual enstrophy density is needed in the minimum enstrophy state, so as to sustain a zonal flow of a certain level.
Non-perturbative analyzes

i) PV-avalanche model

\[ \delta q > 0 \]

\[ \delta q < 0 \]

Key Point: what form does PV flux have s/t satisfy joint-reflection symmetry principle

\[ q(y,t) : PV \text{ profile} \]

\[ q_0(y) : \text{self-organized state} \]

\[ \delta q = q - q_0 : \text{deviation} \]

\[ \Gamma[\delta q] : \text{PV flux} \]

\[ \Gamma[\delta q] \]

Joint-reflection symmetry: \( \Gamma[\delta q] \) invariant under \( y \rightarrow -y \) and \( \delta q \rightarrow -\delta q \)

\[ \Gamma[\delta q] = \sum_l \alpha_l (\delta q)^{2l} + \sum_m \beta_m (\partial_y \delta q)^m + \sum_n \gamma_n (\partial_y^3 \delta q)^n + \ldots \]

large-scale properties: higher-order derivatives neglected
small deviations: higher-order terms in \( \delta q \) neglected

Simplest approximation: \( \Gamma[\delta q] = \frac{\alpha}{2} (\delta q)^2 + \beta \partial_y \delta q + \gamma \partial_y^3 \delta q \)
PV equation: \[ \partial_t \delta q + \alpha \delta q \partial_y \delta q + \beta \partial_y^2 \delta q + \gamma \partial_y^4 \delta q = 0. \]

Avalanche-like transport is triggered by deviation of PV gradient
\[ \delta q \text{ implicitly related to the local PV gradient} \]
\[ \text{transport coefficients (functions of } \delta q \text{) related to the gradient} \]

Convective component of the PV flux can be related to a gradient-dependent effective diffusivity
\[ \Gamma_\delta \sim -D(\partial_y \delta q) \partial_y \delta q \rightarrow -D(\delta q) \delta q \]
\[ \Gamma[\delta q] \sim \delta q^2 \rightarrow -D(\delta q) \delta q, \text{ with } D(\delta q) \rightarrow D_0 \delta q \]
Perturbative analyses of PV flux

i) Modulational instability

- The evolution of perturbation (seed ZF) as a way to look at PV transport

Fluctuations
(broad spectrum)

Modulation
(nonlocal)

Inverse cascade
/scrambling
/PV mixing
(local)

Seed ZF

$\delta \omega$

N.B. Modulational instability (large scale pumping) requires small scale PV mixing $\leftrightarrow$ irreversibility
Revisiting Modulational Instability

ZF evolution determined by Reynolds force

\[
\frac{\partial}{\partial t} \delta V_x = -\frac{\partial}{\partial y} \langle \tilde{v}_x \tilde{u}_y \rangle = \frac{\partial}{\partial y} \sum_{k<} \frac{k_x k_y}{k^4} \tilde{N}_k
\]

vorticity flux

\[N_k = k^2 |\psi_k|^2 / \omega_k\] is wave action density, for Rossby wave it is proportional to the enstrophy density. \(N_k\) is determined by WKE:

\[
\frac{\partial \tilde{N}}{\partial t} + u_g \cdot \nabla \tilde{N} + \delta \omega_k \tilde{N} = \frac{\partial (k_x \delta V_x)}{\partial y} \frac{\partial N_0}{\partial k_y}
\]

→ Turbulent vorticity flux derived

\[
\frac{\partial}{\partial t} \delta V_q = -q^2 \delta V_q \sum_k \left( \frac{k_x^2 k_y}{k^4} \right) \frac{\delta \omega_k}{(\omega - q \cdot u_g)^2 + \delta \omega_k^2} \frac{\partial N_0}{\partial k_y}
\]

\(\kappa(q)\) ≠ const at larger \(q\)

→ scale dependence of PV flux

→ non-Fickian turbulent PV flux
• A simple model from which to view $\kappa(q)$:
  - Defining MFP of wave packets as the critical scale $q_c^{-1} \equiv \nu \delta \omega^{-1}_k$
  - coarse graining scales smaller than $q_c$
  - keeping next order term in expansion of response function

$$q^{-1} \gg q_c^{-1} \Rightarrow \frac{\delta \omega_k}{(q \nu_g)^2 + \delta \omega^2_k} \approx \frac{1}{\delta \omega_k} \left(1 - \frac{q^2}{q_c^2}\right)$$

capture expected and sensible trend!

→ zonal growth evolution:

$$\partial_t \delta V_x = -q^2 D \delta V_x + q^4 H \delta V_x$$

→ negative viscosity and positive hyper-viscosity

$D = \sum_k \frac{k_x^2}{\delta \omega_k k_y^4} \frac{\partial N_0}{\partial k_y} < 0$

$H = -\sum_k q_c^{-2} \frac{k_x^2}{\delta \omega_k k_y^4} \frac{\partial N_0}{\partial k_y} > 0$

(expected wave enstrophy spectrum statistics)

previous calculation of relaxation models:

$$\frac{\partial \langle v_x \rangle}{\partial t} = \Gamma_q = \frac{1}{\partial_y \langle \phi \rangle} \partial_y \left[ \mu \left( \frac{\langle q \rangle \partial_y \langle q \rangle}{(\partial_y \langle \phi \rangle)^2} + \frac{\partial^2_y \langle q \rangle}{\partial_y \langle \phi \rangle} \right) \right]$$

$$\Gamma[q] = \frac{\alpha}{2} (\delta q)^2 + \beta \partial_y \delta q + \gamma \partial_y^3 \delta q,$$
Discussion of $D$ and $H$

- **Roles of negative viscosity and positive hyper-viscosity** (*Real space*)

\[
\frac{\partial}{\partial t} \delta V_x = D\partial_y^2 \delta V_x - H\partial_y^4 \delta V_x
\]

\[
\frac{\partial}{\partial t} \int \frac{1}{2} \delta V_x^2 d^2 x = -D\int (\partial_y \delta V_x)^2 d^2 x - H\int (\partial_y^2 \delta V_x)^2 d^2 x
\]

- $D < 0 \Rightarrow \gamma_{q,D} > 0$  ZF growth (Pumper $D$)
- $H > 0 \Rightarrow \gamma_{q,H} < 0$  ZF suppression (Damper $H$)

$\Rightarrow$  $D$, $H$ as model of spatial PV/momentum flux beyond over-simplified negative viscosity

- $D = Hq^2$ sets the cut-off scale

\[
\Rightarrow l_c^2 = \frac{H}{|D|}
\]

$\ell > l_c$ : ZF energy growth  $\Rightarrow$  $D$ process dominates at large scale

$\ell < l_c$ : ZF energy damping  $\Rightarrow$  $H$ process dominates at small scale

**Minimum enstrophy model**

\[
\Gamma_q = \frac{1}{\partial_y \langle \phi \rangle} \partial_y \mu \partial_y \left( \frac{\partial_y \langle q \rangle}{\partial_y \langle \phi \rangle} \right) \Rightarrow \ell_c \equiv \left( \frac{\partial_y \langle q \rangle}{\partial_y \langle \phi \rangle} \right)^{-\frac{1}{2}}
\]
Perturbative analyses
ii) Parametric instability

### Pseudo-fluid (wave packets) model

<table>
<thead>
<tr>
<th></th>
<th>Pseudo-fluid</th>
<th>Molecular fluid</th>
</tr>
</thead>
<tbody>
<tr>
<td>distribution function</td>
<td>$N_k$</td>
<td>$f$</td>
</tr>
<tr>
<td>scale</td>
<td>$v_g$/$\delta \omega_k$</td>
<td>mean free path</td>
</tr>
<tr>
<td>density</td>
<td>$n^w = \int N_k dk$</td>
<td>$n = \int f dv$</td>
</tr>
<tr>
<td>momentum</td>
<td>$p^w = \int kN_k dk$</td>
<td>$p = \int mv f dv$</td>
</tr>
<tr>
<td>energy</td>
<td>$E^w = \int \omega_k N_k dk$</td>
<td>$E = \int \frac{1}{2}mv^2 f dv$</td>
</tr>
<tr>
<td>velocity</td>
<td>$V^w = \frac{\int v_g N_k dk}{\int N_k dk}$</td>
<td>$V = \frac{\int v f dv}{\int f dv} = \frac{p}{mn}$</td>
</tr>
</tbody>
</table>

mean free path of wave packets
-- pseudo-fluid evolution:
 multiplying the WKE by \( v_{gy} \) and integrating over \( k \)
 normalizing by pseudo-density \( n^\omega \)

\[
\Rightarrow \frac{\partial}{\partial t} V^w_y + V^w_y \frac{\partial}{\partial y} V^w_y = -a \langle v_x \rangle \]

inviscid Burgers’ eq.
 source: zonal shear

\[
a = \int \frac{2\beta k_x^2}{k^4} \left(1 - \frac{4k_y^2}{k^2}\right) N_k dk / \int N_k dk
\]

-- ZF evolution:

\[
\frac{\partial}{\partial t} \langle v_x \rangle = -\frac{\partial}{\partial y} V^w_y P^w_x
\]

\( P^\omega_x \) evolves in the same way as \( \langle v_x \rangle \)

\[
\int k_x WKE \ d^2 k \Rightarrow \frac{\partial}{\partial t} \int k_x N_k d^2 k = -\frac{\partial}{\partial y} \int v_{gy} k_x N_k d^2 k
\]

• ZF growth rate in monochromatic limit:
 linearizing the above two eqs.

\[
\gamma_q = \sqrt{q^2 k_x^2 |\varphi_k|^2 \left(1 - \frac{4k_y^2}{k^2}\right)}
\]

The reality of \( \gamma_q \) requires \( k_x^2 > 3k_y^2 \)

\[
\gamma_q \propto |q| \text{ indicates convective instability}
\]
\[ \frac{\partial \langle v_x \rangle}{\partial t} = \Gamma_q = \frac{1}{\partial_y \langle \phi \rangle} \partial_y \left[ \mu \left( -\frac{\langle q \rangle \partial_y \langle q \rangle}{(\partial_y \langle \phi \rangle)^2} + \frac{\partial^2 \langle q \rangle}{\partial_y \langle \phi \rangle} \right) \right] \]

- **Minimum enstrophy**

- **PV-avalanche**

- **Modulational instab.**

- **Parametric instab.**

\[ \gamma_q = \sqrt{q^2 k_x \varphi_k \left( 1 - \frac{4k_y^2}{k^2} \right)} \]

<table>
<thead>
<tr>
<th>PV flux</th>
<th>convective</th>
<th>viscous</th>
<th>hyper-viscous</th>
<th>coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>(non-perturb.)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Min. enstrophy relaxation</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PV-avalanche relaxation</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(perturbative)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Modulational instability</td>
<td></td>
<td></td>
<td></td>
<td>(D_t(&lt;0), H_t(&gt;0))</td>
</tr>
<tr>
<td>Parametric instability</td>
<td></td>
<td></td>
<td></td>
<td>(\gamma_q(\sim</td>
</tr>
</tbody>
</table>
Summary

• Inhomogeneous PV mixing is identified as the fundamental mechanism for zonal flow formation. This study offered new perspectives and approaches to calculating spatial flux of PV. The structure of PV flux is studied by non-perturbative relaxation principles and perturbative analyses of wave kinetic equation. These are synergetic and complementary approaches.

• Vorticity flux from selective decay of enstrophy shows complex structure of diffusion and higher order diffusion terms. The homogenized quantity in the minimum enstrophy state is the ratio of PV gradient to zonal flow velocity. This is consistent with the structure of the PV staircase.

• Vorticity flux from PV-avalanche model is constrained by the joint reflection symmetry condition, and contains diffusive, hyper-diffusive, and convective terms. The convective transport of PV can be generalized to an effective diffusive transport.

• Transport coefficients are derived using perturbation theory. Both relaxation principle and perturbation theory reveal some critical scales at which ZF growth and ZF damping are equal.